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TOWARD SECOND-ORDER SENSITIVITY ANALYSIS IN SET-VALUED OPTIMIZATION

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Abstract. We study second-order contingent derivatives of perturbation mappings associated with families of set-valued optimization problems. After presenting some general inclusions, we focus on finite-dimensional problems whose data satisfy certain convexity assumptions. Our second-order results recover a number of known results from first order sensitivity analysis as special cases.

1. Introduction

Sensitivity analysis, the quantitative analysis of the perturbation map, is of paramount interest in optimization theory and has applications in several branches of pure and applied mathematics. For example, the stability of the output least-squares for inverse problems is achieved by the classical sensitivity analysis results for scalar optimization problems (see [10]). During the last five decades substantial progress has been made in sensitivity analysis of optimization problems with scalar objectives. On the other hand, the differentiability issues of the perturbation map for vector optimization problems are rather involved, and they require modern tools from variational analysis. The main difficulty here stems from the fact that the perturbation map for a vector optimization problem is, in general, multi-valued.

To fix the ideas, we consider the following parameter dependent vector optimization problem

$$\mathrm{Min}_C f(w,x) \quad \text{subject to } w \in H(x),$$

where X is the parameter space, W and Y are Banach spaces, $f : W \times X \to Y$ is a single-valued map, $H : X \rightrightarrows W$ is a set-valued map, $C \subset Y$ is the ordering cone, and Min_C indicates the minimum with respect to the ordering induced by C. Define a set-valued map $G : X \rightrightarrows Y$ by

$$G(x) := \{y \in Y \mid y = f(w,x) \text{ for some } w \in H(x)\}.$$

The set-valued perturbation map in this setting is

$$P(x) = \mathrm{Min}_C G(x).$$

The primary objective here is to investigate the relationships between the derivatives of P and G.

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Around two decades ago, Tanino [25] (see also [26]), in a seminal work, proved many interesting relationships among the first-order derivatives of the maps P and G. Since then several authors have investigated the differentiability of set-valued perturbation maps (see [7], [18],[19],[27],[28],[29] and cited references therein). Interesting results on sensitivity analysis for non-smooth optimization problems are available in [22],[31],[32],[33], among others. Although there are important developments in sensitivity analysis for set-valued optimization problems, the studies previously have been limited to the use of first-order derivatives.

The primary objective of the present work is to bring forth, in a unified framework, second-order sensitivity analysis for set-valued perturbation maps. In recent years, the second-order contingent derivatives have been used to give second-order optimality conditions in set-valued optimization (see [14]). To obtain second-order analogues of the known first-order sensitivity analysis results, we introduce several new concepts. This includes a second-order analogue of S-derivatives of Shi [27] and second-order directional compactness, among others.

Here is an outline of this paper. Section 2 collects some background material, including various notion of minimality, tangent cones and tangent sets, and derivatives and epiderivatves of set-valued maps. In Section 3, we prove various connections among derivatives of a set-valued map, the derivatives of its profile map, and their various minimal points. Section 4 presents second-order sensitivity analysis for the set-valued perturbation maps. We employ second-order contingent derivatives and epiderivatives for the differentiability of the set-valued perturbation maps.

2. Preliminaries

Let Z be a normed space being partially ordered by a proper, pointed, closed, and convex cone $C \subset Z$. Let $\mathcal{K}$ be the set of all proper, pointed, closed, and convex cones $K \subset Z$ such that $C\backslash\{0\} \subset \operatorname{int}(K)$ ("int" stands for interior). Let $D \subset Z$ and let $y \in D$ be arbitrary.

- ◦ An element y is said to be a minimal point of D if $D \cap (\{y\} - C) = \{y\}$.
- ◦ Assume that the ordering cone C is solid, that is, it has a nonempty interior $\operatorname{int}(C)$. An element y is said to be a weakly minimal point of D if $D \cap (\{y\} - \operatorname{int}(C)) = \emptyset$.
- ◦ An element y is said to be a properly minimal point of D if, for some $K \in \mathcal{K}$, we have $D \cap (\{y\} - K) = \{y\}$, that is, the element y is a minimal point of D with respect to K.

The set of all minimal points, weakly minimal points and properly minimal points of D with respect to C will be denoted by $\mathrm{Min}_C D$, $\mathrm{WMin}_C D$ and $\mathrm{PMin}_C D$, respectively. The following chain of inclusions is then known to hold: $\mathrm{PMin}(D, C) \subset \mathrm{Min}(D, C) \subset \mathrm{WMin}(D, C)$.

We now recall the concepts of tangent cones and tangent sets (see [2],[22] for details).

Definition 2.1. Let Z be a normed space, let $S \subset Z$ be nonempty, and let $w \in Z$.

1. The second-order contingent set $T^2(S, \bar{z}, w)$ of S at $\bar{z} \in S$ in a direction $w \in Z$ is the set of all $z \in Z$ such that there are a sequence $(z_n) \subset Z$ with

$z_n \to z$ and a sequence $(\lambda_n) \subset P := \{t \in R \mid t > 0\}$ with $\lambda_n \downarrow 0$ so that $\bar{z} + \lambda_n w + \lambda_n^2 z_n \in S$.

2. The contingent cone $T(S, \bar{z})$ of S at $\bar{z} \in S$ is the set of all $z \in Z$ such that there are a sequence $(z_n) \subset Z$ with $z_n \to z$ and a sequence $(\lambda_n) \subset P$ with $\lambda_n \downarrow 0$ so that $\bar{z} + \lambda_n z_n \in S$.

Remark 2.2. Equivalently (see Cambini et al. [6]), the second-order contingent set $T^2(S, \bar{z}, w)$ is the set of all $z \in Z$ such that there are sequences $(\alpha_n) \subset \mathbb{P}$, $(\beta_n) \subset \mathbb{P}$ and $(z_n) \subset S$ with $\alpha_n \to \infty$, $\beta_n \to \infty$, $(\beta_n/\alpha_n) \to 1$, $z_n \to \bar{z}$, $\alpha_n(z_n - \bar{z}) \to w$ and $\beta_n(\alpha_n(z_n - \bar{z}) - w) \to z$.

