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# DUALITY FOR A SYSTEM OF MULTIOBJECTIVE PROBLEMS WITH EXPONENTIAL TYPE INVEXITY FUNCTIONS

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Abstract. A system of multiobjective nonsmooth fractional programming problem with exponential $V$-$r$-invexity is characterized. Employing the optimality conditions we construct two kinds of duality models: Wolfe type duality and Mond-Weir type duality. Consequently, the duality theorems are proved under exponential type $V$-$r$-invexity. That is the optimal values of the duality problems are equal to the optimal value of primary problem under the framework of exponential $V$-$r$-invexity for Lipschitz function.

## 1. Introduction

Let $X$ be a separable reflexive Banach space. Let $f_i,\, g_i : X \longrightarrow \mathbb{R}$, $i = 1, 2, ..., k$, and $h : X \longrightarrow \mathbb{R}^m$ be locally Lipschitz functions on $X$. We consider a system of multiobjective nonsmooth fractional programming problem as the following form:

$$(FP) \qquad \text{Minimize} \qquad \phi(x) \equiv \frac{f(x)}{g(x)} \equiv \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_k(x)}{g_k(x)}\right) \equiv (\phi_1(x),\, \phi_2(x),\, \dots,\, \phi_k(x))$$

subject to $x \in X$ with constraint set

$$\mathfrak{F} = \{x \in X \mid h(x) = (h_1, h_2, \cdots, h_m)(x) \in -\mathbb{R}^m_+\}.$$

Without loss of generality, we may assume that $f_i(x) \geq 0$, $g_i(x) > 0$ for all $x \in X$, $i = 1, 2, ..., k$ in the problem $(FP)$. Some other multiobjective nonsmooth fractional or nonfractional programming are studied in $[1-3, 5, 8, 9, 16, 20-24]$.

In the recent years, many authors establish the necessary optimality conditions and employ the necessary optimality conditions with some additional assumptions to establish the sufficient optimality conditions. Consequently as the existence of optimal solutions are established, one could constitute the duality theorems. Since the sufficient optimality conditions are various, so the duality models are also various. Many researchers effort to search the extra assumptions in different view points. Hence several duality happended in different assumptions. For example, some authors consider generalized convexity $[2, 3, 5, 7, 8, 12, 17, 20-22, 24]$, generalized invexity $[1, 4, 9, 10, 13, 16, 19, 23]$ in varous types. Most of programming problems are considered in real variable cases. Actually the optimization problems are not only for real variables, but also in case of complex variables (cf. Lai et al. $[11, 14, 15, 17, 18]$ ect.) as well as in set variable functions (e.g. [13]).

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In 1978, Hiriart-Urruty [7] established necessary optimality conditions in a single nonfractional objective function without differentiability or convexity assumptions. He used the concept of generalized gradient of a locally Lipschitz function to consider local extremization problems in global case for objective functions. For constraint case, it is considered by the way of cone adherent displacements, and optimality conditions in Kuhn-Tuker form under constraint qualification. Furthermore, saddle point theory is developed for the subdifferentiable multiobjective fractional programming problem by Bector et al. [3] in 1993 and the duality results for multiobjective fractional optimization problems under convexity are presented. Later in 1996, Liu [20] proved necessary and sufficient optimality conditions of multiobjective fractional programming problems containing nonsmooth $(F, \rho)$-convex functions. For such optimaization problems, they proved necessary and sufficient optimality conditions for minimization and duality theorems under generalized invex functions. Chen and Lai [4] in 2003 established sufficient optimality conditions and paramertic duality theorems for nondifferentiable fractional variational problems involving generalized $(F, \rho, \theta)$-invexity. They showed that generalized invex functions diagrams with interesting inclusion relations. Lee and Lai [19] proved sufficient optimality conditions for nondifferentiable fractional variational programming under certain specific structure of generalized invexity and employing the sufficient optimality conditions constructed two kinds of parameter-free dual models, namely the Mond-Wier dual type and Wolfe dual type are formulated. Lai [10] employed the sufficient optimality conditions to construct a mixed type dual programming problem and several duality theorems involving generalized invexity are proved. Lai and Huang [13] established the sufficient optimality conditions for a minimax programming problem involving $p$ fractional $n$-set functions under generalized invexity. One of such a notion is a nondifferentiable $V$-$r$-invex for vector function, introduced by Antczak [1]. He used a nondifferentiable $V$-$r$-invexity for a nonsmooth nonfractional multiobjective programming problem to derive Karush-Kuhn-Tuker necessary and sufficient optimality conditions. In [12], Lai and Chen established necessary and sufficient optimality conditions on a nondifferentiable minimax fractional programming problem. Applying the optimality conditions, they constituted two dual models: Mond-Weir type and Wolf type. On these duality types, they proved three duality theorems-weak, strong, and strict converse duality theorems.

Recently there are many authors interesting to establish sufficient optimality conditions and duality results for multiobjective programming problems in different representations. See for instance, in [5], Chinchuluun et al. obtained efficiency conditions and duality theorems for multiobjective fractional programming under $(C, \alpha, \rho, d)$-convexity. Kim et al. [8] considered a class of nondifferentiable multiobjective fractional programs in which each component of the objective function contains a term of support function for a compact convex set. Nobakhtian [23] established the necessary and sufficient optimality conditions under various generalized invexity assumptions. Lai and Ho [16] have established the theorems of sufficient optimality conditions and duality thoerems of parametric dual model under exponential (Exp. for brevity) $V$-$r$-invexity for Problem $(FP)$.

In this paper, we are motivated from the results of Antczak [1], Liu [20], and Lai and Chen [12], to establish duality theorems for a system of multiobjective fractional programmings under nonsmooth Exp. $V$-$r$-invex Lipschitz functions. We construct two kinds of parameter free dual models, the Mond-Wier type and Wolfe type duality form. For convenience, we recall firstly the notations and known results in the next section.

## 2. Definitions and Preliminaries

In Euclidean space $\mathbb{R}^m$, we denote by $\mathbb{R}^m_+$ the order cone. For cone partial order, if $x = (x_1, x_2, ..., x_m)$, $y = (y_1, y_2, ..., y_m)$ in $\mathbb{R}^m$, we denote

(1) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \dots, m$;
(2) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \dots, m$;
(3) $x \geqq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2, \dots, m$;
(4) $x \geq y$ if and only if $x \geqq y$ and $x_i \neq y_i$ for some $i \in \{1, 2, \dots, m\}$.

We shall use the following definitions.

