Journal of Nonlinear and Convex Analysis
Volume 14, Number 1, 2013, 1–20

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ATTRACTIVE POINT THEOREMS AND ERGODIC THEOREMS FOR 2-GENERALIZED NONSPREADING MAPPINGS IN BANACH SPACES

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Abstract. In this paper, using Banach limits, we study attractive points and fixed points of general nonlinear mappings in Banach spaces. Then we obtain attractive point theorems and fixed point theorems for the nonlinear mappings in Banach spaces. Using these results, we prove nonlinear ergodic theorems for 2-generalized nonspreading mappings in Banach spaces.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a smooth Banach space and let J be the duality mapping of E. The function $\phi\colon E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$. Let C be a nonempty subset of E. Let T be a mapping of C into E. Then we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [20] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in E : \phi(z, Tx) \leq \phi(z, x),\ \forall x \in C\}$.

We know from [20] that $A(T)$ is closed and convex. This property is important. In the case when $E = H$ is a real Hilbert space, $A(T)$ is the set of attractive points of T in the sense of Takahashi and Takeuchi [31], i.e.,

$$A(T) = \{z \in H : \|z - Tx\| \leq \|z - x\|,\ \forall x \in C\}.$$

A mapping $T : C \to E$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that if C is a bounded, closed and convex subset of a Hilbert space H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty; see [29]. Furthermore, from Baillon [2] we know the first nonlinear ergodic theorem in a Hilbert space: Let C be a bounded, closed and convex subset of H and let $T : C \to C$ be nonexpansive. Then for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

2010 *Mathematics Subject Classification.* 47H10, 47H09, 47J20.
Key words and phrases. Attractive point, Banach limit, Banach space, fixed point, generalized nonspreading mapping.
This research was supported by the National Science Council of Republic of China.
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converges weakly to $z \in F(T)$. Recently, Kocourek, Takahashi and Yao [14] defined a broad class of generalized hybrid mappings containing the class of nonexpansive mappings in a Hilbert space. A mapping $T : C \to H$ is called *generalized hybrid* [14] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. We call such a mapping an (α, β)-*generalized hybrid* mapping. Then Kocourek, Takahashi and Yao [14] proved a fixed point theorem for such mappings in a Hilbert space. Furthermore, they proved a nonlinear mean convergence theorem of Baillon's type [2] in a Hilbert space. Maruyam, Takahashi and Yao [24] also defined a more broad class of nonlinear mappings called 2-generalized hybrid containing the class of generalized hybrid mappings. Kocourek, Takahashi and Yao [15] and Takahashi, Wong and Yao [32] extended the classes of generalized hybrid mappings and of 2-generalized hybrid mappings in a Hilbert space to classes of nonlinear mappings in a Banach space, respectively. Kocourek, Takahashi and Yao [15] and Takahashi, Wong and Yao [32] called such classes in a Banach space the classes of generalized nonspreading mappings and of 2-generalized nonspreading mappings, respectively and then proved fixed point theorems and nonlinear ergodic theorems in a Banach space. Very recently, Takahashi and Takeuchi [31] proved the following fixed point and mean convergence theorem without convexity in a Hilbert space.

Theorem 1.1. *Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a generalized hybrid mapping from C into itself. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by*

$$v_0 \in C, \quad v_{n+1} = Tv_n, \quad b_n = \frac{1}{n}\sum_{k=0}^{n-1} v_k$$

for all $n \in \mathbb{N} \cup \{0\}$. If $\{v_n\}$ is bounded, then the following hold:

(i) *$A(T)$ is nonempty, closed and convex;*
(ii) *$\{b_n\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n\to\infty} P_{A(T)}v_n$ and $P_{A(T)}$ is the metric projection of H onto $A(T)$.*

Such a theorem was also extended to Banach spaces by Lin and Takahashi [20] in the case when the mappings are generalized nonspreading.

In this paper, using Banach limits, we study attractive points and fixed points of general nonlinear mappings in Banach spaces. Then we obtain attractive point theorems and fixed point theorems for the nonlinear mappings in Banach spaces. Using these results, we prove nonlinear ergodic theorems for 2-generalized nonspreading mappings in Banach spaces.

2. Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the weak convergence of $\{x_n\}$ to x and the strong convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. A Banach space E is said to satisfy *Opial's condition* if $\{x_n\}$ is a sequence in E with $x_n \rightharpoonup x$, then

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||, \ \forall y \in E, \ y \neq x.$$

The duality mapping J from E into E^* is defined by

$$Jx := \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\},\ \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E. Then the space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t}$$

exists for all $x, y \in S(E)$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in S(E)$. A Banach space E is said to be *strictly convex* if $||\frac{x+y}{2}|| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $||\frac{x+y}{2}|| < 1 - \delta$ whenever $x, y \in S(E)$ and $||x - y|| \geq \varepsilon$. Furthermore, we know from [28] that

(i) if E is smooth, then J is single-valued;
(ii) if E is reflexive, then J is onto;
(iii) if E is strictly convex, then J is one-to-one;
(iv) if E is strictly convex, then J is strictly monotone;
(v) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

A Banach space E is said to have *Kadec-Klee property* if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, then $x_n \to x$. It is known that if E uniformly convex, then E has the Kadec-Klee property. Let E be a smooth Banach space and let J be the duality mapping on E. Throughout this paper, define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) := ||x||^2 - 2\langle x, Jy \rangle + ||y||^2,\ \forall x, y \in E.$$

Observe that, in a Hilbert space H, $\phi(x, y) = ||x - y||^2$ for all $x, y \in H$. Furthermore, we know that for each $x, y, z, w \in E$,

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2; \tag{2.1}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle; \tag{2.2}$$

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w). \tag{2.3}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \text{ if and only if } x = y. \tag{2.4}$$

If E is a smooth, strictly convex and reflexive Banach space, then for $x, y \in E$ and $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$,

$$\phi(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)) \leq \lambda\phi(x, y) + (1 - \lambda)\phi(x, z). \tag{2.5}$$

Let $\phi_* : E^* \times E^* \to \mathbb{R}$ be the function defined by

$$\phi_*(x^*, y^*) := ||x^*||^2 - 2\langle J^{-1}y^*, x^* \rangle + ||y^*||^2,\ \forall x^*, y^* \in E^*.$$

We have that

$$\phi(x, y) = \phi_*(Jy, Jx)\, quad \forall x, y \in E. \tag{2.6}$$

Lemma 2.1 (Xu [35])**.** Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$||ax + (1-a)y||^2 \le a||x||^2 + (1-a)||y||^2 - a(1-a)g(||x-y||)$$

for all $x, y \in B_r$ and $a \in [0,1]$, where $B_r := \{z \in E : ||z|| \le r\}$.