Remark 2.3. It is known that the contingent cone $T(S, \bar{z})$ is a nonempty closed cone (cf. [2]). On the other hand, $T^2(S, \bar{z}, w)$ is only a closed set (possibly empty), non-connected in general, and nonempty only if $w \in T(S, \bar{z})$. For other details and examples of these cones and sets, the reader is referred to [2],[6],[11],[22],[23] and the references therein.

Let X and Y be normed spaces, and let $F : X \rightrightarrows Y$ be a set-valued map. The effective domain of F is given by $\mathrm{dom}(F) := \{x \in X \mid F(x) \neq \emptyset\}$, and the graph of F is defined by $\mathrm{gph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$. Given a proper, pointed, and convex cone $C \subset Y$, we define the profile map $F_+(x) = (F + C)(x) = F(x) + C$, for every $x \in \mathrm{dom}(F)$. The epigraph of F is then defined as the graph of F_+. In other words, $\mathrm{epi}(F) = \mathrm{graph}(F_+)$.

We conclude this section by recalling two notions of second-order derivatives.

Definition 2.4. Let $F : X \rightrightarrows Y$ be set-valued, let $(\bar{x}, \bar{y}) \in gph(F)$, and let $(\bar{u}, \bar{v}) \in X \times Y$.

(i) A set-valued map $D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \rightrightarrows Y$ defined by
$$D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \left\{y \in Y \mid \quad (x, y) \in T^2(gph(F), (\bar{x}, \bar{y}), (\bar{u}, \bar{v}))\right\}$$
is called the second-order contingent derivative of F at $(\bar{x}, \bar{y})$ in direction $(\bar{u}, \bar{v})$.

(ii) A set-valued map $D_g^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \rightrightarrows Y$ defined by
$$D_g^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = Min_C D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x), \quad x \in dom(D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$$
is called the second-order generalized contingent epiderivative of F at $(\bar{x}, \bar{y})$ in direction $(\bar{u}, \bar{v})$.

It is clear that if $(\bar{u}, \bar{v}) = (0, 0)$ in the above definition, we recover the contingent derivative $DF(\bar{x}, \bar{y})$, and the generalized contingent epiderivative of F at $(\bar{x}, \bar{y})$, respectively (cf. [2]). Moreover, if $F : X \to Y$ is a single valued map which is twice continuously Fréchet differentiable around $\bar{x} \in \Omega \subset X$, then the second-order contingent derivative of the restriction F_Ω of F to Ω at $\bar{x}$ in a direction $\bar{u}$ is given by the formula (see [2, p. 215]):

$$D^2F_\Omega(\bar{x}, F(\bar{x}), \bar{u}, F'(\bar{x})(\bar{u}))(x) = F'(\bar{x})(x) + F''(\bar{x})(\bar{u}, \bar{u})/2 \text{ for } x \in T^2(\Omega, \bar{x}, \bar{u}).$$

It is empty when $x \notin T^2(\Omega, \bar{x}, \bar{u})$.

3. Auxiliary results

Our primary objective is to study relationships among derivatives of various set-valued perturbation maps. It turns out that some results that hold for general set-valued maps render useful insight for the quantitative analysis of perturbation maps. In the following, we collect a few auxiliary results that will play an instrumental role for the rest of this paper.

Let X and Y be real normed spaces, let $C \subset Y$ be a proper, pointed, closed, and convex cone, and let $F : X \rightrightarrows Y$ be a set-valued map. Our immediate objective is to give connections among second-order derivatives of F, its profile map $F + C$, and their various minimal points. In the following, we assume that $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$, and $(\bar{u}, \bar{v}) \in X \times Y$ is arbitrary.

We begin with the following lemma.

Lemma 3.1. *For every* $x \in \mathrm{dom}(D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$, *the following inclusion holds:*

$$D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) + C \subset D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x). \tag{3.1}$$

Proof. Let $y \in D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$, and let $c \in C$ be chosen arbitrarily. Then there exist sequences $(t_n) \subset \mathbb{P}$ and $(x_n, y_n) \subset X \times Y$ such that $t_n \downarrow 0$, $(x_n, y_n) \to (x, y)$ and $\bar{y} + t_n\bar{v} + t_n^2 y_n \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n)$. For $\bar{y}_n = y_n + c$, we notice that $\bar{y} + t_n\bar{v} + t_n^2\bar{y}_n \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n) + C$. Since $\bar{y}_n$ converges to $y + c$, we conclude that $y + c \in D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. This establishes (3.1). □

The following corollary is useful when working with the second-order epiderivatives (cf. [14],[15]).

Corollary 3.2. *For every* $x \in \mathrm{dom}(D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$, *the following identity holds:*

$$D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) + C = D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x). \tag{3.2}$$

Proof. The inclusion $\supset$ holds trivially and the inclusion $\subset$ follows from Lemma 3.1 and the identity $(F + C)(\cdot) + C = (F + C)(\cdot)$. □

The converse inclusion of (3.1) does not hold in general. Moreover, the effective domains of the two derivatives can be very different. We illustrate this in the context of the first-order derivative (the case where $(\bar{u}, \bar{v}) = (0, 0)$).

Example 3.3. Let $X = Y = \mathbb{R}$ and let $C = \mathbb{R}_+$. Define a map $F : \mathbb{R} \rightrightarrows \mathbb{R}$ as follows:

$$F(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ [1, 2] & \text{if } x = 0. \end{cases}$$

Then $\mathrm{dom}(DF(0, 1)) = \{0\}$, and

$$\begin{aligned} DF(0, 1)(0) &= \mathbb{R}_+, \\ D(F + \mathbb{R}_+)(0, 1)(x) &= \mathbb{R}, \quad \text{for every } x \in \mathbb{R}. \end{aligned}$$

Hence

$$D(F + \mathbb{R}_+)(0, 1)(x) \not\subset DF(0, 1)(x) + C.$$

It is worth noticing that for every $x \in \mathbb{R}$, we have $\text{Min}_{\mathbb{R}_+} D(F + \mathbb{R}_+)(0, 1)(x) = \emptyset$. Also notice that $\text{Min}_{\mathbb{R}_+} DF(0, 1)(0) = \{0\}$. Therefore,

$$\text{Min}_{\mathbb{R}_+} D(F + \mathbb{R}_+)(0, 1)(0) \neq \text{Min}_{\mathbb{R}_+} DF(0, 1)(0),$$

even though F is convex-valued. This reveals an error in Theorem 3.6.6 of [9], which claims that these two sets will be equal whenever $F(x)$ is convex for each x near 0.