**Definition 2.1.** The function $\theta : X \longrightarrow \mathbb{R}$ is said to be locally Lipschitz at $x \in X$ if there exists a positive constant $K \in \mathbb{R}$ and a neighborhood $\Gamma$ of $x \in X$ such that

$$|\theta(y) - \theta(z)| \leqq K\|y - z\| \qquad \text{for all} \quad z, y \in \Gamma,$$

where $\|\cdot\|$ denotes any norm of $X$.

Given a vector $\nu$ in $X$, the *generalized directional derivative* of function $\theta$ at $x \in X$ in the direction $\nu$ (Clarke sense, see [6]) is defined by

$$\theta^\circ(x; \nu) = \limsup_{\substack{y \longrightarrow x \\ \lambda \longrightarrow 0^+}} \frac{\theta(y + \lambda\nu) - \theta(y)}{\lambda}.$$

The *generalized subdifferential* of $\theta$ at $x \in X$ is defined as the subset in the dual space $X^*$ of $X$ by

$$\partial^\circ\theta(x) = \{\xi \in X^* : \theta^\circ(x; \nu) \geq \langle \xi, \nu \rangle \quad \text{for all} \quad \nu \in X\},$$

where $\langle \xi; \nu \rangle$ stands for the dual pair for $X$ and $X^*$.

Evidently, $\theta^\circ(x; \nu) = \max\{\langle \xi; \nu \rangle : \xi \in \partial^\circ\theta(x)\}$ for any $x$ and $\nu$ in $X$.

Let $h : X \to \mathbb{R}^m$ be a local Lipschitz function. For a point $x_0 \in X$, we define a subindex set at $x_0$ by

$$J(x_0) = \{j \in J : h_j(x_0) = 0\}, \qquad J = \{1, 2, \cdots, m\}.$$

Let $\Lambda = \{\nu \in X : h_j^\circ(x_0, \nu) < 0, j \in J(x_0)\}$. If $\Lambda \neq \emptyset$, we say that the problem $(FP)$ has constraint qualification at $x_0$ (cf. [7]).

In order to approve the duality theorems hold with respect to (w.r.t. for bievity) problem $(FP)$, we need the following definition in our framework.

**Definition 2.2** (cf. Antczak [1])**.** Let $r$ be a real number, $\eta : X \times X \longrightarrow X$ with property that $\eta(x, u) = 0$ only if $x = u$, $\theta : X \longrightarrow \mathbb{R}^k$ a locally Lipschitz function on $X$. The function $\theta$ is called an *Exp. $V$-$r$-invex (strictly)* function w.r.t. $\eta$ at $u \in X$ if, for each component $i = 1, 2 \cdots, k$, there exist functions $\alpha_i : X \times X \longrightarrow \mathbb{R}_+\backslash\{0\}$

and $\xi_i \in \partial^\circ\theta_i(u)$, such that the following inequality holds:

(2.1)
$$\frac{1}{r}e^{r\theta_i(x)} - \frac{1}{r}e^{r\theta_i(u)} \quad \begin{matrix} \geqq \\ (>) \end{matrix} \; e^{r\theta_i(u)}\alpha_i(x,u)\langle\xi_i,\ \eta(x,u)\rangle \quad \begin{matrix} \text{for all} \quad x \in X \qquad r \neq 0, \\ (x \neq u, \quad r \neq 0). \end{matrix}$$

As $r \to 0^+$, the above limit reduces another generalized $\eta$-invexity:

$$\theta_i(x) - \theta_i(u) \quad \begin{matrix} \geqq \alpha_i(x,u)\langle\xi_i,\ \eta(x,u)\rangle & & i = 1,2\cdots,k, \\ (>) & (x \neq u) & i = 1,2\cdots,k. \end{matrix} \tag{2.2}$$

If each component $\theta_i$ satisfies inequality (2.2), the function $\theta$ is still called $V$-invex at $u$ in $X$ w.r.t. $\eta$.

If the above inequalities hold for any point $u \in X$, then $\theta = (\theta_1, \theta_2, \cdots, \theta_n)$ is called an *Exp. $V$-$r$-invex (strictly)* w.r.t. $\eta$ on $X$.

It follows from definition 2, that the vector-valued function $\theta : X \longrightarrow \mathbb{R}^k$ is a locally Lipschitz Exp. $V$-$r$-invex function if each component $\theta_i$, $i = 1,2\cdots,k$, is a locally Lipschitz Exp. $r$-invex function, that is, the relations (2.1) holds. If $r = 0$, (2.2) is the special case of (2.1).

**Remark 2.3.** In this paper we will prove only in the case $r \neq 0$. While the case for $r = 0$ it is easily reduced.

A feasible solution $\overline{x}$ to $(FP)$ is said to be an *efficient solution* to $(FP)$ if there is no $x \in \mathfrak{F}$ such that $\phi(x) \leq \phi(\overline{x})$.

In order to establish duality theorems w.r.t. the primal problem $(FP)$, we need the result of necessary optimality conditions for problem $(FP)$. The sufficient optimality conditions of $(FP)$ follows from the converse of necessary conditions by extra assumptions. To approve the existence of solution for $(FP)$, at first, we tread with the subproblems $(SFP_i)$ of $(FP)$ as the following lemma.

**Lemma 2.4** (cf. [3]). *The point $\overline{x}$ is an optimal solution to problem $(FP)$ if and only if $\overline{x}$ solves for the subproblem $(SFP_i)$ which is given as the following:*

$$(SFP_i) \quad \textit{Minimize} \quad \frac{f_i(x)}{g_i(x)},$$

*subject to*

$$x \in M_i = \left\{ x \in X : \frac{f_p(x)}{g_p(x)} \leqq \frac{f_p(\overline{x})}{g_p(\overline{x})} \equiv \phi_p(\overline{x}),\ p \neq i,\ p = 1,2,\cdots,k,\ h(x) \in -\mathbb{R}^m_+ \right\},$$

*equivalently:*

$$M_i = \{x \in X : f_p(x) - \phi_p(\overline{x})g_p(x) \leqq 0,\ p \neq i,\ p = 1,2,\cdots,k,\ h(x) \in -\mathbb{R}^m_+\}.$$

