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THE MAXIMAL IC-COLORINGS OF $K_{1,2,n}$

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ABSTRACT. The IC-index of a connected graph G is denoted by $M(G)$. In this paper, we prove that $M(K_{1,2,n}) = 13 \cdot 2^{n-1} + 1$, where $n \geq 2$ and give all the maximal IC-colorings of $K_{1,2,n}$.

1. Introduction

If f is a positive integer-valued function on the vertex set $V(G)$ of a connected graph G, if $S_f(H)$ denotes the sum $\sum_{u \in V(H)} f(u)$ for IC-subgraphs (induced connected subgraphs) H of G, and if for each integer α with $1 \leq \alpha \leq S_f(G)$ there is an IC-subgraph H of G such that $\alpha = S_f(H)$, then f is called an IC-coloring of G. An IC-coloring of a connected graph G is maximal if it maximizes $S_f(G)$. The IC-index of a connected graph G is the integer $M(G) = S_f(G)$ where f is any maximal IC-coloring of G.

The problem of finding IC-indices and IC-colorings of finite graphs was introduced by Salehi et al. in 2005 [8], and it can be considered as a derived problem of the postage stamp problem in number theory, which has been extensively studies [1, 2, 3, 4, 6]. Penrice proved that $M(K_n) = 2^n - 1$ and $M(K_{1,n}) = 2^n + 2$ for $n \geq 2$ [7]. Salehi et al. showed that $M(K_{2,n}) = 3 \cdot n^2 + 1$ for $n \geq 2$ [8]. Shiue and Fu proved $M(K_{m,n}) = 3 \cdot 2^{m+n-2} - 2^{m-2} + 2$ for $n \geq m \geq 2$ [9]. Liu and Lee showed that $M(K_{1,1,n}) = 3 \cdot 2^n + 1$ for $n \geq 1$ [5]. In this paper, we prove that $M(K_{1,2,n}) = 13 \cdot 2^{n-1} + 1$, where $n \geq 2$, and give all the maximal IC-colorings of $K_{1,2,n}$.

For convenience' sake, we shall restrict our discussion to the complete tripartite graphs $K_{1,2,n}$. It is useful to consider both concepts of sequences of numbers x_1, x_2, $\ldots$ and partial sums $s_0, s_1, s_2, \ldots$, where $s_0 = 0$, $s_i = x_1 + \cdots + x_i$ and $x_i = s_i - s_{i-1}$ for $i = 1, 2, \ldots$. Roughly speaking, finding $M(K_{1,2,n})$ is equivalent to maximizing $s_{n+3} = x_1 + \cdots + x_{n+3}$ subject to the sequences of positive integers $x_1, \ldots, x_3$ with some constraints.

The rest of the paper is organized as follows. In section 2, we introduce the notations that reformulate our notions in terms of number theory as well as graph theory. This section also gives the lower bounds of s_i, in particular, of s_{n+3} by an example of IC-coloring of $K_{1,2,n}$. In section 3, we study some basic properties of colorings, which are also true for any one-to-one coloring of any finite connected graph (see 3.2 and 3.3). In section 4, we discuss the necessary conditions for maximal IC-colorings of $K_{1,2,n}$ and list all the possibilities. In section 5, we give the maximal IC-colorings of $K_{1,2,n}$ which shows that $M(K_{1,2,n}) = 13 \cdot 2^{n-1} + 1$ for $n \geq 2$.

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2. Notations and definitions

We shall fix the following notations and definitions throughout this paper.

Notations.

(a) f is a positive integer-valued function on G;
(b) $G = P \cup Q \cup R$ where P, Q and R are disjoint sets of cardinalities $|P| = 1$, $|Q| = 2$, and $|R| = n$ with $2 \leq n < \infty$;
(c) $\mathcal{B}$ is the collection of all nonempty subsets H of G such that $H \not\subseteq P$, $H \not\subseteq Q$, and $H \not\subseteq R$ whenever $|H| \geq 2$;
(d) $\sim$ is a relation on G such that $u \sim v$ if and only if $\{u, v\} \subseteq P$ or $\{u, v\} \subseteq Q$ or $\{u, v\} \subseteq R$;
(e) S_f is the function on $\mathcal{B}$ defined by $S_f(H) = \sum_{u \in H} f(u)$ for all $H \in \mathcal{B}$.

When f is one-to-one, that is, $f(u) \neq f(v)$ if u, $v \in G$ and $u \neq v$, by identifying u with $f(u)$ for all $u \in G$, we shall write

(f) $f = \langle P, Q, R \rangle$ where P, Q and R are disjoint sets of positive integers;
(g) $G = \{x_1, x_2, \ldots, x_{n+3}\}$ where $G = P \cup Q \cup R$ and $0 < x_1 < x_2 < \cdots < x_{n+3} < \infty$;
(h) $S_f(H) = \sum_{x \in H} x$;
(i) $s_0 = 0$, $s_i = x_1 + \cdots + x_i$ for $1 \leq i \leq n + 3$;
(j) $x_{n+4} = \infty$;
(k) $f^+ = \{x_i \in G \mid x_i = s_{i-1} + 1\} = \{x_{i_1}, x_{i_2}, \ldots, x_{i_{|f^+|}}\}$ where $0 < x_{i_1} < x_{i_2} < \cdots < x_{i_{|f^+|}} < \infty$.

Intuitively, f is a *coloring* on a *complete tripartite graph* $G = K_{1,2,n}$ which has three *partite sets* P, Q and R, $\mathcal{B}$ is the collection of all IC-*subgraphs* (*induced connected subgraphs*) of G, $\{u, v\} \in \mathcal{B}$ means that u and v are *adjacent*, and $u \sim v$ if and only if u and v are *in the same partite set*. We shall use these terminologies freely.

Definitions.

(a) We say that f *produces* α if $\alpha = S_f(H)$ for some $H \in \mathcal{B}$;
(b) We call f an IC-*coloring* of G if f produces all the integers α with $1 \leq \alpha \leq S_f(G)$;
(c) An IC-coloring f of G is *maximal* if it maximizes $S_f(G)$, that is, $S_f(G) = \max\{S_g(G) \mid g \text{ is an IC-coloring of } G\}$;
(d) The IC-*index* of G is the integer $M(G) = S_f(G)$ where f is any maximal IC-coloring of G.

The following example illustrates some notations and definitions introduced above.