To prove the converse inclusion of (3.1), we need to impose certain restrictions on the map F and/or on the ordering cone C. It turns out that a second-order extension of the S-derivative of Shi [27] has an important role to play. We recall that the S-derivative of F at $(\bar{x}, \bar{y}) \in \text{gph}(F)$ is a set-valued map $D_S F(\bar{x}, \bar{y}) : X \rightrightarrows Y$ such that $y \in D_S F(\bar{x}, \bar{y})(x)$ if and only if there are sequences $(t_n) \subset \mathbb{P}$ and $(x_n, y_n) \in \text{gph}(F)$ such that $x_n \to \bar{x}$ and $t_n[(x_n, y_n) - (\bar{x}, \bar{y})] \to (x, y)$.

The following definition proposes an extension of the above notion of S-derivative.

Definition 3.4. The second-order S-derivative $D_S^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ of F at $(\bar{x}, \bar{y}) \in$ *gph*(F) in direction $(\bar{u}, \bar{v}) \in X \times Y$ is a set-valued map $D_S^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \rightrightarrows Y$ such that $y \in D_S^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$ if and only if there are sequences $(\alpha_n), (\beta_n) \subset \mathbb{P}$ and $(x_n, y_n) \in$ *gph*(F) such that

$$\begin{array}{rcl} x_n & \to & \bar{x}, \\ \alpha_n[(x_n, y_n) - (\bar{x}, \bar{y})] & \to & (\bar{u}, \bar{v}) \\ \beta_n[\alpha_n[(x_n, y_n) - (\bar{x}, \bar{y})] - (\bar{u}, \bar{v})] & \to & (x, y). \end{array}$$

We are now ready to prove the converse inclusion of (3.1).

Proposition 3.5. *Assume that the set $B = \{z \in C | \ \|z\| = 1\}$ is compact. Assume that*

$$D_S^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(0) \cap (-C/\{0\}) = \emptyset. \tag{3.3}$$

Then for every $x \in \text{dom}(D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$, the following identity holds

$$D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = D^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) + C. \tag{3.4}$$

Proof. In view of (3.1), it suffices to show that $D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset D^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) + C$. Assume that $y \in D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Therefore, there are sequences $(t_n) \subset \mathbb{P}$, $(c_n) \subset C$, and $(x_n, y_n) \in X \times Y$, with $t_n \downarrow 0$, $(x_n, y_n + c_n) \to (x, y)$ so that $\bar{y} + t_n \bar{v} + t_n^2 y_n \in F(\bar{x} + t_n \bar{u} + t_n^2 x_n)$. Since the set B is compact, and $C = \text{cone}(B)$, we have $c_n = r_n b_n$, for some $r_n > 0$ and $b_n \in B$. Without any loss of generality, we assume that $r_n \to r \in [0, \infty]$ and $b_n \to b$ for some $b \in B$. We claim that $r < \infty$. For the sake of argument, we assume that $r_n \to \infty$. This, in view of the convergence $(y_n + r_n b_n) \to y$, implies that $r_n^{-1} y_n \to -b$, and hence

$$\frac{1}{r_n t_n} \left[\frac{1}{t_n} \left[(\bar{x} + t_n \bar{u} + t_n^2 x_n, \bar{y} + t_n \bar{v} + t_n^2 y_n) - (\bar{x}, \bar{y}) \right] - (\bar{u}, \bar{v}) \right] = \left(\frac{x_n}{r_n}, \frac{y_n}{r_n} \right) \to (0, -b).$$

Therefore $-b \in D_S^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(0) \cap (-C/\{0\})$ which is a contradiction to (3.3). Hence $r < \infty$, implying that $y \in D^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) + C$. The proof is complete. $\square$

The following corollary is immediate.

Corollary 3.6. *If Y is finite dimensional and (3.3) remains valid, then (3.4) holds.*

Proof. For finite dimensional Y, $B = \{z \in C |\ \|z\| = 1\}$ is a compact base for C. $\square$

By setting $(\bar{u}, \bar{v}) = (0, 0)$ in the above result, we deduce the following corollary.

Corollary 3.7. (Shi [27]) *Assume that the set $B = \{z \in C|\ \|z\| = 1\}$ is compact. Assume that*

$$D_S F(\bar{x}, \bar{y})(0) \cap (-C/\{0\}) = \emptyset.$$

Then for every $x \in \mathrm{dom}(D(F + C)(\bar{x}, \bar{y}))$, the following identity holds:

$$D(F + C)(\bar{x}, \bar{y})(x) = DF(\bar{x}, \bar{y})(x) + C.$$

We note that in Example 3.3, $-y \in D_S F(0, 1)(0)$ for all $y > 0$, consistent with Corollary 3.7.

To derive several useful conclusions from (3.4), we need the following result.

Lemma 3.8. *Let A and B be nonempty subsets of Y. If $A + C = B$, then*

(a) $\mathrm{PMin}_C A = \mathrm{PMin}_C B$.
(b) $\mathrm{Min}_C A = \mathrm{Min}_C B$.
(c) $\mathrm{WMin}_C A \subset \mathrm{WMin}_C B$.

Moreover, if $\tilde{C}$ is a proper, closed, and convex cone with $\tilde{C} \subset \mathrm{int}(C) \cup \{0\}$ and $A + \tilde{C} = B$, then

(d) $\mathrm{WMin}_C A = \mathrm{WMin}_C B$,

Proof. (a) We will first prove that $\mathrm{PMin}_C A \subset \mathrm{PMin}_C B$. Let $u \in \mathrm{PMin}_C A$. Then there exists a proper, pointed, closed, and convex cone K (that is, $K \in \mathcal{K}$) such that $C \backslash \{0\} \subset \mathrm{int}(K)$ and $u \in \mathrm{Min}_K A$. Clearly, $u \in A$ and since C is a cone, we deduce that $u \in A + C$ as well. Assume now that $u \notin \mathrm{PMin}_C B$. Then $u \notin \mathrm{Min}_K B$, for every $K \in \mathcal{K}$. Therefore, there exists $v \in B$ such that $u - v \in K \backslash \{0\}$. Since $A + C = B$, there exists $w \in A$ so that $v - w \in C$. Hence, $u - w = (u - v) + (v - w) \in K \backslash \{0\} + C \subset K \backslash \{0\} + K = K \backslash \{0\}$, which contradicts the fact that $u \in \mathrm{Min}_K A$. Therefore $u \in \mathrm{PMin}_C B$.

