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# ERGODIC THEOREMS FOR REVERSIBLE SEMIGROUPS OF NONLINEAR OPERATORS

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ABSTRACT. Let $S$ be a semitopological semigroup and $C$ be a closed convex subset of a Hilbert space $H$ and $\Im = \{T_s : s \in S\}$ be a continuous representation of $S$ as weakly nonexpansive mappings of $C$ into itself. In this paper, we prove that if $S$ is right reversible, $F(\Im)$ is nonempty and the net $\|T_s x - T_{gs}(x)\| \to 0$ for each fixed $g$ in a generating set of $S$, then the net $\{T_s(x) : s \in S\}$ converges weakly to an element in $F(\Im)$. Also if the space $RUC(S)$ of right uniformly continuous functions on $S$ has a left invariant mean and there exist $x \in C$ with bounded orbit, then $C$ contains a common fixed point for $\Im$.

## 1. Introduction

Let $S$ be a *semitopological semigroup* with identity, i.e., $S$ is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from $S$ to $S$ are continuous.

A semitopological semigroup $S$ is *left reversible* if any two closed right ideals of $S$ has nonvoid intersection, i.e., $\overline{aS} \cap \overline{bS} \neq \emptyset, a, b \in S$. In this case, $(S, \preceq)$ is a directed system when the binary relation $'' \preceq ''$ on $S$ is defined by $a \preceq b$ if and only if $\{a\} \cup \overline{aS} \supseteq \{b\} \cup \overline{bS}, a, b \in S$. Left reversible semitopological semigroups include all commutative semigroup. Right reversibility of $S$ is defined similarly. $S$ is called *reversible* if it is both left and right reversible.

Let $S$ be a semitopological semigroup with identity, $H$ be a real Hilbert space and $\Im = \{T_s : s \in S\}$ be a continuous representation of $S$ on a closed convex subset $C$ of $H$ into $C$, i.e.,
(i) $T_e = I$,
(ii) $T_{ab}(x) = T_a T_b(x), a, b \in S, x \in C$,
(iii) the mapping $(s, x) \to T_s(x)$ from $S \times C$ into $C$ is continuous when $S \times C$ has the product topology.

Let $F(\Im)$ denotes the set $\{x \in C : T_s(x) = x$ for all $s \in S\}$ of common fixed points of $\Im$ in $C$. Then, as is well known, $F(\Im)$ is a closed convex subset of $C$.

In [10], Lau proved the following: Let $S$ be a semitopological semigroup and $C$ be a closed convex subset of a Hilbert space $H$ and $\Im = \{T_s : s \in S\}$ be a continuous representation of $S$ as nonexpansive mappings of $C$ into itself. If $S$ is right reversible, $F(\Im)$ is nonempty and the net $\|T_s x - T_{gs}(x)\| \to 0$ for each fixed $g$ in a generating set of $S$, then the net $\{T_s(x) : s \in S\}$ converges weakly to an element in $F(\Im)$. Also if the space $RUC(S)$ of right uniformly continuous functions

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on $S$ has a left invariant mean and there exist $x \in C$ with bounded orbit, then $C$ contains a common fixed point for $\Im$.

In this paper, we shall study results of Lau in [10] on the continuous representation of $S$ as weakly nonexpansive mappings of $C$ into itself. For more recent works on fixed point properties for left reversible semigroups of nonexpansive mapping see [11], [12], [17] and [18].

## 2. Preliminaries

Now, we introduce a new generalized mapping.

**Definition 2.1.** Let $C$ be a nonempty bounded closed convex subset of $H$ and let $\Im = \{T_s : s \in S\}$ be a continuous representation of $S$ on $C$.

(1) The mapping $T_s$ is said to be *nonexpansive* if

$$\|T_s x - T_s y\| \leq \|x - y\|$$

for $x, y \in C$ and $s \in S$.

(2) The mapping $T_s$ is said to be *weakly nonexpansive* if for each $x$ in $C$, there exist functions $\phi : S \times [0,\infty) \to [0,\infty)$ with $\limsup_{s\in S} \phi_s(k) \leq k$, $\gamma : S \times C \to [0,\infty)$ with $\lim_{s\in S} \gamma_s(x) = 0$ and $u : S \to [0,\infty)$ with $\lim_{s\in S} u(s) = 0$ such that

$$\|T_s x - T_s y\| \leq \phi_s(\|x - y\|) + \gamma_s(x) + u(s)$$

for all $y \in C$ and $s \in S$, where $\phi_s(k) = \phi(s,k)$ and $\gamma_s(x) = \gamma(s,x)$.