Lemma 2.2 (Kamimura and Takahashi [13])**.** Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$g(\|x-y\|) \le \phi(x, y)$$

for all $x, y \in B_r$, where $B_r := \{z \in E : ||z|| \le r\}$.

Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called *generalized nonexpansive* [7] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \le \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \ge 0$. A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [6, 8, 7] for more details. The following results are in Ibaraki and Takahashi [7].

Lemma 2.3 (Ibaraki and Takahashi [7])**.** *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.4 (Ibaraki and Takahashi [7])**.** *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) *$z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \le 0$ for all $y \in C$;*
(ii) *$\phi(Rx, z) + \phi(x, Rx) \le \phi(x, z)$.*

In 2007, Kohsaka and Takahashi [16] proved the following results:

Lemma 2.5 (Kohsaka and Takahashi [16])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) *C is a sunny generalized nonexpansive retract of E;*
(b) *C is a generalized nonexpansive retract of E;*
(c) *JC is closed and convex.*

Lemma 2.6 (Kohsaka and Takahashi [16]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) $z = Rx$;
(ii) $\phi(x, z) = min_{y \in C} \phi(x, y)$.

Very recently, Inthakon, Dhompongsa and Takahashi [12] obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping in a Banach space; see also Ibaraki and Takahashi [10].

Lemma 2.7 (Inthakon, Dhompongsa and Takahashi [12]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Lemmas 2.5 and 2.7.

Lemma 2.8 (Inthakon, Dhompongsa and Takahashi [12]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [28].

3. Attractive point theorems

Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. We denote by $A(T)$ the set of attractive points [20] of T, that is, $A(T) = \{u \in E : \phi(u, Tx) \le \phi(u, x),\ \forall x \in C\}$. We know the following lemma.

Lemma 3.1 (Lin and Takahashi [20]). Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then, $A(T)$ is a closed and convex subset of E.

Using the technique developed by Takahashi [26], we prove the following attractive point theorem for mappings in a Banach space.

Theorem 3.2. *Let E be a smooth and reflexive Banach space with the duality mapping J and let C be a nonempty subset of E. Let T be a mapping of C into E. Let $\{x_n\}$ be a bounded sequence of E and let μ be a mean on l^∞. Suppose that*

$$\mu_n \phi(x_n, Ty) \le \mu_n \phi(x_n, y)$$

for all $y \in C$. Then, $A(T)$ is nonempty. In particular, if E is strictly convex, T is a mapping of C into itself, $\{x_n\}$ is a bounded sequence of C and C is closed and convex, then $F(T)$ is nonempty.

Proof. Using a mean μ and a bounded sequence $\{x_n\}$, we define a function $g : E^* \to \mathbb{R}$ as follows:

$$g(x^*) = \mu_n \langle x_n, x^* \rangle, \quad \forall x^* \in E^*.$$

Since μ is linear, g is also linear. Furthermore, putting $K = \sup_{n \in \mathbb{N}} \|x_n\|$, we have

$$\begin{aligned} |g(x^*)| &= |\mu_n \langle x_n, x^* \rangle| \\ &\le \|\mu\| \sup_{n \in \mathbb{N}} |\langle x_n, x^* \rangle| \\ &\le \|\mu\| \sup_{n \in \mathbb{N}} \|x_n\| \|x^*\| \\ &= \sup_{n \in \mathbb{N}} \|x_n\| \|x^*\| \\ &= K \|x^*\| \end{aligned}$$

for all $x^* \in E^*$. Then g is a linear and continuous real-valued function on E^*. Since E is reflexive, there exists a unique element z of E such that

$$g(x^*) = \mu_n \langle x_n, x^* \rangle = \langle z, x^* \rangle$$

for all $x^* \in E^*$. Such an element z is in $D = \overline{co}\{x_n : n \in \mathbb{N}\}$, where $\overline{co}A$ is the closure of the convex hull of A. In fact, if $z \notin D$, then there exists $y^* \in E^*$ by the separation theorem [28] such that

$$\langle z, y^* \rangle < \inf_{y \in D} \langle y, y^* \rangle.$$

Since $\{x_n\} \subset D$ and μ is a mean, we have

$$\langle z, y^* \rangle < \inf_{y \in D} \langle y, y^* \rangle \le \inf_{n \in \mathbb{N}} \langle x_n, y^* \rangle \le \mu_n \langle x_n, y^* \rangle = \langle z, y^* \rangle.$$

This is a contradiction. Then we have $z \in D$. From (2.2) we have that for $y \in C$ and $n \in \mathbb{N}$,

$$\phi(x_n, y) = \phi(x_n, Ty) + \phi(Ty, y) + 2\langle x_n - Ty, JTy - Jy \rangle.$$

Thus we have that for $y \in C$,

$$\begin{aligned} \mu_n \phi(x_n, y) &= \mu_n \phi(x_n, Ty) + \mu_n \phi(Ty, y) + 2\mu_n \langle x_n - Ty, JTy - Jy \rangle \\ &= \mu_n \phi(x_n, Ty) + \phi(Ty, y) + 2\langle z - Ty, JTy - Jy \rangle. \end{aligned}$$

Since, by assumption, $\mu_n \phi(x_n, Ty) \le \mu_n \phi(x_n, y)$ for all $y \in C$, we have

$$\mu_n \phi(x_n, y) \le \mu_n \phi(x_n, y) + \phi(Ty, y) + 2\langle z - Ty, JTy - Jy \rangle.$$

This implies that

$$0 \le \phi(Ty, y) + 2\langle z - Ty, JTy - Jy \rangle \tag{3.1}$$

for all $y \in C$. Then we have from (2.3) that

$$\begin{aligned} 0 &\le \phi(Ty,y)+\phi(z,y)+\phi(Ty,Ty)-\phi(z,Ty)-\phi(Ty,y) \\ &= \phi(z,y)-\phi(z,Ty). \end{aligned}$$

This implies that $\phi(z,Ty) \le \phi(z,y)$ for all $y \in C$. Therefore $z \in A(T)$.