To prove the opposite inclusion $\mathrm{PMin}_C B \subset \mathrm{PMin}_C A$, assume that $u \in \mathrm{PMin}_C B \subset B$. Then there exists $v \in A$ such that $u - v = c \in C$. We claim that $c = 0$. To see this, assume that $c \neq 0$. Then $c \in C \backslash \{0\} \subset K \backslash \{0\}$, contradicting that $u \in \mathrm{Min}_K B$. Hence $u = v \in A$, and since $A \subset B$ and $B \cap (\{u\} - K) = \{u\}$, we have $A \cap (\{u\} - K) = \{u\}$ as well. Therefore $u \in \mathrm{PMin}_C A$.

The proofs of (b), (c) and (d) are based on similar arguments and hence are omitted (see [11, Lemmas 4.7 and 4.13]). $\square$

In view of Lemma 3.8, we obtain the following result.

Theorem 3.9. *If (3.4) holds, then for every $x \in \mathrm{dom}(D^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$, we have*

(a) $\mathrm{PMin}_C D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \mathrm{PMin}_C D^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$.
(b) $\mathrm{Min}_C D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \mathrm{Min}_C D^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$.
(c) $\mathrm{WMin}_C D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset \mathrm{WMin}_C D^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$.

Moreover, if (3.4) holds with C replaced by $\tilde{C}$, a closed, and convex cone with $\tilde{C} \subset \mathrm{int}(C) \cup \{0\}$, then

(d) $\mathrm{WMin}_C D^2(F+\tilde{C})(\bar{x},\bar{y},\bar{u},\bar{v})(x) = \mathrm{WMin}_C D^2F(\bar{x},\bar{y},\bar{u},\bar{v})(x)$.

Proof. The proof follows directly from (3.4) and Lemma 3.8 by setting $A = D^2F(\bar{x},\bar{y},\bar{u},\bar{v})(x)$, and $B = D^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. ☐

Noticing the importance of (3.4), we give two more conditions that would validate it.

For various purposes a number of compactness assumptions have been introduced in optimization. In the following we extend two notions of compactness and employ them to prove (3.4).

Definition 3.10. A map $F : X \rightrightarrows Y$ is called second-order directionally compact at $(\bar{x},\bar{y}) \in gph(F)$ with respect to $(\bar{u},\bar{v}) \in X \times Y$ in a direction $x \in X$, if for every sequence $t_n \downarrow 0$ and every sequence $x_n \to x$, any sequence y_n with $\bar{y} + t_n\bar{v} + t_n^2 y_n \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n)$ contains a convergent subsequence.

Remark 3.11. If $(\bar{u},\bar{v}) = (0,0)$, then the above concept reduces to the known concept of directional compactness at $(\bar{x},\bar{y})$ in the direction x (see [3]). Second-order directional compactness holds, in particular, in the case where F is single-valued, the Hadamard directional derivative

$$\bar{v} := F'(\bar{x};\bar{u}) := \lim_{t\downarrow 0, u\to\bar{u}} (F(\bar{x}+tu) - F(\bar{x}))/t$$

exists, and the parabolic second-order directional derivative

$$F''(\bar{x},\bar{u};x) := \lim_{t\downarrow 0, w\to x} (F(\bar{x}+t\bar{u}+t^2w) - F(\bar{x}) - tF'(\bar{x};\bar{u}))/t^2$$

exists. To see this, suppose that $t_n \downarrow 0$, $x_n \to x$, and $\bar{y} + t_n\bar{v} + t_n^2 y_n \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n)$. Then

$$y_n = (F(\bar{x}+t_n\bar{u}+t_n^2x_n) - F(\bar{x}) - t_nF'(\bar{x};\bar{u}))/{t_n}^2,$$

so that $y_n \to F''(\bar{x},\bar{u};x)$.

With the concept of second-order directional compactness, we derive a generalization of Proposition 5 of Bednarczuk and Song [3].

Proposition 3.12. *Assume that F is second-order directionally compact at $(\bar{x},\bar{y})$ with respect to $(\bar{u},\bar{v})$ in any direction $x \in X$. Then (3.4) holds.*

Proof. In view of (3.1), it suffices to show that

$$D^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v})(x) \subset D^2F(\bar{x},\bar{y},\bar{u},\bar{v})(x) + C.$$

Let $y \in D^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. Then there exist sequences $(t_n) \subset \mathbb{P}$, $(x_n,y_n) \subset X \times Y$ and $c_n \in C$ such that $t_n \downarrow 0$, $(x_n,y_n) \to (x,y)$ and

$$\bar{y} + t_n\bar{v} + t_n^2(y_n - c_n/t_n^2) = \bar{y} + t_n\bar{v} + t_n^2 y_n - c_n \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n).$$

Since F is second-order directionally compact, we may assume that $y_n - c_n/t_n^2$ converges to some $\tilde{y} \in Y$. Hence $\tilde{y} \in D^2F(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ and $c_n/t_n^2 \to (y - \tilde{y}) \in C$, confirming that $y \in D^2F(\bar{x},\bar{y},\bar{u},\bar{v})(x) + C$. The proof is complete. ☐

In the following definition we propose an extension of the notion of compactly approximable multi-functions introduced by Amahroq and Thibault [1].

Definition 3.13. A map $F : X \rightrightarrows Y$ is called second-order compactly approximable at $(\bar{x}, \bar{y}) \in gph(F)$ with respect to $(\bar{u}, \bar{v}) \in X \times Y$, if for each $x' \in X$, there exists a set-valued map R from X into the set of all nonempty compact subsets of Y, a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ in X, and a function $r :]0,1] \times X \to]0, \infty]$ such that

1. $\lim_{(t,x)\to(0,x')} r(t,x) = 0$.
2. For each $x \in X$ and $t \in]0,1]$, we have $F(\bar{x}+t\bar{u}+t^2x) \subset \bar{y}+t\bar{v}+t^2(R(x')+r(t,x)B_Y)$.