Example. Let $P = \{x_2\}$, $Q = \{x_3, x_4\}$, and $R = \{x_1, x_5, \ldots, x_{n+3}\}$, where $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 8$, $x_5 = 12$, $x_i = 13 \cdot 2^{i-5}$ $(6 \leq i \leq n + 3)$. Then $f = \langle P, Q, R \rangle$ represents a one-to-one coloring on the complete tripartite graph $G = P \cup Q \cup R$, and $P \in \mathcal{B}$, $Q \notin \mathcal{B}$, $x_1 \sim x_i$ $(5 \leq i \leq n + 3)$. We shall show that the following are true:

(a) $G = \{x_1, x_2, \ldots, x_{n+3}\}$, $0 < x_1 < x_2 < \cdots < x_{n+3} < x_{n+4} = \infty$.
(b) $s_0 = 0$, $s_1 = 1$, $s_2 = 3$, $s_3 = 7$, $s_4 = 15$, $s_i = 13 \cdot 2^{i-4} + 1$ $(5 \leq i \leq n + 3)$.
(c) $f^+ = \{x_1, x_2, x_3, x_4\}$, $i_1 = 1$, $i_2 = 2$, $i_3 = 3$, and $i_4 = 4$.

(d) $x_i \le s_{i-1} + 1$ $(1 \le i \le n+3)$.
(e) $0 = s_0 < s_1 < \cdots < s_{n+3} = S_f(G) = 13 \cdot 2^{n-1} + 1$.
(f) f produced α for all integer α with $1 \le \alpha \le S_f(G)$.
(g) f is an IC-coloring of G.

It is clear that (a)-(e) are true, and that (f) implies (g), so we need only prove (f). To see (f), let α be an integer such that $1 \le \alpha \le S_f(G)$. Then by (e), there is a unique j_1 such that $s_{j_1-1}+1 \le \alpha \le s_{j_1}$, it follows from this and, by (d) with $i = j_1$, $x_{j_1} \le s_{j_1-1}+1$ that $0 \le \alpha - x_{j_1} \le s_{j_1-1}$. If $\alpha - x_{j_1} > 0$, then, by (d) and (e) again, there is a unique j_2, $j_2 < j_1$, such that $s_{j_2-1}+1 \le \alpha - x_{j_1} \le s_{j_2}$ and $x_{j_2} \le s_{j_2-1}+1$, so that $0 \le \alpha - x_{j_1} - x_{j_2} \le s_{j_2-1}$. Since $s_0 = 0$, by continuing in this way if necessary, we obtain $\alpha = x_{j_1} + \cdots + x_{j_r}$ for some integer $1 \le j_r < \cdots < j_2 < j_1 \le n+3$ with $r \ge 1$. Let $H = \{x_{j_1}, \ldots, x_{j_r}\}$. If $r = 1$ then $H \in \mathcal{B}$ and $\alpha = S_f(H)$, so that f produced α and we are done. Now assume that $r > 1$. If $H \subseteq Q$ then $\alpha = x_3 + x_4 = x_5 = S_f(\{x_5\})$ so that f produces α; if $H \subseteq R$ and if x_i is the smallest integer in H other than x_1, then $i \ge 5$, so that, by $x_5 = x_3 + x_4$ and $x_i = s_{i-1} - 1 = x_2 + \cdots + x_{i-1}$ for $6 \le i \le n+3$, we have $\alpha = S_f(Q \cup (H - \{x_5\}))$ if $i = 5$ and $\alpha = S_f((H - \{x_i\}) \cup \{x_2, x_3, \ldots, x_{i-1}\})$ if $6 \le i \le n+3$; finally, if $H \not\subseteq Q$ and $H \not\subseteq R$, then $H \in \mathcal{B}$ for $H \not\subseteq P$ ($|H| = r > 1$ and $|P| = 1$), so that $\alpha = S_f(H)$ can be produced by f.

Remark. The example shows that $M(G) = M(K_{1,2,n}) \ge 13 \cdot 2^{n-1} + 1$.

3. One-to-one IC-colorings

Proposition 3.1. *Let f be a coloring of G. Then:*

(a) $|\mathcal{B}| = 2^{n+2} + 2^{n+1} + 2^n + n - 1$.
(b) *If $H_1, K_1, \ldots, H_m, K_m$ are $2m$ distinct members of $\mathcal{B}$ such that $S_f(H_i) = S_f(K_i)$ for all $1 \le i \le m$, then*
  (i) $S_f(\mathcal{B} - \{K_1, \ldots, K_m\}) = S_f(\mathcal{B})$.
  (ii) $|\mathcal{B}| - m \ge |S_f(\mathcal{B})|$.
  (iii) $|\mathcal{B}| - m \ge S_f(G)$, *if f is an IC-coloring.*
(c) *If $f(u) = f(v)$, $u \ne v$, and if $\mathcal{A} = \{A \subseteq (G - \{u, v\}) \mid A \cup \{u\} \in \mathcal{B}$ and $A \cup \{v\} \in \mathcal{B}\}$, then exactly one of the following five statements holds:*
  (i) $\{u, v\} \subseteq Q$ *with* $|\mathcal{A}| = 2^{n+1}$.
  (ii) $\{u, v\} \subseteq R$ *with* $|\mathcal{A}| = 2^n + 2^{n-1} + 2^{n-2} + 1$.
  (iii) $P \cap \{u, v\} \ne \varnothing$ *and* $Q \cap \{u, v\} \ne \varnothing$ *with* $|\mathcal{A}| = 2^n + 2^{n-1} + \cdots + 1$.
  (iv) $P \cap \{u, v\} \ne \varnothing$ *and* $R \cap \{u, v\} \ne \varnothing$ *with* $|\mathcal{A}| = 2^n + 2^{n-1} + 1$.
  (v) $Q \cap \{u, v\} \ne \varnothing$ *and* $R \cap \{u, v\} \ne \varnothing$ *with* $|\mathcal{A}| = 2^n + 2^{n-1}$.
(d) *If f is an IC-coloring of G and if $S_f(G) \ge 2^{n+2} + 2^n + 2^{n-1} + n$, then f is one-to-one.*
(e) *If f is a maximal IC-coloring of G then f is one-to-one.*

Proof. (a) Let $\mathcal{B}_1 = \{H \in \mathcal{B} \mid P \subseteq H\}$ and $\mathcal{B}_2 = \{H \in \mathcal{B} \mid P \not\subseteq H\}$. Then $|\mathcal{B}_1| = 2^{|Q \cup R|} = 2^{n+2}$ and, according to whether $|H| = 1$ or not, $|\mathcal{B}_2| = (|Q| + |R|) + (2^{|Q|} - 1) \cdot (2^{|R|} - 1) = (2 + n) + (2^2 - 1) \cdot (2^n - 1)$. Now the desired identity follows from $|\mathcal{B}| = |\mathcal{B}_1| + |\mathcal{B}_2|$.

(b) By assumption, S_f maps $\mathcal{B} - \{K_1, \ldots, K_m\}$ onto its range $S_f(\mathcal{B})$, so that (i) holds. (ii) follows from (i) for S_f may not be one-to-one. If f is an IC-coloring, then $S_f(\mathcal{B}) = \{1, 2, \ldots, S_f(G)\}$, so that $|S_f(\mathcal{B})| = S_f(G)$, and, by (ii), (iii) is true.