In particular, if E is strictly convex, T is a mapping of C into itself, $\{x_n\}$ is a bounded sequence of C and C is closed and convex, then we have from $D = \overline{co}\{x_n : n \in \mathbb{N}\} \subset C$ that z is an element of C. Putting $y = z$ in (3.1), we have that

$$0 \le \phi(Tz,z) + 2\langle z - Tz, JTz - Jz\rangle.$$

Then we have from (2.3) that

$$0 \le \phi(Tz,z)+\phi(z,z)+\phi(Tz,Tz)-\phi(z,Tz)-\phi(Tz,z).$$

Thus we have $0 \le -\phi(z,Tz)$ and hence $0 = \phi(z,Tz)$. Since E is strictly convex, we have $Tz = z$. This completes the proof. □

Let E be a refrexive Banach space. Then for a bounded sequence $\{x_n\}$ in E and a mean μ on l^∞, as in the proof of Theorem 3.2, there exists a unique element z of E such that

$$\mu_n\langle x_n, x^*\rangle = \langle z, x^*\rangle$$

for all $x^* \in E^*$; see also [26] and [4]. We call such a point z the *mean vector* [21] of $\{x_n\}$ for μ. Let E be a smooth Banach space, let C be a nonempty subset of E and let J be the duality mapping from E into E^*. A mapping $T : C \to E$ is called 2-*generalized nonspreading* [32] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1\phi(T^2x,Ty)+\alpha_2\phi(Tx,Ty)+(1-\alpha_1-\alpha_2)\phi(x,Ty) \\ &\qquad +\gamma_1\{\phi(Ty,T^2x)-\phi(Ty,x)\}+\gamma_2\{\phi(Ty,Tx)-\phi(Ty,x)\} \\ &\le \beta_1\phi(T^2x,y)+\beta_2\phi(Tx,y)+(1-\beta_1-\beta_2)\phi(x,y) \\ &\qquad +\delta_1\{\phi(y,T^2x)-\phi(y,x)\}+\delta_2\{\phi(y,Tx)-\phi(y,x)\} \end{aligned} \tag{3.2}$$

for all $x, y \in C$. Observe that if $F(T) \ne \emptyset$, then $\phi(u,Ty) \le \phi(u,y)$ for all $u \in F(T)$ and $y \in C$. Thus we have $F(T) \subseteq A(T)$. Furthermore, if $\alpha_1 = 0, \beta_1 = 0, \gamma_1 = 0$ and $\delta_1 = 0$, from (3.2) we obtain the following:

$$\begin{aligned} &\alpha_2\phi(Tx,Ty)+(1-\alpha_2)\phi(x,Ty)+\gamma_2(\phi(Ty,Tx)-\phi(Ty,x)) \\ &\qquad \le \beta_2\phi(Tx,y)+(1-\beta_2)\phi(x,y)+\delta_2(\phi(y,Tx)-\phi(y,x)) \end{aligned}$$

for all $x, y \in C$. That is, T is a generalized nonspreading mapping [15] in a Banach space. If E is a Hilbert space, then we have $\phi(x,y) = \|x-y\|^2$ for all $x, y \in E$. Thus from (3.2), we obtain the following:

$$\begin{aligned} &\alpha_1\|T^2x-Ty\|^2+\alpha_2\|Tx-Ty\|^2+(1-\alpha_1-\alpha_2)\|x-Ty\|^2 \\ &\qquad +\gamma_1(\|Ty-T^2x\|^2-\|Ty-x\|^2)+\gamma_2(\|Ty-Tx\|^2-\|Ty-x\|^2) \\ &\le \beta_1\|T^2x-y\|+\beta_2\|Tx-y\|^2+(1-\beta_1-\beta_2)\|x-y\|^2 \\ &\qquad +\delta_1(\|y-T^2x\|^2-\|y-x\|^2)+\delta_2(\|y-Tx\|^2-\|y-x\|^2) \end{aligned}$$

for all $x, y \in C$. This implies that

$$
\begin{aligned}
(\alpha_1+\gamma_1) & \| T^2x - Ty \|^2 \\
&+ (\alpha_2+\gamma_2) \| Tx - Ty \|^2 + [1-(\alpha_1+\gamma_1)-(\alpha_2+\gamma_2)] \| x - Ty \|^2 \\
&\leq (\beta_1+\delta_1)\|T^2x - y\| \\
&+ (\beta_2+\delta_2) \| Tx - y \|^2 + [1-(\beta_1+\delta_1)-(\beta_2+\delta_2)] \| x - y \|^2
\end{aligned}
$$

for all $x, y \in C$. That is, T is a 2-generalized hybrid mappings [24] in a Hilbert space. Now, we prove an attractive point theorem for 2-generalized nonspreading mappings in a Banach space.

Theorem 3.3. Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let T be a 2-generalized nonspreading mapping of C into itself. Then the following are equivalent:

(1) $A(T) \neq \emptyset$;
(2) $\{T^n v_0\}$ is bounded for some $v_0 \in C$.

Additionally, if E is strictly convex and C is closed and convex, then the following are equivalent:

(1) $F(T) \neq \emptyset$;
(2) $\{T^n v_0\}$ is bounded for some $v_0 \in C$.

Proof. If $A(T) \neq \emptyset$, then $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in A(T)$ and $x \in C$. So, $\phi(u, T^n x) \leq \phi(u, x)$ for all $n \in \mathbb{N}$ and $x \in C$ and hence $\{T^n x\}$ is bounded. We show the reverse. Since $T : C \to C$ is a 2-generalized nonspreading, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$
\begin{aligned}
&\alpha_1\phi(T^2x, Ty) + \alpha_2\phi(Tx, Ty) + (1-\alpha_1-\alpha_2)\phi(x, Ty) \\
&\qquad + \gamma_1(\phi(Ty, T^2x) - \phi(Ty, x)) + \gamma_2(\phi(Ty, Tx) - \phi(Ty, x)) \\
&\quad \leq \beta_1\phi(T^2x, y) + \beta_2\phi(Tx, y) + (1-\beta_1-\beta_2)\phi(x, y) \\
&\qquad + \delta_1(\phi(y, T^2x) - \phi(y, x)) + \delta_2(\phi(y, Tx) - \phi(y, x))
\end{aligned}
\tag{3.3}
$$

for all $x, y \in C$. Replacing x by $T^n v_0$ in the inequality (3.3), we have that

$$
\begin{aligned}
&\alpha_1\phi(T^{n+2}v_0, Ty) + \alpha_2\phi(T^{n+1}v_0, Ty) + (1-\alpha_1-\alpha_2)\phi(T^n v_0, Ty) \\
&\qquad + \gamma_1(\phi(Ty, T^{n+2}v_0) - \phi(Ty, T^n v_0)) + \gamma_2(\phi(Ty, T^{n+1}v_0) - \phi(Ty, T^n v_0)) \\
&\quad \leq \beta_1\phi(T^{n+2}v_0, y) + \beta_2\phi(T^{n+1}v_0, y) + (1-\beta_1-\beta_2)\phi(T^n v_0, y) \\
&\qquad + \delta_1(\phi(y, T^{n+2}v_0) - \phi(y, T^n v_0)) + \delta_2(\phi(y, T^{n+1}v_0) - \phi(y, T^n v_0)).
\end{aligned}
$$

Since $\{T^n v_0\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. We have that

$$\mu_n\phi(T^n v_0, Ty) \leq \mu_n\phi(T^n v_0, y)$$

for all $y \in C$. Therefore we have Theorem 3.3 from Theorem 3.2. $\square$

Using Theorem 3.3, we have following attractive point theorem for generalized nonspreading mappings in a Banach space which was proved by Lin and Takahashi [20].