**Theorem 2.5** (Necessary Optimality Conditions cf. [20]). *If $\overline{x}$ is an optimal solution of $(FP)$ satisfying constraint qualification for $(SFP_i)$, $i = 1,2,...,k$. Then, there exist $\alpha^* \in \mathbb{R}^k_+, z^* \in \mathbb{R}^m$ such that*

$$0 \in \sum_{i=1}^{k} \alpha_i^*[\partial^\circ f_i(\overline{x}) + \phi_i(\overline{x})\partial^\circ(-g_i)(\overline{x})] + \langle z^*, \partial^\circ h(\overline{x})\rangle_m, \tag{2.3}$$

$$z_j^* h_j(\overline{x}) = 0 \quad \textit{for all} \quad j = 1,2,...,m, \tag{2.4}$$

$$f_i(\overline{x}) - \phi_i(\overline{x})g_i(\overline{x}) = 0 \quad \textit{for all} \quad i = 1,2,...,k, \tag{2.5}$$

(2.6) $$\alpha^* \in I, z^* \in \mathbb{R}^m_+ \ \textit{if} \ \ h_j(x) < 0 \ \textit{for} \ x \in X,$$

*where* $$I = \{\alpha^* \in \mathbb{R}^k \,|\, \alpha^* = (\alpha_1^*, \alpha_2^*, \dots, \alpha_k^*) > 0, \ \textit{and} \ \sum_{i=1}^{k} \alpha_i^* = 1\},$$

*and* $\langle z^*, \partial^\circ h(\overline{x})\rangle_m \equiv \sum_{j=1}^m z_j^* \partial^\circ h_j(\overline{x})$.

We will use the extra condition by Exp. $V$-$r$-invexity to show the following sufficient optimality theorem:

**Theorem 2.6** (Sufficient Optimality Conditions cf. [16]). *Let $\overline{x} \in \mathfrak{F}$ and there exist $\alpha^*$, $z^*$ satisfying the conditions (2.3) $\sim$ (2.6) at the point $\overline{x}$. Furthermore suppose that any one of the following conditions (a) and (b) holds:*

(a) $A_1(x) = \sum_{i=1}^{k} \alpha_i^* [f_i(x) - \phi_i(\overline{x}) g_i(x)] + \sum_{j=1}^{m} z_j^* h_j(x)$ *is an Exp. $V$-$r$-invex function at $\overline{x}$ in $\mathfrak{F}$ w.r.t. $\eta$,*

(b) $A_2(x) = \sum_{i=1}^{k} \alpha_i^* [f_i(x) - \phi_i(\overline{x}) g_i(x)]$ *and* $A_3(x) = \sum_{j=1}^{m} z_j^* h_j(x)$ *are Exp. $V$-$r$-invex functions at $\overline{x}$ in $\mathfrak{F}$ w.r.t. the same function $\eta$.*

*Then, $\overline{x}$ is an efficient solution to problem $(FP)$.*

## 3. Wolfe type duality model

In order to propose Wolfe type dual model, it is convenient to restate the necessary conditions in Theorem 2.5 as the following form. Mainly it is used the expressions (2.3) and (2.5) to get

$$0 \in \sum_{i=1}^{k} \alpha_i^* [\partial^\circ f_i(\overline{x}) + \frac{f_i(\overline{x})}{g_i(\overline{x})} \partial^\circ(-g_i)(\overline{x})] + \langle z^*, \partial^\circ h(\overline{x})\rangle_m.$$

Then putting $\alpha^* = \overline{\alpha}^* g(\overline{x}) \in I$ in the above expression, we obtain

$$0 \in \sum_{i=1}^{k} \overline{\alpha}_i^* g_i(\overline{x}) [\partial^\circ f_i(\overline{x}) + \langle z^*, \partial^\circ h(\overline{x})\rangle_m] + \sum_{i=1}^{k} \overline{\alpha}_i^* f_i(\overline{x}) \partial^\circ(-g_i)(\overline{x}).$$

Consequently, from inequality (2.4), it yields

$$0 \in \sum_{i=1}^{k} \overline{\alpha}_i^* g_i(\overline{x}) [\partial^\circ f_i(\overline{x}) + \langle z^*, \partial^\circ h(\overline{x})\rangle_m] + \sum_{i=1}^{k} \overline{\alpha}_i^* [f_i(\overline{x}) + \langle z^*, h(\overline{x})\rangle_m] \partial^\circ(-g_i)(\overline{x}),$$

where $\langle z^*, h(\overline{x})\rangle_m \equiv \sum_{j=1}^m z_j^* h_j(\overline{x})$. For simplicity, we write $\overline{\alpha}_i^*$ still by $\alpha_i^*$. Then the result of Theorem 2.5 can be restated as the following theorem.

**Theorem 3.1** (Necessary Optimality Conditions). *If $\overline{x}$ is an efficient solution to $(FP)$ and satisfying constraint qualification in $(SFP_i)$, $i = 1, 2, ..., k$. Then, there exist $\alpha^* \in \mathbb{R}^k_+$, $z^* \in \mathbb{R}^m$ such that*

(3.1)
$$0 \in \sum_{i=1}^{k} \alpha_i^* g_i(\overline{x}) [\partial^\circ f_i(\overline{x}) + \langle z^*, \partial^\circ h(\overline{x})\rangle_m] + \sum_{i=1}^{k} \alpha_i^* [f_i(\overline{x}) + \langle z^*, h(\overline{x})\rangle_m] \partial^\circ(-g_i)(\overline{x}),$$

$$z_j^* h_j(\overline{x}) = 0 \quad \text{for all} \quad j = 1, 2, ..., m, \tag{3.2}$$

$$\alpha^* \in I,\ z^* \in \mathbb{R}_+^m \ \text{if}\ h_j(x) < 0 \ \text{for}\ x \in X. \tag{3.3}$$

For convenience we change the elements $(\overline{x}; \alpha^*, z^*)$ in the necessary optimality conditions (3.1) $\sim$ (3.3) by using $(u; \alpha, z)$ as new elements in the relative expressions, where $u \in \mathfrak{F}$ is a feasible solution of problem $(FP)$ and $(\alpha, z)$ are the multipliers in the necessary conditions (3.1) $\sim$ (3.3) which are applied as the constraints of the new duality problems.