Note that weakly nonexpansive mapping is the same almost asymptotically nonexpansive type mapping in [7].

**Example 2.2.** *Let $S = \mathbb{N}$ (the set of all natural numbers with addition) and the closed convex subset $C = [0,1]$. Define $T_s : [0,1] \to [0,1]$ by*

$$T_s(x) = \begin{cases} x + \frac{1}{2s}\sin \pi x, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \\ 0, & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

*for all $x \in [0,1]$ and $s \in S$. Then $T_s$ is weakly nonexpansive in the sense of (2). Indeed, let $\|x - y\| = k \in [0,\infty)$, define $\phi : S \times [0,\infty) \to [0,\infty), \gamma_s(x) : S \to [0,\infty)$ and $u : S \to [0,\infty)$ by*

$$\phi_s(k) = \begin{cases} \dfrac{2^{\frac{s(s+1)}{2}-1}-1}{2^{\frac{s(s+1)}{2}-1}}\, k, & \text{if } k \in [0,\frac{1}{2}], \\ \\ k - \left(\frac{1}{2}\right)^{\frac{s(s+1)}{2}}, & \text{if } k \in [\frac{1}{2},\infty), \end{cases}$$

$$\gamma_s(x) = \frac{1}{s}\, x$$

*and*

$$u(s) = \frac{1}{s},$$

*for all $s \in S$ and $k \in [0,\infty)$.*

Throughout this paper, unless otherwise specified, $S$ denotes a semitopological semigroup and $\Im = \{T_s : s \in S\}$ a continuous representation of $S$ as weakly nonexpansive mappings from a nonempty closed convex subset $C$ of a Hilbert space $H$ into $C$.

## 3. Weak convergence of $\{T_s(x) : s \in S\}$

If $S$ is right reversible and $S$ is directed, then for each $x \in C$, let $\omega(x)$ denote the set of all weak limit points of subnets of the net $\{T_s(x) : s \in S\}$. We will need in our proofs the following modification of Opial's condition for bounded nets in a Hilbert space.

**Lemma 3.1** ([25]). *Let $H$ be a Hilbert space and let $\{x_\alpha\}$ be a bounded net in $H$ converging weakly to $x_0$. Then for any $x \in H, x \neq x_0$,*

$$\liminf_\alpha \|x_\alpha - x\| > \liminf_\alpha \|x_\alpha - x_0\|.$$

*Here $\liminf_\alpha\{s_\alpha\}$ of a bounded net of real number $\{s_\alpha\}$ is the limit of the increasing net $\{t_\beta\}$, where $t_\beta = \inf_{\alpha \succeq \beta} s_\alpha$.*

**Lemma 3.2.** *Assume that $S$ is right reversible and $x \in C$.*

(a) *If $F(\Im)$ is nonempty and $\omega(x) \subseteq F(\Im)$, then the net $\{T_t(x) : t \in S\}$ converges weakly to some $y \in F(\Im)$.*

(b) *Suppose the net $\{T_t(x) : t \in S\}$ converges weakly to some $y \in C$. Then $y \in F(\Im)$ if and only if $\{T_t(x) : t \in S\}$ is bounded.*

*Proof.* (a) If $y \in F(\Im)$, then the net $\{\|T_a x - y\| : a \in S\}$ is bounded. Also if $b \succeq a, b \in \overline{Sa}$, let $\{s_\alpha\}$ be a net in $S$ such that $s_\alpha a \to b$. Then, for each $\alpha$,

$$\|T_{s_\alpha a}(x) - y\| = \|T_{s_\alpha}(T_a x) - T_{s_\alpha} y\| \leq \phi_{s_\alpha}(\|T_a x - y\|) + \gamma_{s_\alpha}(T_a x) + u(s_\alpha).$$

Hence

$$\begin{aligned}\|T_b x - y\| &= \limsup_\alpha \|T_{s_\alpha a}(x) - y\| \\ &\leq \limsup_\alpha \{\phi_{s_\alpha}(\|T_a x - y\|) + \gamma_{s_\alpha}(T_a x) + u(s_\alpha)\} \\ &= \|T_a x - y\|.\end{aligned}$$