Theorem 3.4. Let E be a smooth and reflexive Banach space. Let C be a nonempty subset of E and let T be a generalized nonspreading mapping of C into itselt. Then, the following are equivalent:

(a) $A(T) \neq \emptyset$;
(b) $\{T^n x\}$ is bounded for some $x \in C$.

Additionally, if E is strictly convex and C is closed and convex, then the following are equivalent:

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ is bounded for some $x \in C$.

Proof. Putting $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$ in (3.1), we obtain that

$$\begin{aligned} \alpha_2\phi(Tx, Ty) + (1-\alpha_2)\phi(x, Ty) + \gamma_2\big\{\phi(Ty, Tx) - \phi(Ty, x)\big\} \\ \leq \beta_2\phi(Tx, y) + (1-\beta_2)\phi(x, y) + \delta_2\big\{\phi(y, Tx) - \phi(y, x)\big\} \end{aligned} \tag{3.4}$$

for all $x, y \in C$. Such a mapping is a generalized nonspreading mapping. So, we have the desired result from Theorem 3.3. □

4. Skew-attractive point theorem

Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. We denote by $B(T)$ the set of *skew-attractive points* [20] of T, i.e., $B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z),\ \forall x \in C\}$. The following lemma was by Lin and Takahashi [20].

Lemma 4.1 (Lin and Takahashi[20])**.** Let E be a smooth Banach space and let C be a nonempty subset of H. Let T be a mapping from C into E. Then, $B(T)$ is a closed.

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Define a mapping T^* as follows:

$$T^*x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the *duality mapping* of T; see [33] and [5]. If T is a mapping of C into itself, then T^* is a mapping of JC into JC; see [33].

Lemma 4.2 (Lin and Takahashi [20])**.** Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E and let T^* be the duality mapping of T. Then the following hold:

(1) $JB(T) = A(T^*)$;
(2) $JA(T) = B(T^*)$.

In particular, $JB(T)$ is closed and convex.

Let E be a smooth Banach space and let J be the duality mapping from E into E^*. Let C be a nonempty subset of E. A mapping $T : C \to E$ is called 2-*skew-generalized nonspreading* [32] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$\alpha_1\phi(Ty, T^2x) + \alpha_2\phi(Ty, Tx) + (1-\alpha_1-\alpha_2)\phi(Ty, x)$$

$$
\begin{aligned}
&\quad + \gamma_1(\phi(T^2x, Ty) - \phi(x, Ty)) + \gamma_2(\phi(Tx, Ty) - \phi(x, Ty)) \\
&\leq \beta_1\phi(y, T^2x) + \beta_2\phi(y, Tx) + (1 - \beta_1 - \beta_2)\phi(y, x) \\
&\quad + \delta_1(\phi(T^2x, y) - \phi(x, y)) + \delta_2(\phi(Tx, y) - \phi(x, y))
\end{aligned} \tag{4.1}
$$

for all $x, y \in C$.

Theorem 4.3. Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a 2-skew-generalized nonspreading mapping of C into itself. Then the following are equivalent:

(1) $B(T) \neq \emptyset$;
(2) $\{T^n v_0\}$ is bounded for some $v_0 \in C$.

Additionally, if C is closed and JC is closed and convex, then the following are equivalent:

(1) $F(T) \neq \emptyset$;
(2) $\{T^n v_0\}$ is bounded for some $v_0 \in C$.

Proof. If $B(T) \neq \emptyset$, then $\phi(Ty, u) \leq \phi(y, u)$ for all $u \in B(T)$ and $y \in C$. So, $\phi(T^n y, u) \leq \phi(y, u)$ for all $n \in \mathbb{N}$ and $y \in C$ and then $\{T^n y\}$ is bounded for all $y \in C$. We show the reverse. As in the proof of Theorem 4.3 in [20], we obtain that

$$
\begin{aligned}
&\alpha_1\phi_*(T^{*2}x^*, T^*y^*) + \alpha_2\phi_*(T^*x^*, T^*y^*) + (1 - \alpha_1 - \alpha_2)\phi_*(x^*, T^*y^*) \\
&\qquad + \gamma_1(\phi_*(T^*y^*, T^{*2}x^*) - \phi_*(T^*y^*, x^*)) + \gamma_2(\phi_*(T^*y^*, T^*x^*) - \phi_*(T^*y^*, x^*)) \\
&\quad \leq \beta_1\phi_*(T^{*2}x^*, y^*) + \beta_2\phi_*(T^*x^*, y^*) + (1 - \beta_1 - \beta_2)\phi_*(x^*, y^*) \\
&\qquad + \delta_1(\phi_*(y^*, T^{*2}x^*) - \phi_*(y^*, x^*)) + \delta_2(\phi_*(y^*, T^*x^*) - \phi_*(y^*, x^*))
\end{aligned}
$$

for all $x^*, y^* \in JC$. This implies that T^* is a 2-generalized nonspreading mapping of JC into itself. Since $\|T^n x\| = \|JT^n x\| = \|(JTJ^{-1})^n Jx\| = \|(T^*)^n Jx\|$. Thus if $\{T^n x\}$ is bounded for some $x \in C$, then $\{(T^*)^n Jx\}$ is bounded. By Theorem 3.3, we obtain that $A(T^*)$ is nonempty. From Lemma 4.2, we also know that $A(T^*) = JB(T)$. Therefore $B(T)$ is nonempty. Additionally, assume that C is closed and JC is closed and convex. If $\{T^n x\}$ is bounded for some $x \in C$, then $\{(T^*)^n x\}$ is bounded. By Theorem 3.3, we obtain that $F(T^*)$ is nonempty. Hence $F(T)$ is nonempty. It is obvious that if $F(T) \neq \emptyset$, then $\{T^n u\} = \{u\}$ for $u \in F(T)$, that is, $\{T^n v_0\}$ is bounded for some $v_0 \in C$. This completes the proof. □

Using Theorem 4.3, we have the following attractive point theorem for skew-generalized nonspreading mappings in a Banach space which was proved by Lin and Takahashi [20].

Theorem 4.4. Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty subset of E and let T be a skew-generalized nonspreading mapping of C into itselt. Then, the following are equivalent:

(a) $B(T) \neq \emptyset$;
(b) $\{T^n x\}$ is bounded for some $x \in C$.