(c) If $u \sim v$ then $\{u, v\} \subseteq Q$ or $\{u, v\} \subseteq R$ for $|P| = 1$, and if $\{u, v\} \in \mathcal{B}$ (the negation of $u \sim v$) then u and v belong to different partite sets, so that we have five cases to discuss:

Case 1. $\{u, v\} \subseteq Q$.
Then $|\mathcal{A}| = 2^{|P \cup R|} = 2^{n+1}$ and (i) follows.

Case 2. $\{u, v\} \subseteq R$.
Then $|\mathcal{A}| = (2^{|P \cup Q|} - 1) \cdot 2^{|R - \{u, v\}|} + |\{\varnothing\}| = (2^3 - 1) \cdot 2^{n-2} + 1$ and (ii) follows.

Case 3. $P \cap \{u, v\} \neq \varnothing$ and $Q \cap \{u, v\} \neq \varnothing$.
Then $|\mathcal{A}| = 2^{|Q|-1} \cdot (2^{|R|} - 1) + |\{\varnothing\}| = 2^1 \cdot (2^n - 1) + 1$ and (iii) follows.

Case 4. $P \cap \{u, v\} \neq \varnothing$ and $R \cap \{u, v\} \neq \varnothing$.
Then $|\mathcal{A}| = (2^{|Q|} - 1) \cdot 2^{|R|-1} + |\{\varnothing\}| = (2^2 - 1) \cdot 2^{n-1} + 1$ and (iv) follows.

Case 5. $Q \cap \{u, v\} \neq \varnothing$ and $R \cap \{u, v\} \neq \varnothing$.
Let $\mathcal{A}_1 = \{A \in \mathcal{A} \,|\, P \subseteq A\}$ and $\mathcal{A}_2 = \{A \in \mathcal{A} \,|\, P \not\subseteq A\}$. Then $|\mathcal{A}_1| = 2^{|Q|-1} \cdot 2^{|R|-1} = 2^{2-1} \cdot 2^{n-1}$ and $|\mathcal{A}_2| = (2^{|Q|-1} - 1) \cdot (2^{|R|-1} - 1) + |\{\varnothing\}| = (2^{2-1} - 1)(2^{n-1} - 1) + 1$. Thus $|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2| = 2^n + 2^{n-1}$ and (v) follows.

(d) If f is an IC-coloring of G such that $f(u) = f(v)$ for some distinct u and v in G, that is, we assume that f is not one-to-one. Then, by (c), $|\mathcal{A}| \geq 2^n + 2^{n-1}$. If $\mathcal{A} = \{A_1, \ldots, A_m\}$, where $m = |\mathcal{A}|$, and if $H_i = A_i \cup \{u\}$ and $K_i = A_i \cup \{v\}$ for $1 \leq i \leq m$, then, by (a) and (b), $S_f(G) \leq |\mathcal{B}| - m \leq (2^{n+2} + 2^{n+1} + 2^n + n - 1) - (2^n + 2^{n-1})$. Thus $S_f(G) > 2^{n+2} + 2^n + 2^{n-1} + n - 1$ is impossible.

(e) The example in the previous section shows that if f is maximal then $S_f(G) \geq 13 \cdot 2^{n-1} + 1 (= 2^{n+2} + 2^{n+1} + 2^{n-1} + 1)$, so that $S_f(G) - (2^{n+2} + 2^n + 2^{n-1} + n) \geq 2^n - n + 1 \geq 0$, thus, by (d), f is one-to-one.

□

Remark. Our objective is to obtain the IC-index $M(G) = M(K_{1,2,n})$, thus we want to find a maximal IC-coloring of G. Because of (e) in the above theorem, we shall only consider those one-to-one coloring $f = \langle P, Q, R \rangle$ of G in the rest of this paper.

Proposition 3.2. *Let $f = \langle P, Q, R \rangle$ be a coloring of G and α be an integer. Then:*

(a) *If $s_{j-1} < \alpha < x_{j+1}$ for some $1 \leq j \leq n+3$, then x_j must be used in producing α, that is, if $\alpha = S_f(H)$ then $x_j \in H$. (Note $x_{n+4} = \infty$.)*

(b) *If $s_{j-1} < x_i + x_j < x_{j+1}$ for some $1 \leq i < j \leq n+3$, and $S_f(H) = x_i + x_j$, then*
  (i) *if $x_i > s_{i-1}$ then $H = \{x_i, x_j\}$;*
  (ii) *if $x_i = s_{j-1}$ then $H = \{x_i, x_j\}$ or $H = \{x_1, \ldots, x_{i-1}\} \cup \{x_j\}$.*

(c) *If* $s_j \le \alpha$ *for some* $1 \le j \le n+3$, *and if* $j \le k \le n+3$, *then*
   (i) *if* $x_i \le s_{i-1}$ *for all* $j \le i \le k$ *then* $s_k \le \alpha \cdot 2^{k-j}$*;*
   (ii) *if* $x_i \le s_{i-1}+1$ *for all* $j \le i \le k$ *then* $s_k \le (\alpha+1)\cdot 2^{k-j}-1$.

Proof. (a) $s_{j-1} < \alpha < x_{j+1}$ implies that $H \not\subseteq \{x_1, \ldots, x_{j-1}\}$ and $H \subseteq \{x_1, \ldots, x_j\}$.

(b) By (a), x_j must be used in producing $x_i + x_j$. As $S_f(H) = x_i + x_j$, and as $x_i < x_{i+1} < \cdots < x_{j-1}$, we see that $H \subseteq \{x_1, \ldots, x_i\} \cup \{x_j\}$.
   (i) If $x_i > s_{i-1}$ then $S_f(H) > s_{i-1} + x_j$ and we must have $H = \{x_i, x_j\}$;
   (ii) $x_i = s_{i-1}$ then $S_f(H) = x_i + x_j = (x_1 + \cdots + x_{i-1}) + x_j$, so that $H = \{x_i, x_j\}$ or $H = \{x_1, \ldots, x_{i-1}\} \cup \{x_j\}$.

(c) (i) If $x_i \le s_{i-1}$ $(j \le i \le k)$ then $s_i = s_{i-1} + x_i \le 2 \cdot s_{i-1}$ $(j \le i \le k)$ and $s_j \le \alpha$, from this recursive relation, it follows that $s_k \le \alpha \cdot 2^{k-j}$.
   (ii) If $x_i \le s_{i-1}+1$ $(j \le i \le k)$ then $s_i + 1 = s_{i-1} + x_i + 1 \le 2\cdot(s_{i-1}+1)$ $(j \le i \le k)$ and $s_j + 1 \le \alpha + 1$, it follows that $s_k + 1 \le (\alpha+1)\cdot 2^{k-j}$ so that $s_k < (\alpha+1)\cdot 2^{k-j}$.