The following generalization of Proposition 2.2 of Taa [30] is obtained by the concept of second-order compactly approximable maps.

Proposition 3.14. *If F is second-order compactly approximable at $(\bar{x}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$, then (3.4) holds.*

Proof. Assume $y \in D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Hence there exist sequences $(t_n) \subset \mathbb{P}$ and $(x_n, y_n) \subset X \times Y$ such that $t_n \downarrow 0$, $(x_n, y_n) \to (x, y)$ and

$$\bar{y} + t_n\bar{v} + t_n^2 y_n \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n) + C.$$

Then

$$\bar{y} + t_n\bar{v} + t_n^2 y_n = z_n + c_n \quad \text{with } z_n \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n) \text{ and } c_n \in C.$$

Since F is second-order compactly approximable, there exist $\{k_n\} \subset R(x)$, $r_n := r(t_n, x_n)$, $b_n \in B_Y$ and $n_0 \in \mathbb{N}$ such that $r_n \to 0$, and

$$z_n = \bar{y} + t_n\bar{v} + t_n^2(k_n + r_n b_n), \quad \text{for all } n > n_0.$$

This implies

$$\bar{y} + t_n\bar{v} + t_n^2(k_n + r_n b_n) \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n)$$

for all $n > n_0$. In view of the compactness of $R(x)$, we may assume that $k_n \to k \in R(x)$. Because of the fact that $(k_n + r_n b_n) \to k$, we deduce that $k \in D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Since

$$\bar{y} + t_n\bar{v} + t_n^2 y_n = \bar{y} + t_n\bar{v} + t_n^2(k_n + r_n b_n) + c_n$$

for sufficiently large n, we have $y_n - k_n - r_n b_n = t_n^{-2} c_n \in C$. By passing to the limit, we have $y - k \in C$ and hence $y \in k + C \subset D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) + C$. The proof is complete. $\square$

A variant of Theorem 3.9 can be proved without (3.4), as shown in the following result.

Theorem 3.15. *Assume that the set $B = \{z \in C | \, \|z\| = 1\}$ is compact. Then for every $x \in \mathrm{dom}(D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$, the following inclusion holds*

$$\mathrm{Min}_C D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x). \tag{3.5}$$

Proof. Let $y \in \mathrm{Min}_C D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Then $y \in D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$, and hence there exist sequences $(t_n) \subset \mathbb{P}$, $(x_n, y_n) \subset X \times Y$, and $c_n \in C$ such that $t_n \downarrow 0$, $(x_n, y_n) \to (x, y)$ and

$$\bar{y} + t_n\bar{v} + t_n^2(y_n - c_n) \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n).$$

Since $C = \mathrm{cone}(B)$, we have $c_n = r_n b_n$ for some $r_n > 0$ and $b_n \in B$. Moreover, because B is compact, we may assume that $b_n \to b \in B$. We will show that $r_n \to 0$. If not we may assume, taking a subsequence if necessary, that there exists $\epsilon > 0$

such that $r_n > \epsilon$. Set $\bar{c}_n = \epsilon c_n / r_n$. Notice that $c_n - \bar{c}_n \in C$, from which it follows that

$$\bar{y} + t_n \bar{v} + t_n^2 (y_n - \bar{c}_n) \in F(\bar{x} + t_n \bar{u} + t_n^2 x_n) + C.$$

Since $\bar{c}_n \to \epsilon b$, we have that

$$y - \epsilon b \in D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$$

contradicting the C-minimality of y. Therefore $r_n \to 0$, and this implies that $(y_n - c_n) \to y$. Consequently, $y \in D^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. The proof is complete. □

By setting $(\bar{u}, \bar{v}) = (0, 0)$ in the above result, we recover the following result.

Corollary 3.16 ([25, Theorem 2.1])**.** *Assume that the set $B = \{z \in C \mid \|z\| = 1\}$ is compact. Then for every $x \in \operatorname{dom}(D(F + C)(\bar{x}, \bar{y}))$, the following inclusion holds*

$$\mathrm{Min}_C D(F + C)(\bar{x}, \bar{y})(x) \subset DF(\bar{x}, \bar{y})(x).$$

Example 3.17. As an illustration of Theorem 3.15 and Corollary 3.16, let $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined by

$$F(x) := \{(y_1, y_2) \in \mathbb{R}^2_+ \mid y_1 y_2 = x\},$$

and let $C = \mathbb{R}^2_+$, $\bar{x} = 1$, and $\bar{y} = (1, 1)$. Then for all $x \in \mathbb{R}$,

$$DF(\bar{x}, \bar{y})(x) = \{(y_1, y_2) \mid x = y_1 + y_2\}$$

and

$$D(F + C)(\bar{x}, \bar{y})(x) = \{(y_1, y_2) \mid x \le y_1 + y_2\},$$

so that

$$\mathrm{Min}_C D(F + C)(\bar{x}, \bar{y})(x) = DF(\bar{x}, \bar{y})(x),$$

consistent with Corollary 3.16.

For $\bar{v} = (v_1, v_2)$ and $\bar{u} = v_1 + v_2$, one obtains (e.g., by Proposition 2.3 of [31])

$$D^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \{(y_1, y_2) \mid x = y_1 + y_2 + v_1 v_2\}$$

and

$$D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \{(y_1, y_2) \mid x \le y_1 + y_2 + v_1 v_2\},$$

so that equality holds in (3.5).

4. Sensitivity analysis

Throughout this section, unless stated otherwise, we assume that X and Y are normed spaces and $C \subset Y$ is a proper, pointed, closed, and convex cone. When dealing with weak minimality, we will assume that $\operatorname{int}(C)$ is non-empty. Let $G : X \rightrightarrows Y$ be a given set-valued map. We are interested in the assessment of differentiability properties of the set-valued perturbation maps P, Q, and R, defined for $x \in X$, by

$$P(x) = \mathrm{PMin}_C G(x) \tag{4.1a}$$

$$Q(x) = \mathrm{Min}_C G(x) \tag{4.1b}$$

$$R(x) = \mathrm{WMin}_C G(x). \tag{4.1c}$$

The perturbation maps P, Q, and R will be termed the proper perturbation map, the perturbation map, and the weak perturbation map, respectively. Our overall goal here is to investigate the relationships among the second-order derivatives of the perturbation maps P, Q, and R, the second-order derivative of the map G, and various minimal points of these derivatives. A detailed study of such relationships, involving only the first derivatives of the set-valued perturbation maps, was initiated by Tanino [25] and later developed in [18],[19], among others.