Consequently, the $\lim_a \|T_a x - y\|$ exists and is finite for each $y \in F(\Im)$. Since the net $\{T_a x : a \in S\}$ is bounded, it must contain a subnet $\{T_{a_\alpha}(x)\}$ which converges weakly to some $z \in C$. By assumption, $z \in F(\Im)$. Suppose $\{T_a x : a \in S\}$ does not converge weakly to $z$, then there exists another subnet $\{T_{a_\beta} x\}$ which converge weakly to some $u \in F(\Im)$, $z \neq u$. Now by Lemma 3.1,

$$\begin{aligned}\lim_\alpha \|T_{a_\alpha}(x) - z\| &< \lim_\alpha \|T_{a_\alpha}(x) - u\| \\ &= \lim_a \|T_a x - u\| = \lim_\beta \|T_{a_\beta}(x) - u\| \\ &< \lim_\beta \|T_{a_\beta}(x) - z\|,\end{aligned}$$

which is impossible since

$$\lim_\alpha \|T_{a_\alpha}(x) - z\| = \lim_\beta \|T_{a_\beta}(x) - z\|$$

by convergence of the net $\{\|T_a x - z\| : a \in S\}$.

(b) If $F(\Im)$ is nonempty and $z \in F(\Im)$, then

$$\|T_s x - z\| = \|T_s x - T_s z\| \le \phi_s(\|x - z\|) + \gamma_s(x) + u(s)$$

for each $s \in S, x \in C$. So $\limsup_s \|T_s x - z\| \le \|x - z\|$. Hence $\{T_s x : s \in S\}$ is bounded.

Conversely, if $\{T_s x : s \in S\}$ is bounded, let $a \in S$. If $T_a y \neq y$, then by Lemma 3.1,

$$\rho = \liminf_s \|T_s x - T_a y\| > \liminf_s \|T_s x - y\|.$$

Given $\varepsilon > 0$, choose $b \in S$ such that

$$\inf_{b \preceq s} \|T_s x - T_a y\| > \rho - \varepsilon.$$

In particular,

$$\|T_{as}(x) - T_a y\| > \rho - \varepsilon$$

for all $s \succeq b$. Since $\varepsilon > 0$ is arbitrary, we have

$$\liminf_s \|T_{as}(x) - T_a y\| \ge \rho.$$

On the other hand, for each $x \in C$,

$$\|T_{as}(x) - T_a y\| \le \phi_a(\|T_s x - y\|) + \gamma_a(T_s x) + u(a)$$

for all $a, s \in S$. Fix $s \in S$, we have

$$\begin{aligned} \|T_{as}(x) - T_a y\| &\le \limsup_a \|T_{as}(x) - T_a y\| \\ &\le \limsup_a \{\phi_a(\|T_s x - y\|) + \gamma_a(T_s x) + u(a)\} \\ &\le \|T_s x - y\|. \end{aligned}$$

Now, we conclude

$$\liminf_s \|T_{as}(x) - T_a y\| \le \liminf_s \|T_s x - y\|.$$

This is impossible. Hence $T_a y = y$. □

A subset $G$ of $S$ is called a *generating set* if elements of the form $g_1 g_2 g_3 \cdots g_n$, where $g_1, g_2, \cdots, g_n \in G$ is dense in $S$.

**Theorem 3.3.** *Assume that $S$ is right reversible and $x \in C$. If $F(\Im)$ is nonempty and $\|T_a x - T_{ga}(x)\| \to 0$ for all $g$ in a generating set $G$ of $S$, then the net $\{T_a x : a \in S\}$ converges weakly to an element of $F(\Im)$.*

*Proof.* By Lemma 3.2, it suffices to show that $\omega(x) \subseteq F(\Im)$. Let $\{T_{a_\alpha}(x)\}$ be a subnet of $\{T_a(x) : a \in S\}$ converging weakly to some $y \in C$. Let $g \in G$. If $T_g y \neq y$, then by Lemma 3.1

$$\liminf_\alpha \|T_{a_\alpha}(x) - y\| < \liminf_\alpha \|T_{a_\alpha}(x) - T_g y\|. \tag{3.1}$$