Additionally, if C is closed and JC is closed and convex, then the following are equivalent:

(a) $F(T) \neq \emptyset$;

(b) $\{T^n x\}$ is bounded for some $x \in C$.

Proof. Putting $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0$ in (4.1), we obtain that

$$\alpha_2\phi(Ty,Tx) + (1-\alpha_2)\phi(Ty,Tx) + \gamma_2\big\{\phi(Tx,Ty) - \phi(x,Ty)\big\} \leq \beta_2\phi(y,Tx) + (1-\beta_2)\phi(y,x) + \delta_2\big\{\phi(Tx,y) - \phi(x,y)\big\} \tag{4.2}$$

for all $x, y \in C$. Such a mapping is a skew-generalized nonspreading mapping. So, we have the desired result from Theorem 4.3. □

5. Nonlinear ergodic theorems

In the section, we prove a nonlinear ergodic theorem of Baillon's type [2] without convexity for 2-generalized nonspreading mappings in a Banach space. Before proving it, we need the following four lemmas.

Lemma 5.1. *Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J and let D be a nonempty, closed and convex subset of E. Let $\{x_n\}$ be a bounded sequence in D and let μ be a mean on l^∞. If $g : D \to \mathbb{R}$ is defined by*

$$g(z) = \mu_n\phi(x_n, z), \quad \forall z \in D,$$

then the mean vector z_0 of $\{x_n\}$ for μ is a unique minimizer in D such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

Proof. For a bounded sequence $\{x_n\} \subset D$ and a mean μ on l^∞, we know that a function $g : D \to \mathbb{R}$ defined by

$$g(z) = \mu_n\phi(x_n, z), \quad \forall z \in D$$

is well-defined. We also know from the proof of Theorem 3.2 that there exists the mean vector z_0 of $\{x_n\}$ for μ, that is, there exists $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$ such that

$$\mu_n\langle x_n, y^*\rangle = \langle z_0, y^*\rangle, \quad \forall y^* \in E^*.$$

Since D is closed and convex and $\{x_n\} \subset D$, we have $z_0 \in D$. Furthermore we have from (2.2) and (2.3) that for any $z \in D$,

$$\begin{aligned}
g(z) - g(z_0) &= \mu_n\phi(x_n, z) - \mu_n\phi(x_n, z_0) \\
&= \mu_n\big(\phi(x_n, z) - \phi(x_n, z_0)\big) \\
&= \mu_n\big(\phi(x_n, z) - \phi(x_n, z) - \phi(z, z_0) - 2\langle x_n - z, Jz - Jz_0\rangle\big) \\
&= \mu_n\big(-\phi(z, z_0) - 2\langle x_n - z, Jz - Jz_0\rangle\big) \\
&= -\phi(z, z_0) - 2\langle z_0 - z, Jz - Jz_0\rangle \\
&= -\phi(z, z_0) - \phi(z_0, z_0) - \phi(z, z) + \phi(z_0, z) + \phi(z, z_0) \\
&= \phi(z_0, z).
\end{aligned}$$

Then we have that

$$g(z) = g(z_0) + \phi(z_0, z), \quad \forall z \in D. \tag{5.1}$$

This implies that $z_0 \in D$ is a minimizer in D such that $g(z_0) = \min\{g(z) : z \in D\}$. Furthermore, if $u \in D$ satisfies $g(u) = g(z_0)$, then we have from (5.1) that

$\phi(z_0, u) = 0$. Since E is strictly convex, we have that $z_0 = u$ and hence z_0 is a unique minimizer in D such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

This completes the proof. □

Using Lemma 5.1, we obtain the following result.

Lemma 5.2. *Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J, let C be a nonempty subset of E and let T be a mapping of C into itself. Suppose that $A(T) = B(T)$ is nonempty. Then for any $x \in C$, the sequence $\{T^n x\}$ is bounded and the set*

$$\cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap A(T)$$

consists of one point z_0, where z_0 is a unique minimizer of $A(T)$ such that

$$\lim_{n\to\infty} \phi(T^n x, z_0) = \min\{\lim_{n\to\infty} \phi(T^n x, z) : z \in A(T)\}.$$

Additionally, if C is closed and convex, then the set

$$\cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap F(T)$$

consists of one point z_0.

Proof. Since $A(T) = B(T)$ is nonempty, we have that for any $z \in A(T) = B(T)$ and $x \in C$,

$$\phi(T^{n+1}x, z) \le \phi(T^n x, z) \le \cdots \le \phi(x, z), \quad \forall n \in \mathbb{N}.$$

Thus $\{T^n x\}$ is bounded. Let μ be a Banach limit on l^∞. From Lemma 5.1, a unique minimizer $z_0 \in E$ such that

$$\mu_n \phi(T^n x, z_0) = \min\{\mu_n \phi(T^n x, y) : \ y \in E\}$$

is the mean vector $z_0 \in E$ of $\{T^n x\}$ for μ, that is, a point $z_0 \in E$ such that $z_0 \in \overline{co}\{T^n x : n \in \mathbb{N}\}$ and

$$\mu_n \langle T^n x, y^* \rangle = \langle z_0, y^* \rangle, \quad \forall y^* \in E^*.$$

We also know from the proof of Theorem 3.2 that $z_0 \in A(T)$. Furthermore, this $z_0 \in A(T)$ satisfies that

$$\mu_n \phi(T^n x, z_0) = \min\{\mu_n \phi(T^n x, y) : \ y \in A(T)\}.$$

Let us show that $z_0 \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}$. If not, there exists some $k \in \mathbb{N}$ such that $z_0 \notin \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}$. By the separation theorem, there exists $y_0^* \in E^*$ such that

$$\langle z_0, y_0^* \rangle < \inf\left\{\langle z, y_0^* \rangle : z \in \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}\right\}.$$

Using the property of the Banach limit μ, we have that

$$\begin{aligned} \langle z_0, y_0^* \rangle &< \inf\left\{\langle z, y_0^* \rangle : z \in \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}\right\} \\ &\le \inf\{\langle T^{k+n}x, y_0^* \rangle : n \in \mathbb{N}\} \\ &\le \mu_n \langle T^{k+n}x, y_0^* \rangle \\ &= \mu_n \langle T^n x, y_0^* \rangle \\ &= \langle z_0, y_0^* \rangle. \end{aligned}$$