□

Proposition 3.3. *Let* $f = \langle P, Q, R\rangle$ *be an IC-coloring of* G*. Then:*

(a) $x_i \le s_{i-1} + 1$ *for all* $1 \le i \le n+3$.

(b) *If* $x_i \in f^+$, *if* $x_j \ge s_{j-1}$, *and if* $x_i \sim x_j$ *for some* $1 \le i < j \le n+3$, *then* $x_{j+1} \le x_i + x_j$.

(c) *If* $x_i \in f^+$ *and if* $x_i \sim x_j$ *for some* $1 \le i < j \le n+3$, *then either* $s_j \le 2\cdot s_{j-1} - x_i$ *or* $s_{j+1} \le 3\cdot s_{j-1} + 2 + x_i$.

(d) *If* α *is an integer and if* $s_j < \alpha$ *for some* $1 \le j \le n+3$, *then* $S_f(G) < \alpha \cdot 2^{n+3-j}$.

Proof. (a) Suppose, to get a contradiction, that $x_j > s_{j-1}+1$ for some $1 \le j \le n+3$, then $s_{j-1} < s_{j-1}+1 < x_{j+1}$, so that, by 3.2(a), x_j should be used in producing $s_{j-1}+1$, which contradicts $x_j > s_{j-1}+1$.

(b) Suppose not, we would have, $x_i = s_{i-1}+1$, $x_j \ge s_{j-1}$, $x_i \sim x_j$, and $x_{j+1} > x_i + x_j$, so that $x_i > s_{i-1}$ and $s_{j-1} < x_i + x_j < x_{j+1}$, thus, by 3.2(b)(i), $\{x_i, x_j\} \in \mathcal{B}$ should hold for f is an IC-coloring, which would violate $x_i \sim x_j$.

(c) The contrapositive of 3.2(b)(i) shows that either $x_i + x_j \le s_{j-1}$ or $x_{j+1} \le x_i + x_j$. The first inequality implies that $s_j = s_{j-1} + x_j \le s_{j-1} + (s_{j-1} - x_i) = 2\cdot s_{j-1} - x_i$. The second inequality and (a) imply that $s_{j+1} = s_{j-1} + x_j + x_{j+1} \le s_{j-1} + x_j + (x_i + x_j) \le s_{j-1} + (s_{j-1}+1) + x_i + (s_{j-1}+1) = 3\cdot s_{j-1} + 2 + x_i$.

(d) If α is an integer, $s_j < \alpha$ if and only if $s_j \le \alpha - 1$, so that, by (a) and 3.2(c)(ii) with $k = n+3$ and replacing α by $\alpha - 1$, $S_f(G) = s_{n+3} \le \alpha \cdot 2^{n+3-j} - 1 < \alpha \cdot 2^{n+3-j}$.

□

4. Necessary conditions for maximal IC-colorings

Proposition 3.3(a) shows that, for each $1 \le i \le n+3$, $s_{i-1}+1$ is an upper bound for x_i if f is an IC-coloring. An integer $x_i \in G$ with the property that $x_i = s_{j-1}+1$ if and only if $x_i \in f^+$. In the following, we shall sometimes denote the members of f^+ by boldfaced integers. Recall that $f^+ = \{x_{i_1}, x_{i_2}, \ldots, x_{|f^+|}\}$, where $0 < x_{i_1} < x_{i_2} < \cdots < x_{|f^+|} < \infty$.

Proposition 4.1. *Let f be a maximal IC-coloring. Then:*

(a) $s_i \geq 13 \cdot 2^{i-4}$ *for all* $4 \leq i \leq n+3$.
(b) $(x_1, x_2) = (\mathbf{1}, \mathbf{2})$ *and* $x_3 = 3$ *or* $\mathbf{4}$.
(c) (i) *If* $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, 3)$ *then* $x_4 = \mathbf{7}$ *and* $13 \leq x_5 \leq \mathbf{14}$.
 (ii) *If* $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, \mathbf{4})$ *then* $6 \leq x_4 \leq \mathbf{8}$.
(d) (i) *If* $(x_1, x_2, x_3, x_4) = (\mathbf{1}, \mathbf{2}, 3, \mathbf{7})$ *then* $\{x_1, x_3\} \in \mathcal{B}$, $\{x_2, x_3\} \in \mathcal{B}$, $\{x_1, x_4\} \in \mathcal{B}$ *and* $\{x_2, x_4\} \in \mathcal{B}$.
 (ii) *If* $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, \mathbf{4})$ *then* $\{x_1, x_2\} \in \mathcal{B}$ *and* $\{x_1, x_3\} \in \mathcal{B}$.

Proof. (a) If not, then $s_i < 13 \cdot 2^{j-4}$ for some $4 \leq j \leq n+3$, by 3.3(d), we would have $S_f(G) < 13 \cdot 2^{n-1} < M(K_{1,2,n})$.

(b) follows from $s_0 = 0$, $0 < x_1 < x_2 < x_3$ and $x_i \leq s_{i-1} + 1$ for i =1, 2, 3 (3.3(a)).

(c) (i) We have $s_3 = 6$ and, by (a), $s_4 \geq 13$. As $x_4 = s_4 - s_3$ and $x_4 \leq s_3 + 1$, $x_4 = \mathbf{7}$. Similarly, $s_4 = 13$, $s_5 \geq 26$, $x_5 = s_5 - s_4$ and $x_5 \leq s_4 + 1$ imply $13 \leq x_5 \leq \mathbf{14}$.
 (ii) We have $s_3 = 7$. Again, from $s_4 \geq 13$, $x_4 = s_4 - s_3$ and $x_4 \leq s_3 + 1$, we have $6 \leq x_4 \leq \mathbf{8}$.

(d) We observe that $x_i > s_{i-1}$ for $i = 1$, 2.
 (i) As $s_2 < x_i + x_3 < x_4$ and as $s_3 < x_i + x_4 < x_5$ for $i = 1$, 2, we have, by 3.2(b)(i), $\{x_i, x_j\} \in \mathcal{B}$ for $1 \leq i \leq 2$ and $3 \leq j \leq 4$.
 (ii) As $s_1 < x_1 + x_2 < x_3$ and $s_2 < x_1 + x_3 < x_4$, we have, $\{x_1, x_k\} \in \mathcal{B}$ for $2 \leq k \leq 3$.

□

Proposition 4.2. *Let f be a maximal IC-coloring. Then* $|f^+| \geq 4$.

Proof. To get a contradiction, we assume $|f^+| \leq 3$. According to 4.1, we have two cases to discuss.

Case 1. $(x_1, x_2, x_3, x_4) = (\mathbf{1}, \mathbf{2}, 3, \mathbf{7})$.
Then $s_4 = 13$. As $|f^+| \leq 3$, by 3.2(c)(i) with $j = 4$ and $k = n+3$, we would have $S_f(G) = s_{n+3} \leq 13 \cdot 2^{n-1} < M(K_{1,2,n})$, and f could not be maximal.