Since the constraint function $h(u) \in -\mathbb{R}_+^m$ in problem $(FP)$ and the multiplier $z \in \mathbb{R}_+^m$, thus $\langle z, h(u) \rangle_m = \sum_{j=1}^m z_j h_j(u) \leq 0$. By adding $\langle z, h(u) \rangle_m$ into the numerator of the fractional component in the multiobjective of $(FP)$. Consequently, it becomes the following new systems of multiobjective fractional functions in a maximization problem $(WD)$, namely Wolfe type dual problem. Precisely, we introduce the Wolfe type dual $(WD)$ with w.r.t. the primal problem $(FP)$ as follows

$$(WD) \qquad \text{Maximize} \qquad \Psi(u, z) \equiv \left( \frac{f_1(u) + \langle z, h(u) \rangle_m}{g_1(u)}, \ldots, \frac{f_k(u) + \langle z, h(u) \rangle_m}{g_k(u)} \right) \equiv (\Psi_1(u, z), \Psi_2(u, z), ..., \Psi_k(u, z))$$

subject to

$$0 \in \sum_{i=1}^k \alpha_i g_i(u)[\partial^\circ f_i(u) + \langle z, \partial^\circ h(u) \rangle_m] + \sum_{i=1}^k \alpha_i [f_i(u) + \langle z, h(u) \rangle_m] \partial^\circ(-g_i)(u), \tag{3.4}$$

$$u \in X,\ \alpha \in I,\ z \in \mathbb{R}_+^m. \tag{3.5}$$

Here we may assume that $f_i(u) + \langle z, h(u) \rangle_m \geqq 0$ and $g_i(u) > 0$, for all $i = 1, 2, ..., k$.

In order to show problem $(WD)$ is surely a dual problem w.r.t. the problem $(FP)$, we denote $\mathfrak{D}_1$ by the set of feasible points $(u; \alpha, z) \in X \times \mathbb{R}_+^k \times \mathbb{R}^m$ of $(WD)$ satisfying the expressions (3.4) and (3.5) of $(WD)$. Moreover, denote by elements satisfying the necessary optimality conditions of $(FP)$ which is defined by the projective-like of the feasible solutions of problem $(FP)$:

$$pr_X \mathfrak{D}_1 = \{u \in X |\ (u; \alpha, z) \in \mathfrak{D}_1\}.$$

**Theorem 3.2** (Weak Duality)**.** *Let $x$ and $(u; \alpha, z)$ be $(FP)$-feasible and $(WD)$-feasible, respectively. Denote a function $A_4 : X \to \mathbb{R}$, by*

$$A_4(\cdot) = \sum_{i=1}^k \alpha_i g_i(u)[f_i(\cdot) + \langle z, h(\cdot) \rangle_m] - \sum_{i=1}^k \alpha_i g_i(\cdot)[f_i(u) + \langle z, h(u) \rangle_m]$$

*with $A_4(u) = 0$. Suppose that $A_4(\cdot)$ is an Exp. $V$-$r$-invex function at $u \in \mathfrak{F} \cap pr_X \mathfrak{D}_1$ w.r.t. $\eta$.*
*Then $\phi(x) \not\leq \Psi(u, z)$.*

*Proof.* Let $x$ and $(u; \alpha, z)$ be $(FP)$ and $(WD)$-feasible solutions, respectively. According to expression (3.4), there exist $\xi_i \in \partial^\circ f_i(u)$, $\zeta_i \in \partial^\circ(-g_i)(u)$, $i = 1, 2, ..., k$, and $\rho_j \in \partial^\circ h_j(u)$, $j = 1, 2, ..., m$, such that

$$\langle a_4 \rangle \equiv \sum_{i=1}^{k} \alpha_i g_i(u)[\xi_i + \langle z, \rho \rangle_m] + \sum_{i=1}^{k} \alpha_i [f_i(u) + \langle z, h(u) \rangle_m]\zeta_i = 0 \text{ in } X^*,$$

that is, $\langle a_4 \rangle$ is a zero vector of $X^*$, where $\langle z, \rho \rangle_m \equiv \sum_{j=1}^{m} z_j \rho_j$ and $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$.

Since $\eta(x, u) \in X$ and $\langle a_4 \rangle \in X^*$, the dual pair $\langle a_4 \rangle$ and $\eta(x, u)$ of $\langle X^*, X \rangle$ reduces

$$\langle \langle a_4 \rangle,\ \eta(x, u) \rangle = 0. \tag{3.6}$$

Since $A_4$ is an Exp. $V$-$r$-invex function w.r.t. $\eta$ at $u \in \mathfrak{F} \cap pr_X \mathfrak{D}_1$, there exists a function $a_4 : (\mathfrak{F} \cap pr_X \mathfrak{D}_1) \times (\mathfrak{F} \cap pr_X \mathfrak{D}_1) \longrightarrow \mathbb{R}_+ \setminus \{0\}$ such that

$$\frac{1}{r} e^{rA_4(x)} - \frac{1}{r} e^{rA_4(u)} \geqq e^{rA_4(u)} a_4(x, u) \cdot \langle \langle a_4 \rangle,\ \eta(x, u) \rangle = 0 \text{ (by (3.6))},$$

that is,

$$\frac{1}{r} e^{rA_4(x)} - \frac{1}{r} e^{rA_4(u)} \geqq 0.$$

This deduces to

$$A_4(x) \geq A_4(u) = 0. \tag{3.7}$$

We want to prove that $\phi(x) \nleq \Psi(u, z)$.

Suppose on the contrary that $\phi(x) \leq \Psi(u, z)$. Then

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(u) + \langle z, h(u) \rangle_m}{g_i(u)} \qquad \text{for all} \qquad i = 1, 2, ..., k,$$

but $h_j(u) \leq 0$ for any $j \in \{1, 2, \cdots, m\}$, thus

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(u) + \langle z, h(u) \rangle_m}{g_t(u)} \qquad \text{for some} \qquad t \in \underline{k} = \{1, 2, \ldots, k\}.$$

It follows that

$$\sum_{i=1}^{k} \alpha_i [f_i(x) g_i(u)] < \sum_{i=1}^{k} \alpha_i g_i(x)[f_i(u) + \langle z, h(u) \rangle_m].$$

That is, (Remove the right hand side to left hand side in the above inequality, it becomes to the form as the following expression.)

$$\begin{aligned} &\sum_{i=1}^{k} \alpha_i g_i(u)[f_i(x) + \langle z, h(x) \rangle_m] - \sum_{i=1}^{k} \alpha_i g_i(x)[f_i(u) + \langle z, h(u) \rangle_m] = A_4(x) \\ &< 0 + \sum_{i=1}^{k} \alpha_i g_i(u) \langle z, h(x) \rangle_m. \end{aligned} \tag{3.8}$$

Since $g_i(u) > 0$ and $h_j(x) \leq 0$, it yields $\sum_{i=1}^{k} \alpha_i g_i(u)\langle z, h(x)\rangle_m \leq 0$.