This is a contradiction. Thus we have that $z_0 \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}$. Next we show that $\cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap A(T)$ consists of one point z_0. Assume that $z_1 \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap A(T)$. Since $z_1 \in A(T) = B(T)$, we have that

$$\phi(T^{n+1}x, z_1) \leq \phi(T^n x, z_1), \quad \forall n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} \phi(T^n x, z_1)$ exists. Furthermore, we know from the property of a Banach limit μ that

$$\mu_n \phi(T^n x, z_1) = \lim_{n\to\infty} \phi(T^n x, z_1).$$

In general, since $\lim_{n\to\infty} \phi(T^n x, z)$ exists for every $z \in A(T)$, we define a function $g : A(T) \to \mathbb{R}$ as follows:

$$g(z) = \lim_{n\to\infty} \phi(T^n x, z), \quad \forall z \in A(T).$$

Since

$$\phi(z_0, z_1) = \phi(T^n x, z_1) - \phi(T^n x, z_0) - 2\langle T^n x - z_0, Jz_0 - Jz_1\rangle$$

for every $n \in \mathbb{N}$, we have

$$\begin{aligned}\phi(z_0, z_1) &+ 2 \lim_{n\to\infty} \langle T^n x - z_0, Jz_0 - Jz_1\rangle \\ &= \lim_{n\to\infty} \phi(T^n x, z_1) - \lim_{n\to\infty} \phi(T^n x, z_0) \\ &\geq 0.\end{aligned}$$

Let $\epsilon > 0$. Then we have that

$$2 \lim_{n\to\infty} \langle T^n x - z_0, Jz_0 - Jz_1\rangle > -\phi(z_0, z_1) - \epsilon.$$

Hence there exists $n_0 \in \mathbb{N}$ such that

$$2\langle T^n x - z_0, Jz_0 - Jz_1\rangle > -\phi(z_0, z_1) - \epsilon$$

for every $n \in \mathbb{N}$ with $n \geq n_0$. Since $z_1 \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}$, we have

$$2\langle z_1 - z_0, Jz_0 - Jz_1\rangle \geq -\phi(z_0, z_1) - \epsilon.$$

We have from (2.3) that

$$\phi(z_1, z_1) + \phi(z_0, z_0) - \phi(z_1, z_0) - \phi(z_0, z_1) \geq -\phi(z_0, z_1) - \epsilon$$

and hence $\phi(z_1, z_0) \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\phi(z_1, z_0) = 0$. Since E is strictly convex, we have $z_0 = z_1$. Therefore

$$\{z_0\} = \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap A(T).$$

Additionally, if C is closed and convex, then we have that

$$z_0 \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap F(T).$$

Since $\cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap A(T)$ consists of one point z_0, we have that

$$\cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap F(T) = \{z_0\}.$$

This completes the proof. □

Lemma 5.3. *Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let T be a 2-generalized nonspreading mapping of C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Define*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x, \quad \forall n \in \mathbb{N}.$$

If a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ converges weakly to a point u, then $u \in A(T)$. Additionally, if E is strictly convex and C is closed and convex, then $u \in F(T)$.

Proof. Let T be a 2-generalized nonspreading mapping of C into itself. Then, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1\phi(T^2 z, Ty) + \alpha_2\phi(Tz, Ty) + (1-\alpha_1-\alpha_2)\phi(x, Ty) \\ &\quad + \gamma_1\{\phi(Ty, T^2 z) - \phi(Ty, z)\} + \gamma_2\{\phi(Ty, Tz) - \phi(Ty, z)\} \\ &\leq \beta_1\phi(T^2 z, y) + \beta_2\phi(Tz, y) + (1-\beta_1-\beta_2)\phi(z, y) \\ &\quad + \delta_1\{\phi(y, T^2 z) - \phi(y, z)\} + \delta_2\{\phi(y, Tz) - \phi(y, z)\} \end{aligned} \tag{5.2}$$

for all $z, y \in C$. Let $\{T^n x\}$ be a bounded sequence. Replacing z by $T^k x$ in (5.2), we have that for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} &\alpha_1\phi(T^{k+2}x, Ty) + \alpha_2\phi(T^{k+1}x, Ty) + (1-\alpha_1-\alpha_2)\phi(T^k x, Ty) \\ &\quad + \gamma_1\{\phi(Ty, T^{k+2}x) - \phi(Ty, T^k x)\} + \gamma_2\{\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)\} \\ &\leq \beta_1\phi(T^{k+2}x, y) + \beta_2\phi(T^{k+1}x, y) + (1-\beta_1-\beta_2)\phi(T^k x, y) \\ &\quad + \delta_1\{\phi(y, T^{k+2}x) - \phi(y, T^k x)\} + \delta_2\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\} \\ &= \beta_1\{\phi(T^{k+2}x, Ty) + \phi(Ty, y) + 2\langle T^{k+2}x - Ty, JTy - Jy\rangle\} \\ &\quad + \beta_2\{\phi(T^{k+1}x, Ty) + \phi(Ty, y) + 2\langle T^{k+1}x - Ty, JTy - Jy\rangle\} \\ &\quad + (1-\beta_1-\beta_2)\{\phi(T^k x, Ty) + \phi(Ty, y) + 2\langle T^k x - Ty, JTy - Jy\rangle\} \\ &\quad + \delta_1\{\phi(y, T^{k+2}x) - \phi(y, T^k x)\} + \delta_2\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\}. \end{aligned} \tag{5.3}$$

This implies that

$$\begin{aligned} 0 &\leq (\beta_1 - \alpha_1)\{\phi(T^{k+2}x, Ty) - \phi(T^k x, Ty)\} \\ &\quad + (\beta_2 - \alpha_2)\{\phi(T^{k+1}x, Ty) - \phi(T^k x, Ty)\} + \phi(Ty, y) \\ &\quad + 2\langle \beta_1 T^{k+2}x + \beta_2 T^{k+1}x + (1-\beta_1-\beta_2)T^k x - Ty, JTy - Jy\rangle \\ &\quad - \gamma_1\{\phi(Ty, T^{k+2}x) - \phi(Ty, T^k x)\} - \gamma_2\{\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)\} \\ &\quad + \delta_1\{\phi(y, T^{k+2}x) - \phi(y, T^k x)\} + \delta_2\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\} \\ &= (\beta_1 - \alpha_1)\{\phi(T^{k+2}x, Ty) - \phi(T^k x, Ty)\} \\ &\quad + (\beta_2 - \alpha_2)\{\phi(T^{k+1}x, Ty) - \phi(T^k x, Ty)\} + \phi(Ty, y) \\ &\quad + 2\langle T^k x - Ty + \beta_1(T^{k+2}x - T^k x) + \beta_2(T^{k+1}x - T^k x), JTy - Jy\rangle \\ &\quad - \gamma_1\{\phi(Ty, T^{k+2}x) - \phi(Ty, T^k x)\} - \gamma_2\{\phi(Ty, T^{k+1}x) - \phi(Ty, T^k x)\} \\ &\quad + \delta_1\{\phi(y, T^{k+2}x) - \phi(y, T^k x)\} + \delta_2\{\phi(y, T^{k+1}x) - \phi(y, T^k x)\}. \end{aligned} \tag{5.4}$$