On the other hand,

$$\begin{aligned}\|T_{a_\alpha}(x) - T_g y\| &\leq \|T_{a_\alpha}(x) - T_{ga_\alpha}(x)\| + \|T_{ga_\alpha}(x) - T_g(y)\| \\ &\leq \|T_{a_\alpha}(x) - T_{ga_\alpha}(x)\| + \phi_g(\|T_{a_\alpha}(x) - y\|) \\ &\quad + \gamma_g(T_{a_\alpha}(x)) + u(g),\end{aligned}$$

for all $g \in G$. Fix $a_\alpha \in S$, we have

$$\begin{aligned}\|T_{a_\alpha}(x) - T_g(y)\| &\leq \limsup_g \|T_{a_\alpha}(x) - T_g(y)\| \\ &\leq \limsup_g \|T_{a_\alpha}(x) - T_{ga_\alpha}(x)\| + \|T_{a_\alpha}(x) - y\|.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\liminf_\alpha \|T_{a_\alpha}(x) - T_g y\| \\ &\leq \liminf_\alpha (\limsup_g \|T_{a_\alpha}(x) - T_{ga_\alpha}(x)\|) + \liminf_\alpha \|T_{a_\alpha}(x) - y\|.\end{aligned}$$

In particular, since $\lim_\alpha \|T_{a_\alpha}(x) - T_{ga_\alpha}(x)\| = 0$, we have

$$\liminf_\alpha \|T_{a_\alpha}(x) - T_g y\| \leq \liminf_\alpha \|T_{a_\alpha}(x) - y\|.$$

This contradicts to (3.1). Hence $T_g(y) = y$. Since $g \in G$ is arbitrary, it follows that $y \in F(\Im)$. □

**Theorem 3.4.** *Assume that $S$ is right reversible and $F(\Im)$ is nonempty. Let $P$ be the metric projection of $H$ onto $F(\Im)$. Then for each $x \in C$, the net $\{PT_a(x) : a \in S\}$ converges in norm to some $z \in F(\Im)$. Furthermore, if the net $\{PT_a(x) : a \in S\}$ converges weakly to some $y \in F(\Im)$, then $y = z$.*

*Proof.* Observe that

$$\|P(T_a x) - T_a x\| \leq \|P(T_b x) - T_a x\| \tag{3.2}$$

for any $a, b \in S$. If $a \succeq b$ and $a \neq b$, let $s_\alpha b$ be a net converging to $a$. Then for each $\alpha$,

$$\begin{aligned}\|P(T_b x) - T_{s_\alpha b}(x)\| &= \|T_{s_\alpha} P(T_b x) - T_{s_\alpha}(T_b x)\| \\ &\leq \phi_{s_\alpha}(\|P(T_b x) - T_b x\|) + \gamma_{s_\alpha}(PT_b x) + u(s_\alpha).\end{aligned}$$

It follows that

$$\begin{aligned}&\limsup_\alpha \|P(T_b x) - T_{s_\alpha b}(x)\| \\ &\leq \limsup_\alpha \{\phi_{s_\alpha}(\|P(T_b x) - T_b x\|) + \gamma_{s_\alpha}(PT_b x) + u(s_\alpha)\} \\ &\leq \|P(T_b x) - T_b x\|,\end{aligned}$$

i.e.,

$$\|P(T_b x) - T_a x\| \leq \|P(T_b x) - T_b x\| \tag{3.3}$$

for $a \succeq b$. Hence if $a \succeq b$, by (3.2) and (3.3), then

$$\|P(T_a x) - T_a x\| \leq \|P(T_b x) - T_b x\|. \tag{3.4}$$

Let $u \in F(\Im)$ and $v \in H$. By property of $P$, we have

$$\|Pv - u\|^2 \leq \|v - u\|^2 - \|Pv - v\|^2.$$

Put $v = T_a x$ and $u = PT_b x$. For $a \succeq b$, since (3.3), we obtain

$$\begin{aligned}\|P(T_a x) - P(T_b x)\|^2 &\leq \|T_a x - P(T_b x)\|^2 - \|P(T_a x) - T_a x\|^2 \\ &\leq \|P(T_b x) - T_b x\|^2 - \|P(T_a x) - T_a x\|^2.\end{aligned}$$