Therefore, from (3.8),

$$A_4(x) = \sum_{i=1}^{k} \alpha_i g_i(u)[f_i(x) + \langle z, h(x)\rangle_m] - \sum_{i=1}^{k} \alpha_i g_i(x)[f_i(u) + \langle z, h(u)\rangle_m] < 0.$$

This contradicts the inequality (3.7). Hence the proof is complete. □

**Theorem 3.3** (Strong Duality)**.** *Let $\overline{x}$ be the efficient solution of problem $(FP)$ satisfying the constraint qualification at $\overline{x}$. Then there exist $\alpha^* \in \mathbb{R}^k$, and $z^* \in \mathbb{R}^m$ such that $(\overline{x}; \alpha^*, z^*) \in (WD)$-feasible. If the hypotheses of Theorem 3.2 are fulfilled, then $(\overline{x}; \alpha^*, z^*)$ is an efficient solution to problem $(WD)$.*

*Proof.* Let $\overline{x}$ be an efficient solution to problem $(FP)$. Then there exist $\alpha^*$, $z^*$ such that $(\overline{x}; \alpha^*, z^*)$ satisfies (3.1) $\sim$ (3.3), and so $(\overline{x}; \alpha^*, z^*)$ is a feasible point of $(WD)$. Actually $(\overline{x}; \alpha^*, z^*)$ is also an efficient solution of $(WD)$.

If $(\overline{x}; \alpha^*, z^*)$ is not an efficient solution to $(WD)$, then there must have some feasible solution $(x; \alpha, z)$ of $(WD)$, such that

$$\frac{f_i(\overline{x}) + \langle z^*, h(\overline{x})\rangle_m}{g_i(\overline{x})} \leqq \frac{f_i(x) + \langle z, h(x)\rangle_m}{g_i(x)} \quad \text{for all} \quad i = 1, 2, ..., k,$$

and by the constraint qualification of $(FP)$, there is some index $t \in \underline{k}$ to satisfy:

$$\frac{f_t(\overline{x}) + \langle z^*, h(\overline{x})\rangle_m}{g_t(\overline{x})} < \frac{f_t(x) + \langle z, h(x)\rangle_m}{g_t(x)}.$$

It follows from the above inequalities and equality 0 of (3.2) that $\phi(\overline{x}) \leq \Psi(x, z)$ which is a contradiction with Theorem 3.2. Hence $(\overline{x}; \alpha^*, z^*)$ is an efficient solution of $(WD)$. □

**Theorem 3.4** (Strict Converse Duality)**.** *Let $\overline{x}$ and $(u^*; \alpha^*, z^*)$ be the efficient solutions to $(PF)$ and $(WD)$, respectively. Denote a function $A_5 : X \to \mathbb{R}$, by*

$$A_5(\cdot) = \sum_{i=1}^{k} \alpha_i^* g_i(u^*)[f_i(\cdot) + \langle z^*, h(\cdot)\rangle_m] - \sum_{i=1}^{k} \alpha_i^* g_i(\cdot)[f_i(u^*) + \langle z^*, h(u^*)\rangle_m]$$

*with $A_5(u^*) = 0$. Assume that $A_5(\cdot)$ is a strictly Exp. $V$-$r$-invex function at $u^* \in \mathfrak{F} \cap pr_X \mathfrak{D}_1$ w.r.t. $\eta$ for all optimal vectors $\overline{x}$ for $(FP)$ and $(u^*; \alpha^*, z^*)$ for $(WD)$, respectively. Then $\overline{x} = u^*$, and the efficient values of $(FP)$ and $(WD)$ are equal.*

*Proof.* Suppose that $\overline{x} \neq u^*$. From the relation (3.4), there exist $\xi_i \in \partial^\circ f_i(u^*)$, $\zeta_i \in \partial^\circ(-g_i)(u^*)$, $i = 1, 2, ..., k$, and $\rho_j \in \partial^\circ h_j(u^*)$, $j = 1, 2, ..., m$, such that

$$\langle a_5\rangle \equiv \sum_{i=1}^{k} \alpha_i^* g_i(u^*)[\xi_i + \langle z^*, \rho\rangle_m] + \sum_{i=1}^{k} \alpha_i^*[f_i(u^*) + \langle z^*, h(u^*)\rangle_m]\zeta_i = 0 \in X^*,$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$. It follows that the dual pair:

$$\langle\langle a_5\rangle,\ \eta(\overline{x}, u^*)\rangle = 0. \tag{3.9}$$

By Theorem 3.3, there exist $\overline{\alpha}$ and $\overline{z}$ so that $(\overline{x}; \overline{\alpha}, \overline{z})$ becomes an efficient solution of $(WD)$ and

$$\frac{f_i(\overline{x}) + \langle \overline{z}, h(\overline{x})\rangle_m}{g_i(\overline{x})} = \frac{f_i(u^*) + \langle z^*, h(u^*)\rangle_m}{g_i(u^*)}. \tag{3.10}$$

On the other hand, from (3.2) and (3.10), we obtain

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} = \frac{f_i(u^*) + \langle z^*, h(u^*)\rangle_m}{g_i(u^*)}.$$

Hence

$$f_i(\overline{x})g_i(u^*) = [f_i(u^*) + \langle z^*, h(u^*)\rangle_m]g_i(\overline{x}).$$

That is,

$$f_i(\overline{x})g_i(u^*) - [f_i(u^*) + \langle z^*, h(u^*)\rangle_m]g_i(\overline{x}) = 0. \tag{3.11}$$

From (3.2), (3.5), and (3.11), we obtain

$$\sum_{i=1}^{k} \alpha_i^* g_i(u^*)[f_i(\overline{x}) + \langle z^*, h(\overline{x})\rangle_m] - \sum_{i=1}^{k} \alpha_i^* g_i(\overline{x})[f_i(u^*) + \langle z^*, h(u^*)\rangle_m] = A_5(\overline{x})$$
$$= \sum_{i=1}^{k} \alpha_i^* g_i(u^*)\langle z^*, h(\overline{x})\rangle_m.$$