We will need the following concept of function dominance.

Definition 4.1. Let $K, L : X \rightrightarrows Y$ be two set-valued maps. The map L is said to be C-dominated by K near $\bar{u} \in X$, if there exists a neighborhood $\mathcal{N}(\bar{u})$ of $\bar{u}$ such that

$$L(u) \subset K(u) + C, \quad \text{for every } u \in \mathcal{N}(\bar{u}).$$

For the set-valued map $G : X \rightrightarrows Y$, assume that $(\bar{x}, \bar{y}) \in \mathrm{gph}(G)$, and $(\bar{u}, \bar{v}) \in X \times Y$ is arbitrary. Moreover, for $x \in X$, $\mathcal{N}(x)$ will stand for a neighborhood of x.

We have the following implication of (3.5).

Theorem 4.2. *Assume that the set $B = \{z \in C | \, \|z\| = 1\}$ is compact. Assume that G is C-dominated by Q near $\bar{x}$. Then*

$$\mathrm{Min}_C D^2(G + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset D^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x), \quad \text{for every } x \in \mathcal{N}(\bar{x}) \tag{4.2}$$

Proof. Since $Q(x) \subset G(x)$ and G is C-dominated by Q near $\bar{x}$, we have $G(x) + C = Q(x) + C$ for every $x \in \mathcal{N}(\bar{x})$. Consequently

$$D^2(G + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = D^2(Q + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x), \quad \text{for every } x \in \mathcal{N}(\bar{x}),$$

which implies that

$$\mathrm{Min}_C D^2(G{+}C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \mathrm{Min}_C D^2(Q{+}C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x), \quad \text{for every } x \in \mathcal{N}(\bar{x}). \tag{4.3}$$

Because B is compact, Theorem 3.15 ensures that

$$\mathrm{Min}_C D^2(Q + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset D^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x), \quad \text{for every } x \in \mathcal{N}(\bar{x}).$$

Therefore, (4.2) follows from the above inclusion and (4.3). The proof is complete. □

Remark 4.3. Notice that (4.3) implies that $D_g^2 G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = D_g^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$.

Example 4.4. For the mapping F defined in Example 3.17, let $G = F$, and let $C = \mathbb{R}^2_+$, $\bar{x} = 1$, and $\bar{y} = (1, 1)$. Since $Q(x) = G(x)$ for $x > 0$, G is C-dominated by Q near $\bar{x}$. As in Example 3.17, for $\bar{v} = (v_1, v_2)$ and $\bar{u} = v_1 + v_2$, we have

$$D^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \{(y_1, y_2) \mid x = y_1 + y_2 + v_1 v_2\}$$

and

$$D^2(G + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = \{(y_1, y_2) \mid x \leq y_1 + y_2 + v_1 v_2\},$$

so that both sides of (4.2) are equal.

In our next result we exploit analogues of (3.4).

Theorem 4.5. *Assume that one of the following conditions holds.*

(a) $D_S^2 G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(0) \cap (-C/\{0\}) = \emptyset$.

(b) *Q is second-order directionally compact at $(\bar{x}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$ in any direction $x \in X$.*

(c) *Q is second-order compactly approximable at $(\bar{x}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$.*

Then the following are valid:

(1) *If G is dominated by Q near $\bar{x}$, then for every $x \in \mathcal{N}(\bar{x})$, we have*

$$\mathrm{Min}_C D^2 G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset D^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x). \tag{4.4}$$

(2) *If G is $\tilde{C}$-dominated by R near $\bar{x}$, then for every $x \in \mathcal{N}(\bar{x})$, we have*

$$\mathrm{WMin}_C D^2 G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset D^2 R(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x). \tag{4.5}$$

(3) *If G is C-bounded and C-closed, Y is finite dimensional (see* [26]*), and G is dominated by Q near $\bar{x}$, then for every $x \in \mathcal{N}(\bar{x})$, we have*

$$\mathrm{PMin}_C D^2 G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset D^2 P(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x). \tag{4.6}$$

Proof. We begin by deducing some simple implications of (a)-(c). Due to the chain of inclusions

$$D_S^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(0) \subset D_S^2 R(\bar{x}, \bar{y}, \bar{u}, \bar{v})(0) \subset D_S^2 G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(0),$$

we obtain

$$\begin{aligned} D_S^2 G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(0) \cap (-C/\{0\}) = \emptyset \quad &\Rightarrow \quad D_S^2 R(\bar{x}, \bar{y}, \bar{u}, \bar{v})(0) \cap (-C/\{0\}) = \emptyset \\ &\Rightarrow \quad D_S^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(0) \cap (-C/\{0\}) = \emptyset. \end{aligned}$$

Therefore, the following formula holds for $F = G, Q, R$.

$$D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = D^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) + C,$$
$$\text{for all } x \in \mathrm{dom}(D^2(F + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})).$$

In fact, the above formula also remains valid under (b) and (c), because when Q is either second-order directionally compact or second-order compactly approximable, then so are the maps Q and R.