It follows from (3.4) that $\{P(T_a x) : x \in S\}$ is a norm Cauchy net in $H$. Hence it must converge to some $z \in F(\Im)$. If $\{T_a x : a \in S\}$ converges weakly to some $y \in F(\Im)$, then, by property of $P$, we have

$$Re\langle T_a x - P(T_a x), u - PT_a x\rangle \leq 0$$

for all $u \in F(\Im), a \in S$. Hence

$$\langle y - z, u - z\rangle \leq 0$$

for all $u \in F(\Im)$. In particular $y = z$. This completes the proof. □

*Remark* 3.5. Theorem 3.3 and Theorem 3.4 are generalization or improvement of Belluce and Kirk ([2]), Lau ([10]) and Pazy ([26]) for Hilbert spaces.

## 4. Semigroup of weakly nonexpansive mappings

Given a nonempty set $S$, we denoted by $l^\infty(S)$ the Banach space of all bounded real valued functions on $S$ with supremum norm. Let $S$ be a semigroup. Then a subspace $X$ of $l^\infty(S)$ is left (respectively right) translation invariant if $l_a(X) \subseteq X$ (respectively $r_a(X) \subseteq X$) for all $a \in S$, where $(l_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa), s \in S$.

We denote by $CB(S)$ the closed subalgebra of $l^\infty(S)$ consisting of continuous functions.

Let $AP(S)$ denote the space of almost periodic functions $f$ in $CB(S)$, i.e., all $f \in CB(S)$ such that $\{l_a f : a \in S\}$ is relatively compact in the norm topology of $CB(S)$, or equivalently $\{r_a f : a \in S\}$ is relatively compact in the norm topology of $CB(S)$.

Let $LUC(S)$ (respectively $RUC(S)$) be the space of left (respectively right) uniformly continuous functions on $S$, i.e., all $f \in CB(S)$ such that the mapping from $S$ into $CB(S)$ defined by $s \to l_s f$ (respectively $s \to r_s f$) is continuous when $CB(S)$ has the sup norm topology.

*Remark* 4.1 ([1], [23]). The followings are well-known.
(1) $AP(S) \subseteq LUC(S) \cap RUC(S)$.
(2) $LUC(S)$ and $RUC(S)$ are left and right translation invariant closed subalgebras of $CB(S)$, respectively, containing constants.

Let $X$ be a subspace of $l^\infty(S)$ containing constants. Then $\mu \in X^*$ is called a *mean* on $X$ if $\|\mu\| = \mu(1) = 1$. As is well known, $\mu$ is a mean on $X$ if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s) \quad \text{for all } f \in X.$$

Let $X$ be $l_s$-invariant then, a mean $\mu$ on $X$ is *left invariant* if $\mu(l_s f) = \mu(f)$ for all $s \in S$ and $f \in X$. Similarly we can define right invariant mean. $\mu$ is called an *invariant mean* if it is left and right invariant mean. The value of a mean $\mu$ at $f \in X$ will be denoted by $\mu(f), \langle\mu, f\rangle$ or $\mu_t f(t)$.

A semigroup $S$ which has a left (respectively right) invariant mean on $l^\infty(S)$ is called *left (respectively right) amenable.* A semigroup which has an invariant mean is called *amenable.*

**Lemma 4.2.** *If $x \in C$ with relatively compact orbit and $y \in H$, the the following functions are in $AP(S)$ :*

(a) $g_x(s) = \langle y, T_s x\rangle$,
(b) $f_x(s) = \|T_s x - y\|$.

*Proof.* (a) It follows from [9].
(b) It is clear that $f \in CB(S)$. To see that $f$ is almost periodic, for each $z \in C$, let

$$f_z(s) = \|y - T_s z\|.$$

Then

$$r_a f_x(s) = f_x(sa) = \|y - T_{sa}(x)\| = \|y - T_s T_a x\| = f_w(s),$$

where $w = T_a x$. Let $\tau : z \to f_z, z \in S$. If we can show that $\tau$ is continuous when $CB(S)$ has the sup norm topology, then $\overline{\tau(O(x))}$ is a compact subset of $CB(S)$ containing $\{r_a f : a \in S\}$. In particular, $f \in AP(S)$. To see that $\tau$ is continuous, let $\{z_n\}$ be a sequence in $C$, $z_n \to z$, then