Case 2. $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, \mathbf{4})$ and $6 \leq x_4 \leq \mathbf{8}$.
If $x_4 = 6$ then $s_4 = 13$, as discussed above, f could not be maximal. If $x_4 = 8$ then $|f^+| \geq 4$ violating our assumption. Thus $(x_1, x_2, x_3, x_4) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, 7)$ should be true. We claim that $x_5 \leq 11$. If not, we would have $s_3 < x_i + x_4 < x_5$, so that, by 3.2(b)(i), $\{x_i, x_4\} \in \mathcal{B}$ for all $1 \leq i \leq 3$, and there would be *at least two distinct members of x_1, x_2, x_3 in the same partite set*; on the other hand, $s_1 < x_1 + x_2 < x_3$, $s_2 < x_k + x_3 < x_4$ $(1 \leq k \leq 2)$ would imply $\{x_i, x_j\} \in \mathcal{B}$ for all $1 \leq i < j \leq 3$, so that x_1, x_2, x_3 *should be in different partite sets.* Thus $x_5 \leq 11$ should be true. But then $s_5 < 26$ and, by 4.1(a), f could not be maximal.

□

Proposition 4.3. *Let f be a maximal IC-coloring such that $s_4 = 13$. Then:*

(a) $x_j = 13 \cdot 2^{j-5}$ $(5 \le j \le i_4 - 1)$ *and* $x_{i_4} = \mathbf{13 \cdot 2^{i_4-5} + 1}$.
(b) $s_j = 13 \cdot 2^{j-4}$ $(4 \le j \le i_4 - 1)$ *and* $s_{i_4} = 13 \cdot 2^{i_4-4} + 1$.
(c) $\{x, x_j\} \in \mathcal{B}$ *for all* $x \in \{x_{i_1}, x_{i_2}, x_{i_3}\}$ *and all* $5 \le j \le i_4$.

Proof. The existence of i_4 follows from 4.2. We observe that $s_4 = 13$ implies $(x_1, x_2, x_3, x_4) = (\mathbf{1}, \mathbf{2}, 3, \mathbf{7})$ or $(\mathbf{1}, \mathbf{2}, \mathbf{4}, 6)$ so that $i_4 \ge 5$. Now, (a) follows from $s_4 = 13$, $x_j = s_j - s_{j-1}$, $s_j \ge 13 \cdot 2^{j-4}$ (4.1(a)), $x_j \le s_{j-1}$ if $5 \le j \le i_4 - 1$ and $x_{i_4} \in f^+$. (b) follows from $s_4 = 13$ and (a). We prove (c) by a contradiction. If $x \sim x_j$ for some $x \in \{x_{i_1}, x_{i_2}, x_{i_3}\}$ and some $5 \le j \le i_4$, then, by (a), (b) and 3.3(b), we would have $x_{j+1} \le x + x_j \le 7 + (13 \cdot 2^{j-5} + 1) \le 13 \cdot 2^{j-5} + 13 \cdot 2^{j-5} - 5 = 13 \cdot 2^{j-4} - 5$, and, by (b) and 4.1(a), $x_{j+1} = s_{j+1} - s_j \ge 13 \cdot 2^{j-3} - (13 \cdot 2^{j-4} + 1) = 13 \cdot 2^{j-4} - 1 > 13 \cdot 2^{j-4} - 5$ (note that $x_{j+1} = \infty$ if $j = i_4 = n + 3$). □

Proposition 4.4. *Let f be a maximal with* $(x_1, x_2, x_3, x_4) = (\mathbf{1}, \mathbf{2}, 3, \mathbf{7})$. *Then:*

(a) *If $n = 2$, then the partite sets, if they exist, are*
  (i) $\{\mathbf{7}\}$, $\{\mathbf{1}, \mathbf{2}\}$, $\{3, \mathbf{14}\}$, *or*
  (ii) $\{\mathbf{14}\}$, $\{\mathbf{1}, \mathbf{2}\}$, $\{3, \mathbf{7}\}$.
(b) *If $n \ge 3$, then the partite sets, if they exist, are* $\{\mathbf{7}\}$, $\{\mathbf{1}, \mathbf{2}\}$, $\{3, 13, \ldots, \mathbf{13 \cdot 2^{n-2} + 1}\}$.

Proof. By 4.1(d)(i), we have four cases to discuss depending on whether $\{x_1, x_2\} \in \mathcal{B}$, $\{x_3, x_4\} \in \mathcal{B}$ or not.

Case 1. $\{x_1, x_2\} \in \mathcal{B}$ and $\{x_3, x_4\} \in \mathcal{B}$.
Then, by 4.1(d)(i), $\{x_i, x_j\} \in \mathcal{B}$ for all $1 \le i < j \le 4$, there would be more than three partite sets. Thus, this case can not occur.

Case 2. $\{x_1, x_2\} \in \mathcal{B}$ and $x_3 \sim x_4$.
Then, by 4.1(d)(i), $\mathbf{1}$, $\mathbf{2}$, 3 are in different partite sets, so are $\mathbf{1}$, $\mathbf{2}$, $\mathbf{7}$ for $x_3 \sim x_4$. As $s_4 = 13$, by 4.3, $\{x, x_5\} \in \mathcal{B}$ for all $x \in \{\mathbf{1}, \mathbf{2}, \mathbf{7}\}$, which is impossible for we have only three partite sets.

Case 3. $x_1 \sim x_2$ and $\{x_3, x_4\} \in \mathcal{B}$.
Then by 4.1(d)(i), x_1, x_3, x_4 are in different partite sets, so that P, Q, R partition $\{\mathbf{1}, \mathbf{2}, 3, \mathbf{7}\}$ into

$$\{\mathbf{7}\}, \{\mathbf{1}, \mathbf{2}\}, \{3\}.$$

By 4.3(c), we have $3 \sim x_5 \sim x_6 \sim \cdots \sim x_{i_4}$.

Case 3.1 $\{3, x_5, x_6, \ldots, x_{i_4}\} \subseteq Q$.
Then $P = \{\mathbf{7}\}$, $Q = \{3, \mathbf{14}\}$, $R = \{\mathbf{1}, \mathbf{2}, x_6, \ldots, x_{n+3}\}$. We claim that $n = 2$ and (i) will be obtained. If $n > 2$, we would have $x_2 \sim x_6$, so that, by 3.3(c), either $s_6 \le 2 \cdot s_5 - x_2 = 2 \cdot 27 - \mathbf{2} = 52 = 13 \cdot 2^2$ or $s_7 \le 3 \cdot s_5 + 2 + x_2 = 3 \cdot 27 + 2 + \mathbf{2} = 85 < 13 \cdot 2^3$, thus, by 4.1(a), we would have $s_6 = 13 \cdot 2^2$ and $x_6 = s_6 - s_5 = 25$. As $s_6 < 13 \cdot 2^2 + 1 \le M(K_{1,2,3})$, x_7 should exist, and, by 3.3(c) again, we would have either $s_7 \le 2 \cdot s_6 - x_2 = 2 \cdot 13 \cdot 2^2 - \mathbf{2} < 13 \cdot 2^3$ or $s_8 \le 3 \cdot s_6 + 2 + x_2 = 3 \cdot 13 \cdot 2^2 + 2 + \mathbf{2} < 13 \cdot 2^4$, which would violate 4.1(a). Hence (i) is obtained.