Summing up these inequalities (5.4) with respect to $k = 0, 1, \dots, n-1$ and deviding by n, we have that

$$
\begin{aligned}
0 \le{}& \frac{1}{n}(\beta_1 - \alpha_1)\{\phi(T^{n+1}x, Ty) + \phi(T^n x, Ty) - \phi(Tx, Ty) - \phi(x, Ty)\} \\
&+ \frac{1}{n}(\beta_2 - \alpha_2)\{\phi(T^n x, Ty) - \phi(x, Ty)\} + \phi(Ty, y) \\
&+ 2\langle S_n x - Ty, JTy - Jy\rangle \\
&+ \frac{2}{n}\langle \beta_1(T^{n+1}x + T^n x - Tx - x) + \beta_2(T^n x - x), JTy - Jy\rangle \\
&- \frac{\gamma_1}{n}\{\phi(Ty, T^{n+1}x) + \phi(Ty, T^n x) - \phi(Ty, Tx) - \phi(Ty, x)\} \\
&- \frac{\gamma_2}{n}\{\phi(Ty, T^n x) - \phi(Ty, x)\} \\
&+ \frac{\delta_1}{n}\{\phi(y, T^{n+1}x) + \phi(y, T^n x) - \phi(y, Tx) - \phi(y, x)\} \\
&+ \frac{\delta_2}{n}\{\phi(y, T^n x) - \phi(y, x)\},
\end{aligned} \tag{5.5}
$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Since $\{S_{n_i} x\}$ converges weakly to a point u, we obtain that

$$
0 \le \phi(Ty, y) + 2\langle u - Ty, JTy - Jy\rangle, \tag{5.6}
$$

as $n_i \to \infty$ in (5.5). Using (2.3), we obtain

$$
\begin{aligned}
0 &\le \phi(Ty, y) + 2\langle u - Ty, JTy - Jy\rangle \\
&= \phi(Ty, y) + \phi(u, y) + \phi(Ty, Ty) - \phi(u, Ty) - \phi(Ty, y) \\
&= \phi(u, y) - \phi(u, Ty).
\end{aligned} \tag{5.7}
$$

Hence $\phi(u, Ty) \le \phi(u, y)$ and then $u \in A(T)$. Additionally, assume that E is strictly convex and C is closed and convex. Since $\{S_{n_i} x\} \subset C$ and $\{S_{n_i} x\}$ converges weakly to a point u, then $u \in C$ because C is weakly closed. Putting $y = u$ in (5.6), we obtain

$$
\begin{aligned}
0 &\le \phi(Tu, u) + 2\langle u - Tu, JTu - Ju\rangle \\
&= \phi(Tu, u) + \phi(u, u) + \phi(Tu, Tu) - \phi(u, Tu) - \phi(Tu, u) \\
&= -\phi(u, Tu).
\end{aligned} \tag{5.8}
$$

Hence $\phi(u, Tu) = 0$. Since E is strictly convex, we have $u \in F(T)$. This completes the proof. □

Lemma 5.4. *Let E be a uniformly convex and smooth Banach space. Let C be a nonempty subset of E and let $T : C \to C$ be a mapping such that $B(T) \neq \emptyset$. Then, there exists a unique sunny generalized nonexpansive retraction R of E onto $B(T)$. Furthermore, for any $x \in C$, $\lim_{n\to\infty} RT^n x$ exists in $B(T)$.*

Proof. We have from Lemmas 4.1 and 4.2 that $B(T)$ is closed and $JB(T)$ is closed and convex. Then from Lemmas 2.5 and 2.3 , there exists a unique sunny generalized nonexpansive retraction R of E onto $B(T)$. From Lemma 2.4, we know that

$$
0 \le \langle v - Rv, JRv - Ju\rangle, \quad \forall u \in B(T),\ v \in C. \tag{5.9}
$$

We have from (5.9) and (2.3) that

$$\begin{aligned} 0 &\le 2\langle v - Rv, JRv - Ju\rangle \\ &= \phi(v,u) + \phi(Rv,Rv) - \phi(v,Rv) - \phi(Rv,u) \\ &= \phi(v,u) - \phi(v,Rv) - \phi(Rv,u). \end{aligned}$$

Hence we have that

$$\phi(Rv,u) \le \phi(v,u) - \phi(v,Rv), \quad \forall u \in B(T),\ v \in C. \tag{5.10}$$

Since $\phi(Tz,u) \le \phi(z,u)$ for any $u \in B(T)$ and $z \in C$, it follows that

$$\begin{aligned} \phi(T^n x, RT^n x) &\le \phi(T^n x, RT^{n-1}x) \\ &\le \phi(T^{n-1}x, RT^{n-1}x). \end{aligned}$$

Hence the sequence $\phi(T^n x, RT^n x)$ is nonincreasing. Putting $u = RT^n x$ and $v = T^m x$ with $n \le m$ in (5.10), we have from Lemma 2.2 that

$$\begin{aligned} g(\|RT^m x - RT^n x\|) &\le \phi(RT^m x, RT^n x) \\ &\le \phi(T^m x, RT^n x) - \phi(T^m x, RT^m x) \\ &\le \phi(T^n x, RT^n x) - \phi(T^m x, RT^m x), \end{aligned}$$

where g is a strictly increasing, continuous and convex real-valued function with $g(0) = 0$. From the properties of g, $\{RT^n x\}$ is a Cauchy sequence. Therefore $\{RT^n x\}$ converges strongly to a point $q \in B(T)$. This completes the proof. □

Using Lemmas 5.2, 5.3 and 5.4, we prove the following nonlinear ergodic theorem for 2-generalized nonspreading mappings in a Banach space.

Theorem 5.5. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let $T : C \to C$ be a 2-generalized nonspreading mapping such that $A(T) = B(T) \neq \emptyset$ and let $R_{B(T)}$ be the sunny generalized nonexpansive retraction of E onto $B(T)$. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $z_0 \in A(T)$, where $z_0 = \lim_{n\to\infty} R_{B(T)} T^n x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to $z_0 \in F(T)$.