$$\begin{aligned} |\tau(z_n)(s) - \tau(z)(s)| &= |f_{z_n}(s) - f_z(s)| \\ &= |\|y - T_s(z_n)\| - \|y - T_s z\|| \\ &\le |\|T_s z_n - T_s z\|| = \|T_s z_n - T_s z\| \\ &\le \phi_s(\|z_n - z\|) + \gamma_s(z_n) + u(s), \end{aligned}$$

by weakly nonexpansive of $T_s, s \in S$. Hence

$$\begin{aligned} \limsup_s |\tau(z_n)(s) - \tau(z)(s)| &\le \limsup_s \{\phi_s(\|z_n - z\|) + \gamma_s(z_n) + u(s)\} \\ &\le \|z_n - z\|. \end{aligned}$$

Since $z_n \to z$, we have $\|z_n - z\| \to 0$. Therefore $\|\tau(z_n) - \tau(z)\| \to 0$. □

Let $x \in C$ with relatively compact orbit, and $\mu \in AP(S)^*$ be a mean. Then $\psi(y) = \mu(g_x)$, where $g_x(s) = \langle y, T_s x\rangle$ defines a bounded linear functional on $H$. Hence, by the Riesz representation theorem, there exists $z \in H$ such that $\psi(y) = \langle y, z\rangle$ for all $y \in H$. Since $\mu$ is the weak* limit of finite means of the form $\sum_{i=1}^n \lambda_i \delta_{a_i}, \lambda_i \ge 0, \sum_{i=1}^n \lambda_i = 1, a_i \in S$, where $\delta_a(h) = h(a), h \in AP(S)$, and $C$ is weakly closed, it follows that $z \in C$. Write $T_\mu(x) = z$. Then $T_\mu$ is a nonexpansive mapping from $C$ into $C$ and $F(T_\mu) \supseteq F(\Im)$ as readily checked.

**Lemma 4.3.** *If $\mu$ is a left invariant mean on $AP(S)$, then $T_\mu(x)$ is a common fixed point for $\{T_t : t \in S\}$.*

*Proof.* Let $a \in S$ and $z = T_\mu(x)$. For each $s \in S$,

$$\begin{aligned} \|T_a z - T_{as}(x)\| &= \|T_a z - T_a T_s x\| \\ &\le \phi_a(\|z - T_s x\|) + \gamma_a(z) + u(a). \end{aligned}$$

Hence

$$\|T_a z - T_{as}(x)\|^2 \le \{\phi_a(\|z - T_s x\|)\}^2 + (\gamma_a(z) + u(a))^2 + 2\phi_a(\|z - T_s x\|)(\gamma_a(z) + u(a)), \tag{4.1}$$

for $a, s \in S$. Fix $s \in S$, we apply $\limsup_a$ to the inequality (4.1),

$$\begin{aligned} &\limsup_a \|T_a z - T_{as}(x)\|^2 \\ &\le \limsup_a \{\phi_a(\|z - T_s x\|)\}^2 + \limsup_a (\gamma_a(z) + u(a))^2 \\ &\quad + 2\limsup_a \phi_a(\|z - T_s x\|)(\gamma_a(z) + u(a)) \\ &\le \|z - T_s x\|^2 \\ &\le \|z - T_a z\|^2 + \|T_a z - T_s x\|^2 + 2Re\langle z - T_a z, T_a z - T_s x\rangle. \end{aligned} \tag{4.2}$$

Now for $w \in H$, the function $f_x(s) = \|w - T_s x\|^2$ is in $AP(S)$ by Lemma 4.2. Hence if $w = T_a z$, then $l_a f_x(s) = f_x(as) = \|T_a z - T_{as} x\|^2$ is in $AP(S)$. We apply the left invariant mean $\mu$ to the inequality (4.2),

$$\mu_s(\limsup_a \|T_a z - T_{as}(x)\|^2) \le \|z - T_a z\|^2 + \mu_s(\|T_a z - T_s x\|^2) + \mu_s(2Re\langle z - T_a z, T_a z - T_s x\rangle).$$