Case 3.2 $\{3, x_5, x_6, \ldots, x_{i_4}\} \not\subseteq Q$.

Then $P = \{\mathbf{7}\}$, $Q = \{\mathbf{1}, \mathbf{2}\}$, $R = \{3, x_5, x_6, \ldots, x_{n+3}\}$. We claim that $i_4 = n+3$. If not, we would have $x_{i_4} \sim x_{i_4+1}$, so that, by 3.3(c) with $i = i_4$ and $j = i_4+1$ and by 4.3, either $s_{i_4+1} \le 2 \cdot s_{i_4} - x_{i_4} = 2 \cdot (13 \cdot 2^{i_4-4} + 1) - (\mathbf{13} \cdot \mathbf{2}^{i_4-5} + \mathbf{1}) = 13 \cdot 2^{i_4-3} - 13 \cdot 2^{i_4-5} + 1 < 13 \cdot 2^{i_4-3}$ or $s_{i_4+2} \le 3 \cdot s_{i_4} + 2 + x_{i_4} = 3 \cdot (13 \cdot 2^{i_4-4} + 1) + 2 + (\mathbf{13} \cdot \mathbf{2}^{i_4-5} + \mathbf{1}) = 3 \cdot 13 \cdot 2^{i_4-4} + 13 \cdot 2^{i_4-5} + 6 < 13 \cdot 2^{i_4-2}$, thus, by 4.1(a), f could not be maximal. Hence (b) is obtained. (We observe that (i) is also obtained in this case if $n = 2$.)

Case 4. $x_1 \sim x_2$ and $x_3 \sim x_4$.

Then by 4.3(c) P must be $\{x_{i_4}\}$, so we have the following two cases: $P = \{\mathbf{14}\}$, $Q = \{\mathbf{1}, \mathbf{2}\}$, $R = \{3, \mathbf{7}, x_6, \ldots, x_{n+3}\}$ or $P = \{\mathbf{14}\}$, $Q = \{3, \mathbf{7}\}$, $R = \{\mathbf{1}, \mathbf{2}, x_6, \ldots, x_{n+3}\}$. A similar argument in Case 3.1 shows that, in each case, $n = 2$, so that (ii) is the only possibility.

□

Proposition 4.5. *Let f be maximal and $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, \mathbf{4})$. Then:*

(a) *If $x_2 \sim x_3$, then $x_4 = 6$ and*
- (i) *$x_5 = 13$ if $i_4 > 5$,*
- (ii) *$x_5 = \mathbf{14}$, if $i_4 = 5$,*
- (iii) *$\{x_2, x_4\} \in \mathcal{B}$.*

(b) *If $\{x_2, x_3\} \in \mathcal{B}$ then*
- (i) *x_1, x_2, x_3 are in different partite sets,*
- (ii) *$x_4 = \mathbf{8}$,*
- (iii) *$x_3 \sim x_4$,*
- (iv) *$x_5 = 12$,*
- (v) *$\{x_3, x_5\} \in \mathcal{B}$.*

Proof. (a) If $x_2 \sim x_3$, by 3.3(b) and 4.1(c)(ii), $x_4 \le x_2 + x_3 = 6$ and $6 \le x_4 \le \mathbf{8}$, so that $x_4 = 6$. From 4.3, (i) and (ii) follow. (iii) follows from 3.2(b)(i) for $x_2 \in f^+$ and $s_3 < x_2 + x_4 < x_5$.

(b) (i) follows from $\{x_2, x_3\} \in \mathcal{B}$ and 4.1(d)(ii).

(ii) By 4.1(c)(ii), it suffices to prove that $x_4 \ne 6$ and $x_4 \ne 7$. If $x_4 = 6$, then, by 4.3(c), $\{x, x_5\} \in \mathcal{B}$ for all $x \in \{\mathbf{1}, \mathbf{2}, \mathbf{4}\}$ which contradicts (i). If $x_4 = 7$, then, by 3.3(b) with $1 \le i \le 3$ and $j = 4$, $x_5 \le x_3 + x_4 = 11$, so that $s_5 < 26$ violating 4.1(a).

(iii) By (i), it suffices to prove that $\{x_1, x_4\} \in \mathcal{B}$ and $\{x_2, x_4\} \in \mathcal{B}$. Suppose, to the contrary, that $x_1 \sim x_4$ or $x_2 \sim x_4$, then, by 3.3(b) again, $x_5 \le x_2 + x_4 = 10$, so that $s_5 < 26$, again, violating 4.1(a).

(iv) By (iii) and 3.3(b), we have $x_5 \le x_3 + x_4 = 12$. By 4.1(a) with $i = 5$, we have $x_5 = s_5 - s_4 \ge 26 - 15 = 11$. It follows that $11 \le x_5 \le 12$. To prove $x_5 = 12$, let us assume $x_5 = 11$, namely, $(x_1, x_2, x_3, x_4, x_5) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{11})$. Then $x_6 \in G$ for otherwise we would have $S_f(G) = s_5 = 26 < M(K_{1,2,2})$. By (i), $x_i \sim x_6$ for some $1 \le i \le 3$, so that, by 3.3(c) with $j = 6$, either $s_6 \le 2 \cdot s_5 - x_i < 2 \cdot s_5 = 13 \cdot 2^2$ or

$s_7 \leq 3 \cdot s_5 + 2 + x_i \leq 3 \cdot 26 + 2 + \mathbf{4} < 13 \cdot 2^3$, thus f could not be maximal by 4.1(a). Hence $x_5 = 12$.

(v) By 4.1(a), we have $x_6 = s_5 - s_5 \geq 13 \cdot 2^2 - 27 = 25$, so that $s_4 < x_3 + x_5 < x_6$, thus, by 3.2(b)(i), $\{x_3, x_5\} \in \mathcal{B}$.

□

Proposition 4.6. *Let f be maximal, $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, \mathbf{4})$ and $x_1 \in R$. Then:*

(a) $4 \leq i_4 \leq 5$.
(b) $P = \{\mathbf{2}\}$, $Q = \{\mathbf{4}, \mathbf{8}\}$, $\{\mathbf{1}, 12\} \subseteq R$ *if* $i_4 = 4$.
(c) $P = \{\mathbf{14}\}$, $Q = \{\mathbf{2}, \mathbf{4}\}$, $\{\mathbf{1}, 6\} \subseteq R$ *if* $i_4 = 5$.
(d) $x_1 \sim x_j$ $(6 \leq j \leq n + 3)$.
(e) $s_j = 13 \cdot 2^{j-4} + 1$ $(5 \leq j \leq n + 3)$.
(f) $x_j = 13 \cdot 2^{j-5}$ $(6 \leq j \leq n + 3)$.