Since $h(\overline{x}) \in -\mathbb{R}^m_+$, (3.5) and $g_i(u^*) > 0$, we see that

$$A_5(\overline{x}) \leqq 0$$

and $A_5$ at the point $u^*$ is 0, thus

$$A_5(\overline{x}) \leqq 0 = A_5(u^*). \tag{3.12}$$

Since $A_5$ is a strictly Exp. $V$-$r$-invex function w.r.t. $\eta$ at $u^* \in \mathfrak{F} \cap pr_X\mathfrak{D}_1$, there exists a function $a_5 : (\mathfrak{F} \cap pr_X\mathfrak{D}_1) \times (\mathfrak{F} \cap pr_X\mathfrak{D}_1) \longrightarrow \mathbb{R}_+ \setminus \{0\}$ such that

$$\frac{1}{r}e^{rA_5(\overline{x})} - \frac{1}{r}e^{rA_5(u^*)} > e^{rA_5(u^*)}a_5(\overline{x}, u^*) \cdot \langle\langle a_5\rangle,\ \eta(\overline{x}, u^*)\rangle . \tag{3.13}$$

Therefore, the inequalities (3.12) and (3.13) yield

$$\langle\langle a_5\rangle,\ \eta(\overline{x}, u^*)\rangle < 0,$$

which contradicts the equality (3.9). Hence, the proof is complete. □

## 4. Mond-Weir type duality model

For any $u \in \mathfrak{F}$, if use $(\alpha, z) \in \mathbb{R}^k \times \mathbb{R}^m$ instead of $(\alpha^*, z^*) \in \mathbb{R}^k \times \mathbb{R}^m$ satisfying the necessary conditions (3.1) $\sim$ (3.3) as the constraints of a new dual problem. Namely, Mond-Weir type dual $(MWD)$, then it constitutes by a maximization programming problem with the same objective function as the problem $(FP)$, and use the necessary optimality conditions of $(FP)$ as the constraint of the new problem $(MWD)$. Precisely, we can state this dual problem as the maximization problem in the following form:

$$(MWD) \qquad \text{Maximize} \qquad \Phi(u) \equiv \left(\frac{f_1(u)}{g_1(u)},\ \frac{f_2(u)}{g_2(u)}, ...,\ \frac{f_k(u)}{g_k(u)}\right) \equiv (\Psi_1(u),\ \Psi_2(u), ...,\ \Phi_k(u))$$

subject to $u \in X$ and

$$(4.1)\quad 0 \in \sum_{i=1}^{k} \alpha_i g_i(u)[\partial^\circ f_i(u) + \langle z, \partial^\circ h(u)\rangle_m] + \sum_{i=1}^{k} \alpha_i \partial^\circ(-g_i)(u)[f_i(u) + \langle z, h(u)\rangle_m],$$

$$(4.2)\qquad \langle z, h(u)\rangle_m = 0,$$

$$(4.3)\qquad u \in X, \alpha \in I, z \in \mathbb{R}^m_+.$$

Let $\mathfrak{D}_2$ be the constraint set $\{u; \alpha, z\}$ of $(MWD)$ satisfying (4.1) $\sim$ (4.3) which are the necessary optimality conditions of $(FP)$. For convenience we denote the projective-like set by:

$$pr_X\mathfrak{D}_2 = \{u \in X |\ (u; \alpha, z) \in \mathfrak{D}_2\}.$$

**Theorem 4.1** (Weak Duality)**.** *Let $x$ and $(u; \alpha, z)$ be $(FP)$-feasible and $(MWD)$-feasible,respectively. Denote a function $A_6 : X \to \mathbb{R}$, by*

$$A_6(\cdot) = \sum_{i=1}^{k} \alpha_i g_i(u)[f_i(\cdot) + \langle z, h(\cdot)\rangle_m] - \sum_{i=1}^{k} \alpha_i g_i(\cdot)[f_i(u) + \langle z, h(u)\rangle_m]$$

*with $A_6(u) = 0$. Suppose that $A_6(\cdot)$ is an Exp. $V$-$r$-invex function at $u \in \mathfrak{F} \cap pr_X\mathfrak{D}_2$ w.r.t. $\eta$.*
*Then $\phi(x) \nleq \Phi(u)$.*

*Proof.* Let $x$ and $(u; \alpha, z)$ be $(FP)$ and $(MWD)$-feasibles, respectively. From the expression (4.1), there exist $\xi_i \in \partial^\circ f_i(u)$, $\zeta_i \in \partial^\circ(-g_i)(u)$, $i = 1, 2, ..., k$, and $\rho_j \in \partial^\circ h_j(u)$, $j = 1, 2, ..., m$, to satisfy

$$\langle a_6\rangle \equiv \sum_{i=1}^{k} \alpha_i g_i(u)[\xi_i + \langle z, \rho\rangle_m] + \sum_{i=1}^{k} \alpha_i [f_i(u) + \langle z, h(u)\rangle_m]\zeta_i = 0 \in X^*,$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$.
It follows from $\eta(x, u) \in X$ that the dual pair of $\langle X^*, X\rangle$ yields

$$(4.4)\qquad \langle\langle a_6\rangle,\ \eta(x, u)\rangle = 0.$$

Since $A_6$ is an Exp. $V$-$r$-invex function w.r.t. $\eta$ at $u \in \mathfrak{F} \cap pr_X\mathfrak{D}_2$, there exists a function
$a_6 : (\mathfrak{F} \cap pr_X\mathfrak{D}_2) \times (\mathfrak{F} \cap pr_X\mathfrak{D}_2) \longrightarrow \mathbb{R}_+ \setminus \{0\}$ such that

$$\frac{1}{r}e^{rA_6(x)} - \frac{1}{r}e^{rA_6(u)} \ \geq\ e^{rA_6(u)}a_6(x, u) \cdot \langle\langle a_6\rangle,\ \eta(x, u)\rangle = 0. \text{ (by (4.4))}$$

Hence we obtain

$$\frac{1}{r}e^{rA_6(x)} - \frac{1}{r}e^{rA_6(u)} \geq 0.$$

It follows that

$$(4.5)\qquad A_6(x) \geq A_6(u) = 0.$$

We want to prove that $\phi(x) \nleq \Phi(u)$.