Since G is C-dominated by Q near $\bar{x}$, for each $x \in \mathcal{N}(\bar{x})$, we have

$$\begin{aligned} \mathrm{Min}_C D^2 G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) &= \mathrm{Min}_C D^2(G + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \\ &= \mathrm{Min}_C D^2(Q + C)(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \\ &= \mathrm{Min}_C D^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \\ &\subset D^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x), \end{aligned}$$

establishing (4.4). Analogously, since G is $\tilde{C}$-dominated by R near $\bar{x}$, for each $x \in \mathcal{N}(\bar{x})$, we have

$$\begin{aligned} \mathrm{WMin}_C D^2 G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) &= \mathrm{WMin}_C D^2(G + \tilde{C})(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \\ &= \mathrm{WMin}_C D^2(R + \tilde{C})(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \\ &= \mathrm{WMin}_C D^2 R(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \\ &\subset D^2 R(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x), \end{aligned}$$

proving (4.5). Finally, for (4.6), we begin by noticing that $D^2 P(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset D^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Moreover, due to a known relationship between minimal and properly minimal points we also have that $Q(x) \subset \mathrm{cl}(P(x))$ for every $x \in \mathcal{N}(\bar{x})$, which implies that $D^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset D^2 P(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Therefore,

$$D^2 Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = D^2 P(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x).$$

Consequently

$$\begin{aligned}\mathrm{PMin}_C D^2G(\bar{x},\bar{y},\bar{u},\bar{v})(x) &= \mathrm{PMin}_C D^2(G+C)(\bar{x},\bar{y},\bar{u},\bar{v})(x)\\ &= \mathrm{PMin}_C D^2(Q+C)(\bar{x},\bar{y},\bar{u},\bar{v})(x)\\ &= \mathrm{PMin}_C D^2Q(\bar{x},\bar{y},\bar{u},\bar{v})(x)\\ &= \mathrm{PMin}_C D^2P(\bar{x},\bar{y},\bar{u},\bar{v})(x)\\ &\subset D^2P(\bar{x},\bar{y},\bar{u},\bar{v})(x),\end{aligned}$$

confirming (4.6). The proof is complete. □

We now present conditions under which the opposite inclusion in (4.4) is satisfied. We begin by recalling that for a given normed space X, the (negative) polar M° of $M \subset X$ is a subset of X^* defined by

$$M^\circ = \{l \in X^* : l(x) \le 0 \text{ for every } x \in M\}.$$

The following notion of normally minimal points will be used in the sequel.

Definition 4.6 (Tanino [26])**.** Assume that S is a nonempty subset of Y such that $S + C$ is convex. A point $z \in Min_C S$ is called a normally C-minimal point of S if $T(S+C,z)^\circ \subset int(C^\circ) \cup \{0\}$.

We also need to recall the notion of derivable sets.

Definition 4.7. Let Z be a normed space. The second-order adjacent set $A^2(S,\bar{z},w)$ of $S \subset Z$ at $\bar{z} \in S$ in a direction $w \in Z$ is the set of all $z \in Z$ such that for every sequence $(\lambda_n) \subset P$ with $\lambda_n \downarrow 0$ there exists $(z_n) \subset Z$ with $z_n \to z$ so that $\bar{z} + \lambda_n w + \lambda_n^2 z_n \in S$. The set S is said to be second-order derivable at $\bar{z}$ in direction w if $A^2(S,\bar{z},w) = T^2(S,\bar{z},w)$. S is said to be derivable at $\bar{z}$ if S is second-order derivable at $\bar{z}$ in the direction $w = 0$.

Example 4.8. *To illustrate these definitions, let* $S = \{(x_1,x_2) \in \mathbb{R}^2_+ \mid x_1x_2 = 1\}$ *and* $\bar{z} = (1,1)$*. Then*

$$T(S,\bar{z}) = \{(z_1,z_2) \in \mathbb{R}^2 \mid z_2 = -z_1\},$$

and for $w = (w_1,-w_1)$*,*

$$T^2(S,\bar{z},w) = A^2(S,\bar{z},w) = \{(z_1,z_2) \in \mathbb{R}^2 \mid z_2 = -z_1 + {w_1}^2\},$$

which means that S is second-order derivable at $\bar{z}$ in direction w. For $C = \mathbb{R}^2_+$*,*

$$T(S+C,\bar{z}) = \{(z_1,z_2) \in \mathbb{R}^2 \mid z_2 \ge -z_1\},$$

so that

$$T(S+C,\bar{z})^\circ = \{(z_1,z_1) \mid z_1 \le 0\}.$$

Hence $\bar{z}$ is a normally C-minimal point of S. □

Theorem 4.9. *Assume that X and Y are finite dimensional. Assume that* $\bar{x} \in \mathrm{int}(\mathrm{dom}(G))$ *and* $\mathrm{gph}(G+C)$ *is convex. Assume that* $\mathrm{gph}(G)$ *is derivable at* $(\bar{x},\bar{y})$ *in direction* $(\bar{u},\bar{v})$*. If $\bar{y}$ is a normally C-minimal point of $G(\bar{x})$, then*

$$D^2Q(\bar{x},\bar{y},\bar{u},\bar{v})(x) \subset \mathrm{Min}_C D^2G(\bar{x},\bar{y},\bar{u},\bar{v})(x). \tag{4.7}$$

Proof. Assume that $y \in D^2Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$ which implies that $y \in D^2G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Now suppose $y \notin \mathrm{Min}_C D^2G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$, and consequently there exists $\tilde{y} \in D^2G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$ such that $y - \tilde{y} \in C\backslash\{0\}$. Since $\tilde{y} \in D^2G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$, there are sequences $(s_n) \subset \mathbb{P}$ and $(\tilde{x}_n, \tilde{y}_n) \subset X \times Y$ such that $s_n \downarrow 0$, $(\tilde{x}_n, \tilde{y}_n) \to (x, \tilde{y})$, and

$$\bar{y} + s_n\bar{v} + s_n^2\tilde{y}_n \in G(\bar{x} + s_n\bar{u} + s_n^2\tilde{x}_n). \tag{4.8}$$

Analogously, because $y \in D^2Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$, there are sequences $(t_n) \subset \mathbb{P}$ and $(x_n, y_n) \subset X \times Y$ such that $t_n \downarrow 0$, $(x_n, y_n) \to (x, y)$, and

$$\bar{y} + t_n\bar{v} + t_n^2y_n \in Q(\bar{x} + t_n\bar{u} + t_n^2x_n) = \mathrm{Min}_C G(\bar{x} + t_n\bar{u} + t_n^2x_n).$$

Therefore, $(\bar{x} + t_n\bar{u} + t_n^2x_n, \bar{y} + t_n\bar{v} + t_n^2y_n)$ is a boundary point of the convex set $\mathrm{gph}(G + C)$. By a standard separation argument, there are nonzero vectors $(\lambda_n, \mu_n) \in X \times Y$ such that

$$\langle\lambda_n, \bar{x}+t_n\bar{u}+t_n^2x_n\rangle+\langle\mu_n, \bar{y}+t_n\bar{v}+t_n^2y_n\rangle \geq \langle\lambda_n, \hat{x}\rangle+\langle\mu_n, \hat{y}\rangle \quad \text{for all } (\hat{x}, \hat{y}) \in \mathrm{gph}(G+C). \tag{4.9}$$