Proof. (a) By 4.5, it suffices to show that $(x_1, x_2, x_3, x_4, x_5) \neq (\mathbf{1}, \mathbf{2}, \mathbf{4}, 6, 13)$. If not, then $s_4 = 13$, so that, by 4.1 and 4.5, $\{x_1, x_2\} \in \mathcal{B}$ and $x_2 \sim x_3$, and, by 4.3 and 4.5, $\{x_1, x_{i_4}\} \in \mathcal{B}$, $\{x_2, x_{i_4}\} \in \mathcal{B}$ and $x_5 \sim x_6 \sim \cdots \sim x_{i_4}$ with $i_4 \geq 6$, we would have

(i) x_1, x_2, x_{i_4} are in different partite sets,
(ii) $x_2 \sim x_3$ and $x_5 \sim x_{i_4}$,

thus $P = \{x_1\}$, which contradicts $x_1 \in R$.

(b) If $i_4 = 4$, then, by 4.5, $(x_1, x_2, x_3, x_4, x_5) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, 12)$, so that, by 4.5(b) and $x_1 \in R$, $P = \{\mathbf{2}\}$, $Q = \{\mathbf{4}, \mathbf{8}\}$ and $\{\mathbf{1}, 12\} \subseteq R$.
(c) If $i_4 = 5$, then, by 4.5, $(x_1, x_2, x_3, x_4, x_5) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, 6, \mathbf{14})$, so that, by 4.1(d)(ii), 4.3(c), 4.5(a) and $x_1 \in R$, $P = \{\mathbf{14}\}$, $Q = \{\mathbf{2}, \mathbf{4}\}$ and $\{\mathbf{1}, 6\} \subseteq R$.
(d) follows from (a)-(c).
(e) By (a)-(c), $s_5 = 27$. Suppose we have $s_{j-1} \leq 13 \cdot 2^{j-5} + 1$ for some $6 \leq j \leq n+3$, then, by (d) and 3.3(c), either $s_j \leq 2 \cdot s_{j-1} - x_1 \leq 2 \cdot (13 \cdot 2^{j-5} + 1) - \mathbf{1} = 13 \cdot 2^{j-4} + 1$ or $s_{j+1} \leq 3 \cdot s_{j-1} + 2 + x_1 \leq 13 \cdot 2^{j-4} + 13 \cdot 2^{j-5} + 6 < 13 \cdot 2^{j-3}$, so that, by 4.1(a) with $i = j+1$, we have $s_j \leq 13 \cdot 2^{j-4} + 1$ and the equality holds only if $s_{j-1} = 13 \cdot 2^{j-5} + 1$. As f is maximal, $s_{n+3} = M(K_{1,2,n}) \geq 13 \cdot 2^{n-1} + 1$, we must have $s_{n+3} = 13 \cdot 2^{n-1} + 1$ and each $s_j = 13 \cdot 2^{n-4} + 1$ $(5 \leq j \leq n+3)$.
(f) follows from (e) and $x_j = s_j - s_{j-1}$.

□

Proposition 4.7. *Let f be a maximal IC-coloring and $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, \mathbf{4})$. Then the partite sets, if they exist, are in the following list.*

(a) *If $n = 2$, then there are four possibilities:*
  (i) $\{\mathbf{1}\}$, $\{\mathbf{2}, \mathbf{4}\}$, $\{6, \mathbf{14}\}$.
  (ii) $\{\mathbf{14}\}$, $\{\mathbf{2}, \mathbf{4}\}$, $\{\mathbf{1}, 6\}$.
  (iii) $\{\mathbf{1}\}$, $\{\mathbf{4}, \mathbf{8}\}$, $\{\mathbf{2}, 12\}$.
  (iv) $\{\mathbf{2}\}$, $\{\mathbf{4}, \mathbf{8}\}$, $\{\mathbf{1}, 12\}$.
(b) *If $n \geq 3$, then there are three possibilities:*
  (i) $\{\mathbf{1}\}$, $\{\mathbf{2}, \mathbf{4}\}$, $\{6, 13, 13 \cdot 2, \ldots, 13 \cdot 2^{n-3}, \mathbf{13 \cdot 2^{n-2} + 1}\}$.
  (ii) $\{\mathbf{14}\}$, $\{\mathbf{2}, \mathbf{4}\}$, $\{\mathbf{1}, 6, 13 \cdot 2, \ldots, 13 \cdot 2^{n-2}\}$.
  (iii) $\{\mathbf{2}\}$, $\{\mathbf{4}, \mathbf{8}\}$, $\{\mathbf{1}, 12, 13 \cdot 2, \ldots, 13 \cdot 2^{n-2}\}$.

Consequently, $S_f(G) = 13 \cdot 2^{n-1} + 1$.

Proof. By 4.5, we have two cases to discuss.

Case 1. $x_2 \sim x_3$.

Then $(x_1, x_2, x_3, x_4) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, 6)$ and $s_4 = 13$. By 4.1(d)(ii), 4.2 and 4.3(c), $\{x_1, x_2\} \in \mathcal{B}$, $\{x_1, x_{i_4}\} \in \mathcal{B}$ and $\{x_2, x_{i_4}\} \in \mathcal{B}$, so that x_1, x_2, x_{i_4} are in different partite sets and, by 4.5(a)(iii), $\{x_2, x_4\} \in \mathcal{B}$, thus P, Q, R partition the set $\{\mathbf{1}, \mathbf{2}, \mathbf{4}, 6, x_{i_4}\}$ into (note that we are now in the case $\mathbf{2} \sim \mathbf{4}$)

$$\{\mathbf{1}\},\ \{\mathbf{2}, \mathbf{4}\},\ \{6, x_{i_4}\} \text{ or } \{x_{i_4}\},\ \{\mathbf{1}, 6\},\ \{\mathbf{2}, \mathbf{4}\}.$$

It follows that $P = \{\mathbf{1}\}$ or $P = \{x_{i_4}\}$.

Case 1.1 $P = \{1\}$, and $n = 2$.

Then, by 4.5(a)(ii), we obtain (a)(i).

Case 1.2 $P = \{1\}$, and $n \geq 3$.

Then, by 4.3 and $i_4 = n + 3$, we obtain (b)(i). (The proof of $Q = \{\mathbf{2}, \mathbf{4}\}$ and $i_4 = n+3$ is similar to the proof of $i_4 = n+3$ in the case 3 contained in the proof of 4.4.)

Case 1.3 $P = \{x_{i_4}\}$, and $n = 2$.

Then, by 4.5(a)(ii), we obtain (a)(ii).