This implies

$$\begin{aligned} \limsup_a \|T_a z - z\|^2 &\le \|z - T_a z\|^2 + \|T_a z - z\|^2 + 2Re\langle z - T_a z, T_a z - z\rangle \\ &= 0. \end{aligned}$$

Hence $T_a z = z$. Since $a \in S$ is arbitrary, $z \in F(\Im)$. □

**Theorem 4.4.** *If $AP(S)$ has a left invariant mean, $x \in C$ such that $\{T_s x : s \in S\}$ is relatively compact, then $C$ contains a common fixed point for $\Im$. Furthermore, if $\mu$ is a left invariant mean on $AP(S)$, then $F(\Im) = F(T_\mu)$.*

*Proof.* This follows easily from Lemma 4.3. □

*Remark* 4.5. If $S$ is a left reversible semitopological semigroup, then $AP(S)$ has a left invariant mean. However there exists semitopological semigroup $S$ which is not left reversible but $CB(S)$ has a left invariant mean(see [4], [5], [6], [8], [9]).

*Remark* 4.6. Theorem 4.4 should be compared with Corollary 3.3 in [13].

Similarly, we can obtain the following results(see [13], p.1213).

**Lemma 4.7.** *If $x \in C$ with bounded orbit and $y \in H$, then the following functions are in $RUC(S)$ :*

(a) $g_x(s) = \langle y, T_s x\rangle$,
(b) $f_x(s) = \|T_s x - y\|$.

**Theorem 4.8.** *If $RUC(S)$ has a left invariant mean and there exist $x \in C$ such that $\{T_s x : s \in S\}$ is bounded, then $\Im$ has a common fixed point in $C$. Furthermore, if $\mu$ is a left invariant mean on $RUC(S)$, then $F(\Im) = F(T_\mu)$.*

*Remark* 4.9. Theorem 4.4 and Theorem 4.8 are generalization or improvement of Belluce and Kirk ([3]), Holmes and Lau ([4]) and Lim ([19], [20]) for Hilbert spaces.

## 5. Some open problems

**Definition 5.1.** (1) Let $WAP(S)$ denote the space of weakly almost periodic functions $f$ in $CB(S)$, i.e., all $f \in CB(S)$ such that $\{l_a f : a \in S\}$ is relatively compact in the weak topology of $CB(S)$, or equivalently $\{r_a f : a \in S\}$ is relatively compact in the weak topology of $CB(S)$.
(2) A function $f \in CB(S)$ is called *asymptotically left uniformly continuous* if for any $s \in S, \varepsilon > 0$, there exist a neighbourhood $U$ of $s$ and a right ideal $J$ of $S$ such that

$$\|l_u f - l_s f\|_J = \sup\{|f(ut) - f(st)| : t \in J\} < \varepsilon$$

for all $u \in U$. The closed linear span of the set of *asymptotically left uniformly continuous* functions on $S$ is denoted by $ALUC(S)$. Similarly we define the closed subspace $ARUC(S)$ of $CB(S)$ with left and right interchanged.

*Remark* 5.2. ([21], [22]) The following are well-known:
(1) $AP(S) \subseteq WAP(S)$.
(2) $WAP(S) \not\subset RUC(S)$ and $RUC(S) \not\subset WAP(S)$.
When $S$ is a group, then $WAP(S) \subseteq LUC(S) \cap RUC(S)$.

*Remark* 5.3. ([14]) $ALUC(S) \supseteq LUC(S)$ and $ARUC(S) \supseteq RUC(S)$.

**Definition 5.4** ([24])**.** A real valued function $\mu$ on $X$ is called a *submean* on $X$ if the following conditions are satisfies:
(i) $\mu(f+g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$.
(ii) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \geq 0$.
(iii) for $f, g \in X, f \leq g$ implies $\mu(f) \leq \mu(g)$.
(iv) $\mu(c) = c$ for every constant function $c$.

*Remark* 5.5. Clearly every mean is a submean(see also [15] and [16]).

**Open Problems:**
1. Are the functions $g$ and $f$ defined in Lemma 4.7 in $WAP(S)$?
2. Is Theorem 4.8 true with mean replaced by submean?
3. Is Theorem 4.8 true with $RUC(S)$ replaced by $WAP(S)$ or $ARUC(S)$?
4. Is Theorem 4.8 true for weakly compact convex subset of a Banach space? (see [18]).

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