Proof. Let $x \in C$. Since $A(T)$ is nonempty, the sequence $\{T^n x\}$ is bounded. So, $\{S_n x\}$ is bounded. We know from Theorem 5.2 that the set

$$\cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap A(T)$$

consists of one point. To prove that $\{S_n x\}$ converges weakly to a point z_0 in $A(T)$, it is sufficient to show that if $S_{n_i} x \rightharpoonup v$, then $v \in A(T)$ and

$$v \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}.$$

From Lemma 5.3, we have that $v \in A(T)$. Next, we show that

$$v \in \cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}.$$

Fix $k \in \mathbb{N}$. We have that for any $n_i \in \mathbb{N}$ with $n_i > k+1$,

$$S_{n_i}x = \frac{1}{n_i}(x + Tx + \cdots + T^k x) + \frac{n_i - (k+1)}{n_i} \cdot \frac{1}{n_i - (k+1)}(T^{k+1}x + \cdots + T^{n_i - 1}).$$

Thus from $S_{n_i}x \rightharpoonup v$, we have

$$\frac{1}{n_i - (k+1)}(T^{k+1}x + \cdots + T^{n_i - 1}) \rightharpoonup v$$

and hence $v \in \overline{co}\{T^{k+n}x : n \in \mathbb{N}\}$. Since $k \in \mathbb{N}$ is arbitrary, we have that $v \in \cap_{k=1}^{\infty}\overline{co}\{T^{k+n}x : n \in \mathbb{N}\}$. Therefore $\{S_n x\}$ converges weakly to a point z_0 of $A(T)$. Additionally, assume that E is strictly convex and C is closed and convex. Then $z_0 \in C$ because C is weakly closed. From $\phi(Tz_0, z_0) \leq \phi(z_0, z_0) = 0$, we have $\phi(Tz_0, z_0) = 0$ and hence $z_0 \in F(T)$. Therefore $\{S_n x\}$ converges weakly to $z_0 \in F(T)$.

We have from Lemma 5.4 that there exists the sunny generalized nonexpansive retraction $R = R_{B(T)}$ of E onto $B(T)$ and $\{RT^n x\}$ converges strongly to a point $q \in B(T)$. Rewriting the characterization of the retraction R, we have that

$$0 \leq \left\langle T^k x - RT^k x, JRT^k x - Ju \right\rangle, \quad \forall u \in B(T)$$

and hence

$$\begin{aligned} \left\langle T^k x - RT^k x, Ju - Jq \right\rangle &\leq \left\langle T^k x - RT^k x, JRT^k x - Jq \right\rangle \\ &\leq \|T^k x - RT^k x\| \cdot \|JRT^k x - Jq\| \\ &\leq K\|JRT^k x - Jq\|, \end{aligned}$$

where K is an upper bound for $\|T^k x - RT^k x\|$. Summing up these inequalities for $k = 0, 1, \ldots, n-1$ and deviding by n, we arrive to

$$\left\langle S_n x - \frac{1}{n}\sum_{k=0}^{n-1} RT^k x, Ju - Jq \right\rangle \leq \frac{K}{n}\sum_{k=0}^{n-1}\|JRT^k x - Jq\|,$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Letting $n \to \infty$ and remembering that J is continuous, we get that

$$\langle z_0 - q, Ju - Jq\rangle \leq 0.$$

This holds for any $u \in B(T)$. Putting $u = z_0$, we have $\langle z_0 - q, Jz_0 - Jq\rangle \leq 0$. Since J is monotone, we have $\langle z_0 - q, Jz_0 - Jq\rangle = 0$. Since E is strictly convex, we have $Z_0 = q$. Thus $z_0 = \lim_{n\to\infty} R_{B(T)}T^n x$. □

Since a generalized nonspreading mapping is a 2-generalized nonspreading mapping, from Theorem 5.5 we can obtain the following nonlinear ergodic theorem obtained by Lin and Takahashi [20] in a Banach space.

Theorem 5.6. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let $T : C \to C$ be a generalized* *nonspreading mapping such that $A(T) = B(T) \neq \emptyset$. Let R be the sunny generalized nonexpansive retraction of E onto $B(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $z_0 \in A(T)$, where $z_0 = \lim_{n\to\infty} RT^n x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to $z_0 \in F(T)$.

Furthermore, using Theorem 5.5, we have the nonlinear ergodic theorem obtained by Lin and Takahashi [21] in a Hilbert space.

Theorem 5.7 (Lin and Takahashi [21])**.** Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a 2-generalized hybrid mapping from C into itself. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by

$$v_1 \in C,\ v_{n+1} = Tv_n,\ b_n = \frac{1}{n} \sum_{k=1}^{n} v_k$$

for all $n \in \mathbb{N}$. If $A(T) \neq \emptyset$, then $\{b_n\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim\limits_{n\to\infty} P_{A(T)} v_n$.

Proof. Putting $\gamma_1 = \gamma_2 = 0, \delta_1 = \delta_2 = 0$ and $\phi(x, y) = \|x - y\|^2$ in (3.2), we obtain that for any $x, y \in C$,

$$\begin{aligned} \alpha_1\|T^2x{-}Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2. \end{aligned}$$

So, a 2-generalized nonspreading mappings is a 2-generalized hybrid mappings in the sense of Maruyama, Takahashi and Yao [24]. Furthermore, it is obvious that $A(T) = B(T)$. It follows from Lemma 3.1 that $A(T)$ is nonempty, closed and convex. Hence, there exists the metric projection of H onto $A(T)$. In a Hilbert, the metric projection of H onto $A(T)$ is equivalent to the sunny generalized nonexpansive retraction of E onto $B(T)$. So, we have the desired result from Theorem 5.5. □

As in the proof of Theorem 5.5, we have the nonlinear ergodic theorem obtained by Takahashi, Wong and Yao [32].

Theorem 5.8. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex sunny generalized nonexpansive retract of E. Let $T : C \to C$ be a 2-generalized nonspreading mapping with $F(T) \neq \emptyset$ such that $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in C$ and $u \in F(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $z_0 \in F(T)$, where $z_0 = \lim_{n\to\infty} RT^n x$.

Proof. Since there exists a sunny generalized nonexpansive retraction of E onto C, we have from Lemma 2.5 that JC is closed and convex. Since E is a smooth, strictly convex and reflexive and C is closed and convex, we have from Lemma 5.2 that for any $x \in C$,

$$\cap_{k=1}^{\infty} \overline{co}\{T^{k+n}x : n \in \mathbb{N}\} \cap F(T)$$

consists of one point z_0, where z_0 is the mean vector of $\{T^n x\}$ for any Banach limit μ. Thus $\{S_n x\}$ converges weakly to $z_0 \in F(T)$. Furthermore, we know from Lemma 2.8 that there exists a sunny generalized nonexpansive retraction of E onto $F(T)$. Thus as in the proof of Lemma 5.4, we have that $z_0 = \lim_{n\to\infty} RT^n x$. □

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Manuscript received October 26, 2011
revised December 28, 2011

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