Case 1.4 $P = \{x_{i_4}\}$, and $n \geq 3$.

Then, by 4.3(c), $x_{i_4} = x_5$, so that, $(x_1, x_2, x_3, x_4, x_5) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, 6, \mathbf{14})$. We claim that $\mathbf{1} \sim x_6$. If not, then $\mathbf{4} \sim x_6$, so that, by 3.3(c), either $s_6 \leq 2 \cdot s_5 - \mathbf{4} = 2 \cdot 27 - \mathbf{4} < 13 \cdot 2^2$ or $s_7 \leq 3 \cdot s_5 + 2 + \mathbf{4} = 3 \cdot 27 + 6 < 13 \cdot 2^3$, which contradicts 4.1(a). Thus $\mathbf{1} \in R$, and, by 4.6(f), we obtain (b)(ii).

Case 2. $\{x_2, x_3\} \in \mathcal{B}$.

Then, by 4.5(b), $(x_1, x_2, x_3, x_4, x_5) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, 12)$, x_1, x_2, x_3 are in different partite sets, and $x_3 \sim x_4$ and $\{x_3, x_5\} \in \mathcal{B}$, so that P, Q, R partition $\{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, 12\}$ into

$$\{1\},\ \{\mathbf{4}, \mathbf{8}\},\ \{\mathbf{2}, 12\} \text{ or } \{\mathbf{2}\},\ \{\mathbf{4}, \mathbf{8}\},\ \{\mathbf{1}, 12\}$$

It follows that $P = \{\mathbf{1}\}$ or $P = \{\mathbf{2}\}$.

Case 2.1 $P = \{1\}$, and $n = 2$.

Then (a)(iii) is obtained.

Case 2.2 $P = \{1\}$, and $n \geq 3$.

Then, $x \sim x_6$ for some $x \in \{\mathbf{2}, \mathbf{4}, \mathbf{8}\}$ so that, by 3.3(c), either $s_6 \leq 2 \cdot s_5 - x \leq 2 \cdot 27 - \mathbf{2} = 13 \cdot 2^2$ or $s_7 \leq 3 \cdot s_5 + 2 + x \leq 3 \cdot 27 + 2 + \mathbf{8} < 13 \cdot 2^3$, thus, by 4.1(a), we have $s_6 = 13 \cdot 2^2$. As $M(K_{1,2,3}) \geq 13 \cdot 2^2 + 1$, we see that $n \geq 4$, otherwise we would have $S_f(G) = s_6 < M(K_{1,2,3})$, and that $x \sim x_7$ for some $x \in \{\mathbf{2}, \mathbf{4}, \mathbf{8}\}$ for $P = \{\mathbf{1}\}$. By 3.3(c) again, either $s_7 \leq 2 \cdot s_6 - x \leq 2 \cdot 13 \cdot 2^2 - \mathbf{2} < 13 \cdot 2^3$ or $s_8 \leq 3 \cdot s_6 + 2 + x \leq 3 \cdot 13 \cdot 2^2 + 2 + \mathbf{8} < 13 \cdot 2^4$, which contradicts 4.1(a). Therefore, this case can not occur.

Case 2.3 $P = \{\mathbf{2}\}$, and $n = 2$.

Then, (a)(iv) is obtained.

Case 2.4 $P = \{\mathbf{2}\}$, and $n \geq 3$.
By the same argument in case 1.4, we have $\mathbf{1} \sim x_6$ and $\mathbf{1} \in R$, so that, by 4.6(f), we get (b)(iii).

□

5. IC-indices with their maximal IC-colorings

If $n = 2$, that is, $|Q| = |R| = 2$, it is clear that if $\langle P, Q, R\rangle$ is a maximal IC-coloring then so is (P, R, Q) and conversely, thus we shall identify $\langle P, Q, R\rangle$ with (P, R, Q) if $|Q| = |R|$. The following theorem is our main result.

Theorem 5.1.

(a) *The IC-index* $M(K_{1,2,n})$ *of the complete tripartite graph* $K_{1,2,n}$ $(n \geq 2)$ *is* $M(K_{1,2,n}) = 13 \cdot 2^{n-1} + 1$.

(b) *When* $n \geq 3$*, there are exactly four maximal IC-colorings of* $K_{1,2,n}$*:*
- (i) $\langle\{\mathbf{7}\},\ \{\mathbf{1},\ \mathbf{2}\},\ \{3,\ 13,\ \ldots,\ 13 \cdot 2^{n-3},\ \mathbf{13 \cdot 2^{n-2} + 1}\}\rangle$,
- (ii) $\langle\{\mathbf{1}\},\ \{\mathbf{2},\ \mathbf{4}\},\ \{6,\ 13,\ \ldots,\ 13 \cdot 2^{n-3},\ \mathbf{13 \cdot 2^{n-2} + 1}\}\rangle$,
- (iii) $\langle\{\mathbf{14}\},\ \{\mathbf{2},\ \mathbf{4}\},\ \{\mathbf{1},\ 6,\ 13 \cdot 2,\ \ldots,\ 13 \cdot 2^{n-2}\}\rangle$,
- (iv) $\langle\{\mathbf{2}\},\ \{\mathbf{4},\ \mathbf{8}\},\ \{\mathbf{1},\ 12,\ 13 \cdot 2,\ \ldots,\ 13 \cdot 2^{n-2}\}\rangle$,

and, there are exactly six maximal IC-colorings of $K_{1,2,2}$*:*
- (i) $\langle\{\mathbf{7}\}, \{\mathbf{1}, \mathbf{2}\}, \{3, \mathbf{14}\}\rangle$,
- (ii) $\langle\{\mathbf{1}\}, \{\mathbf{2}, \mathbf{4}\}, \{6, \mathbf{14}\}\rangle$,
- (iii) $\langle\{\mathbf{14}\}, \{\mathbf{2}, \mathbf{4}\}, \{\mathbf{1}, 6\}\rangle$,
- (iv) $\langle\{\mathbf{2}\}, \{\mathbf{4}, \mathbf{8}\}, \{\mathbf{1}, 12\}\rangle$,
- (v) $\langle\{\mathbf{14}\},\ \{\mathbf{1},\ \mathbf{2}\},\ \{3,\ \mathbf{7}\}\rangle$,
- (vi) $\langle\{\mathbf{1}\},\ \{\mathbf{2},\ 12\},\ \{\mathbf{4},\ \mathbf{8}\}\rangle$.

Proof. (a) It follows from 4.4 and 4.7 that $S_f(G) = 13 \cdot 2^{n-1} + 1$ if f is a maximal IC-coloring. The existence of maximal IC-colorings may follow from our example in section 2.

(b) A similar argument in our example can be used to prove that each partite sets listed in 4.4 and 4.7 is an IC-coloring and hence maximal. They are the maximal IC-colorings of $K_{1,2,n}$.

□

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