Suppose on the contrary that $\phi(x) \leq \Phi(u)$. Then

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(u)}{g_i(u)} \qquad \text{for all} \qquad i = 1, 2, ..., k,$$

and there is some index $t \in \underline{k}$, such that

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(u)}{g_t(u)}.$$

Then, by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha) \in I$, we have

$$\sum_{i=1}^{k} \alpha_i f_i(x) g_i(u) < \sum_{i=1}^{k} \alpha_i g_i(x) f_i(u). \tag{4.6}$$

Since $h(x) \in -\mathbb{R}^m_+$, it follows from (4.2), (4.3), and (4.6) that

$$\sum_{i=1}^{k} \alpha_i g_i(u)[f_i(x) + \langle z, h(x)\rangle_m] < \sum_{i=1}^{k} \alpha_i g_i(x)[f_i(u) + \langle z, h(u)\rangle_m].$$

This implies

$$A_6(x) = \sum_{i=1}^{k} \alpha_i g_i(u)[f_i(x) + \langle z, h(x)\rangle_m] - \sum_{i=1}^{k} \alpha_i g_i(x)[f_i(u) + \langle z, h(u)\rangle_m] < 0,$$

which contradicts the inequality (4.5) and the proof of theorem is complete. □

**Theorem 4.2** (Strong Duality). *Let $\overline{x}$ be the efficient solution of problem $(FP)$ satisfying the constraint qualification at $\overline{x}$. Then there exist $\alpha^* \in \mathbb{R}^k$, and $z^* \in \mathbb{R}^m$ such that $(\overline{x}; \alpha^*, z^*) \in (MWD)$-feasible. If the hypotheses of Theorem 4.1 are fulfilled, then $(\overline{x}; \alpha^*, z^*)$ is an efficient solution to problem $(MWD)$. Furthermore the efficient values of $(FP)$ and $(MWD)$ are equal.*

*Proof.* Let $\overline{x}$ be an efficient solution to problem $(FP)$. Then there exist $\alpha^*$,$z^*$ such that $(\overline{x}; \alpha^*, z^*)$ satisfies (4.1) $\sim$ (4.3), that is, $(\overline{x}; \alpha^*, z^*) \in \mathfrak{D}_2$ is a feasible solution for the problem $(MWD)$. Actually $(\overline{x}; \alpha^*, z^*)$ is also an efficient solution of $(MWD)$.
Suppose on the contrary that if $(\overline{x}; \alpha^*, z^*)$ were not an efficient solution to $(MWD)$. Then there exists a feasible solution $(x; \alpha, z)$ of $(MWD)$ such that

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} \leqq \frac{f_i(x)}{g_i(x)} \quad \text{for all} \quad i = 1, 2, ..., k,$$

and there is a $t \in \underline{k}$,

$$\frac{f_t(\overline{x})}{g_t(\overline{x})} < \frac{f_t(x)}{g_t(x)}.$$

It follows that $\phi(\overline{x}) \leq \Phi(x)$ which contradicts the weak duality Theorem 4.1. Hence $(\overline{x}; \alpha^*, z^*)$ is an efficient solution of $(MWD)$ and the efficient values of $(FP)$ and $(MWD)$ are clearly equal. □

**Theorem 4.3** (Strict Converse Duality)**.** *Let $\overline{x}$ and $(u^*;\alpha^*,z^*)$ be the efficient solutions of $(FP)$ and $(MWD)$, respectively. Denote a function $A_7 : X \to \mathbb{R}$, by*

$$A_7(\cdot) = \sum_{i=1}^{k} \alpha_i^* g_i(u^*)[f_i(\cdot) + \langle z^*, h(\cdot)\rangle_m] - \sum_{i=1}^{k} \alpha_i^* g_i(\cdot)[f_i(u^*) + \langle z^*, h(u^*)\rangle_m].$$

*with $A_7(u^*) = 0$. If $A_7(\cdot)$ is a strictly Exp. $V$-$r$-invex function at $u^* \in \mathfrak{F} \cap pr_X\mathfrak{D}_2$ w.r.t. $\eta$ for all optimal vectors $\overline{x}$ in $(FP)$ and $(u^*;\alpha^*,z^*)$ in $(MWD)$, respectively. Then $\overline{x} = u^*$, and the efficient values of $(FP)$ and $(MWD)$ are equal.*

*Proof.* Suppose that $\overline{x} \neq u^*$. From the expression (4.1), there exist $\xi_i \in \partial^\circ f_i(u^*)$, $\zeta_i \in \partial^\circ(-g_i)(u^*)$, $i = 1, 2, ..., k$, and $\rho_j \in \partial^\circ h_j(u^*)$, $j = 1, 2, ..., m$, such that

$$\langle a_7\rangle \equiv \sum_{i=1}^{k} \alpha_i^* g_i(u^*)[\xi_i + \langle z^*, \rho\rangle_m] + \sum_{i=1}^{k} \alpha_i^*[f_i(u^*) + \langle z^*, h(u^*)\rangle_m]\zeta_i = 0 \in X^*,$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$.
It follows that the dual pair in $\langle X^*, X\rangle$ becomes

$$\langle\langle a_7\rangle,\ \eta(\overline{x}, u^*)\rangle = 0. \tag{4.7}$$

From Theorem 4.2, we see that there exist $\overline{\alpha}$ and $\overline{z}$ such that $(\overline{x};\overline{\alpha},\overline{z})$ is the efficient solution of $(MWD)$ and

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} = \frac{f_i(u^*)}{g_i(u^*)} \quad \text{for all} \quad i = 1, 2, \ldots, k. \tag{4.8}$$

By inequality (4.2) and equality (4.8), it becomes

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} = \frac{f_i(u^*) + \langle z^*, h(u^*)\rangle_m}{g_i(u^*)}. \tag{4.9}$$

Eliminate the dominators in (4.9), we get

$$f_i(\overline{x})g_i(u^*) = [f_i(u^*) + \langle z^*, h(u^*)\rangle_m]g_i(\overline{x}).$$

or

$$f_i(\overline{x})g_i(u^*) - [f_i(u^*) + \langle z^*, h(u^*)\rangle_m]g_i(\overline{x}) = 0.$$

According to the above equality and by the property (4.3), $A_7(\overline{x})$ reduces to
(4.10)

$$\sum_{i=1}^{k} \alpha_i^* g_i(u^*)[f_i(\overline{x}) + \langle z^*, h(\overline{x})\rangle_m] - \sum_{i=1}^{k} \alpha_i^* g_i(\overline{x})[f_i(u^*) + \langle z^*, h(u^*)\rangle_m] = A_7(\overline{x})$$
$$= \sum_{i=1}^{k} \alpha_i^* g_i(u^*)\langle z^*, h(\overline{x})\rangle_m.$$

From relations $h(\overline{x}) \in -\mathbb{R}_+^m$, (4.3), (4.10), and $g_i(u^*) > 0$, we obtain

$$A_7(\overline{x}) \leqq\ 0\ = A_7(u^*).$$

Hence

$$\frac{1}{r}e^{rA_7(\overline{x})} - \frac{1}{r}e^{rA_7(u^*)} \leqq 0 \qquad \text{for any} \quad r \neq 0. \tag{4.11}$$

Since $A_7$ is a strictly Exp. $V$-$r$-invex function w.r.t. $\eta$ at $u^* \in \mathfrak{F} \cap pr_X \mathfrak{D}_2$, there exists a mapping $a_7 : (\mathfrak{F} \cap pr_X \mathfrak{D}_2) \times (\mathfrak{F} \cap pr_X \mathfrak{D}_2) \longrightarrow \mathbb{R}_+ \setminus \{0\}$ such that

$$\frac{1}{r} e^{rA_7(\overline{x})} - \frac{1}{r} e^{rA_7(u^*)} \;>\; e^{rA_7(u^*)} a_7(\overline{x}, u^*) \cdot \langle \langle a_7 \rangle,\ \eta(\overline{x}, u^*) \rangle . \tag{4.12}$$

From (4.11) and (4.12), we obtain

$$\langle \langle a_7 \rangle,\ \eta(x^*, u^*) \rangle < 0.$$

This contradicts the equality (4.7). Hence, the proof of theorem is complete. $\square$

**Remark 4.4.** Finally, it is remarkable that one can easily see that the feasible sets of $\mathfrak{D}_1$ of $(WD)$ and $\mathfrak{D}_2$ of $(MWD)$ are essentially equivalent.

**Remark 4.5.** There is a plausible problem can be derived for minimax programming problem. For example, if the finite index set $\underline{k}$ in $(FP)$ is replaced by a compact Banach space $Y$, then problem $(FP)$ could formate to be a minimax programming problem with objective function $\phi(x, y)$. Thus problem becomes

$$\min_{x \in X} \max_{y \in Y} \quad \phi(x, y) = \frac{f(x, y)}{g(x, y)}$$

such that a cone constraint set in another Banach space $Z$ with order cone $C$, and $h : X \to (Z, C)$ satisfying $-h(x) \in C$,

where $f$ and $g$ need some reasonable conditions.

## References

[1] T. Antczak, *Optimality and duality for nonsmooth multiobjective programming problems with $V$-$r$-invexity*, J. Glob. Optim. **45** (2009), 319–334.

[2] D. Bahatia, and P. K. Garg, *Duality for nonsmooth nonlinear fractional multiobjective programs via $(F, \rho)$-convexity*, Optimization **43** (1998), 185–197.

[3] C. R. Bector, S. Chandra and I. Husain, *Optimality conditions and duality in subdifferentiable multiobjective fractional programming*, J. Optim.Theory Appl. **79** (1993), 105–125.

[4] J. C. Chen and H. C. Lai, *Fractional programming for variational problem with $(F, \rho, \theta)$-invexity*, J. Nonlinear Convex Anal. **4** (2003), 25–41.

[5] A. Chinchuluun, D. Yuan and P. M. Pardalos, *Optimality conditions and duality for nondifferentiable multiobjective fractional programming with generalized convexity*, Ann. Oper. Res. **154** (2007), 133–147.

[6] F. H. Clarke, *Optimization and Non-smooth Analysis*, Wiley-Interscience, New York. 1983.

[7] J. B. Hiriart-Urruty, *On optimality conditions in nondifferentiable programming*, Math. Prog. **14** (1978), 73–86.

[8] D. S. Kim, S. J. Kim and M. H. Kim, *Optimality and duality for a class of nondifferentiable multiobjective fractional programming problems*, J. Optim. Theory Appl. **129** (2006), 131–146.

[9] D. S. Kim and S. Schaible, *Optimality and duality for invex nonsmooth multiobjective programming problems*, Optimization **53** (2004), 165–176.

[10] H. C. Lai, *Sufficiency and duality of variational programming with generalized invexity*, Taiwanese J. Math. **10** (2006), 1685–1695.

[11] H. C. Lai and J. C. Chen, *Optimality conditions for minimax programming of analytic functions*, Taiwanese J. Math. **8** (2004), 673–686.

[12] H. C. Lai and H. M. Chen, *Duality on a nondifferentiable minimax fractional programming*, J. Glob. Optim. DOI:10,1007/s-10898-010-9631-8, (2010).

[13] H. C. Lai and T. Y. Huang, *Minimax fractional programming for n-set functions and mixed type duality under generalized invexity*, J. Optim. Theory Appl. **139** (2008), 295–313.

[14] H. C. Lai and T. Y. Huang, *Optimality conditions for a nondifferentiable minimax programming in complex spaces*, Nonlinear Anal. **71** (2009), 1205–1212.

[15] H. C. Lai and T. Y. Huang, *Complex analysis methods related an optimization problem with complex variables*, Eur. J. Pure Appl. Math. **3** (2010), 989–1005.

[16] H. C. Lai and S. C. Ho, *Optimality and duality for nonsmooth multiobjective fractional programmings involving exponential $V$-$r$-invexity*, Nonlinear Anal. (2011), doi:10.1016/j.na.2011.12.013.

[17] H. C. Lai, J. C. Lee and S. C. Ho, *Parametric duality on minimax programming involving generealized convexity in complex spaces*, J. Math. Anal. Appl. **323** (2006), 1104–1115.

[18] H. C. Lai, J. C. Liu and S. Schaible, *Complex minimax fractional programming of analytic functions*, J. Optim. Theory Appl. **137** (2008), 171–184.

[19] J. C. Lee and H. C. Lai, *Parameter-free dual moddels for fractional programming with generalized invexity*, Annals Oper. Res. **133** (2005), 47–61.

[20] J. C. Liu, *Optimality and duality for multiobjectional fractional programming involving nonsmooth $(F, \rho)$-convex functions*, Optimization **36** (1996), 333–346.

[21] S. Liu and E. Feng, *Optimality conditions and duality for a class of nondifferentiable multiobjective fractional programming problems*, J. Glob. Optim. **38** (2007), 653–666.

[22] S. K. Mishra and R. N. Mukherjee, *On generalized convex multi-objective nonsmooth programming*, J. Austral. Math. Soc. Ser. **B 38** (1996) 140–148.

[23] S. Nobakhtian, *Optimality and duality for nonsmooth multiobjective fractional programming with mixed constraints*, J. Glob Optim. **41** (2008), 103–115.

[24] C. Singh, and M. A. Hanson, *Saddle point theory for nondifferentiable multiobjective fractional programming*, J. Inform. Optim. Sci. **7** (1986), 41–48.

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