We normalize these vectors so that $\|(\lambda_n, \mu_n)\| = 1$, and assume that $(\lambda_n, \mu_n) \to (\lambda, \mu)$. Passing to the limit as $n \to \infty$ in (4.9), we obtain

$$\langle\lambda, \bar{x}\rangle + \langle\mu, \bar{y}\rangle \geq \langle\lambda, \hat{x}\rangle + \langle\mu, \hat{y}\rangle \quad \text{for all } (\hat{x}, \hat{y}) \in \mathrm{gph}(G + C). \tag{4.10}$$

Let $z \in G(\bar{x})+C$ be arbitrary. The map $G+C$ being convex is lower-semicontinuous at $\bar{x}$ (see [26]). Therefore, there exists a sequence $(z_n) \subset Y$ such that $z_n \to z$ and $z_n \in G(\bar{x} + t_n\bar{u} + t_n^2x_n) + C$. By setting $(\hat{x}, \hat{y}) = (\bar{x} + t_n\bar{u} + t_n^2x_n, z_n)$ in (4.9), we obtain

$$\langle\lambda_n, \bar{x} + t_n\bar{u} + t_n^2x_n\rangle + \langle\mu_n, \bar{y} + t_n\bar{v} + t_n^2y_n\rangle \geq \langle\lambda_n, \bar{x} + t_n\bar{u} + t_n^2x_n\rangle + \langle\mu_n, z_n\rangle.$$

Passing to the limit as $n \to \infty$ in the above inequality, we obtain

$$\langle\mu, \bar{y}\rangle \geq \langle\mu, z\rangle.$$

Because $z \in G(\bar{x}) + C$ was chosen arbitrarily, the above inequality confirms that $\mu \in T(G(\bar{x}) + C, \bar{y})^\circ$. Since $\bar{y}$ is a normally C-minimal point, we deduce that $\mu \in (\mathrm{int}(C^\circ) \cup \{0\})$. However, due to the assumption that $\bar{x} \in \mathrm{int}(\mathrm{dom}(G))$, we have $\mu \neq 0$, ensuring that $\mu \in \mathrm{int}(C^\circ)$. This, in view of the fact that $y - \tilde{y} \in C\backslash\{0\}$, confirms that

$$\langle\mu, y\rangle < \langle\mu, \tilde{y}\rangle. \tag{4.11}$$

Due to the assumption that G is derivable, we can set $t_n = s_n$. Then using (4.8) in (4.9), we have

$$\langle\lambda_n, \bar{x}+t_n\bar{u}+t_n^2x_n\rangle+\langle\mu_n, \bar{y}+t_n\bar{v}+t_n^2y_n\rangle \geq \langle\lambda_n, \bar{x}+t_n\bar{u}+t_n^2\tilde{x}_n\rangle+\langle\mu_n, \bar{y}+t_n\bar{v}+t_n^2\tilde{y}_n\rangle,$$

which after simplifying yields

$$\langle\lambda_n, x_n\rangle + \langle\mu_n, y_n\rangle \geq \langle\lambda_n, \tilde{x}_n\rangle + \langle\mu_n, \tilde{y}_n\rangle.$$

Passing to the limit as $n \to \infty$ in the above inequality, we obtain $\langle\mu, y\rangle \geq \langle\mu, \tilde{y}\rangle$. This, however, is a contradiction to (4.11). Consequently $y \in \mathrm{Min}_C D^2G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$, and the proof is complete. $\square$

By setting $(\bar{u}, \bar{v}) = (0, 0)$, we obtain the following result.

Corollary 4.10. [26, Theorem 5.2]. *Assume that* $\bar{x} \in \mathrm{int}(\mathrm{dom}(G))$ *and* $\mathrm{gph}(G+C)$ *is convex. If* $\bar{y}$ *is a normally* C*-minimal point of* $G(\bar{x})$*, then*

$$DQ(\bar{x},\bar{y})(x) \subset \mathrm{Min}_C DG(\bar{x},\bar{y})(x).$$

Remark 4.11. We observe that the hypotheses of Theorem 4.9 are satisfied in the case of Example 4.4.

The following results show that when dealing with weak-perturbation maps, we may dispense with the requirement that $\bar{y}$ is a normally C-minimal point of $G(\bar{x})$.

Theorem 4.12. *Assume that* X *and* Y *are finite dimensional. Assume that* $\bar{x} \in \mathrm{int}(\mathrm{dom}(G))$ *and* $\mathrm{gph}(G+C)$ *is convex. Assume that* $\mathrm{gph}(G)$ *is derivable at* $(\bar{x},\bar{y})$ *in direction* $(\bar{u},\bar{v})$*. Then*

$$D^2R(\bar{x},\bar{y},\bar{u},\bar{v})(x) \subset \mathrm{WMin}_C D^2G(\bar{x},\bar{y},\bar{u},\bar{v})(x). \tag{4.12}$$

Proof. Assume that $y \in D^2R(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ but $y \notin \mathrm{WMin}_C D^2G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$. Therefore, there exists $\tilde{y} \in D^2G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ such that $y - \tilde{y} \in \mathrm{int}(C)$. As in the proof of (4.10), we prove that

$$\langle \lambda, \bar{x}\rangle + \langle \mu, \bar{y}\rangle \geq \langle \lambda, \hat{x}\rangle + \langle \mu, \hat{y}\rangle \quad \text{for all } (\hat{x},\hat{y}) \in \mathrm{gph}(G+C),$$

which, as above, will confirm that $\mu \in C^\circ\backslash\{0\}$. However, now we have $y-\tilde{y} \in \mathrm{int}(C)$, which implies $\langle \mu, y-\tilde{y}\rangle < 0$. (In fact, the normality assumption was required for this inequality). By the same approach as used in the proof of (4.7), we obtain a contradiction to $\langle \mu, y-\tilde{y}\rangle < 0$. The proof is complete. □

For another proof of (4.12), we recall the notion of second-order lower Dini derivative.

Definition 4.13. The second-order lower Dini derivative of $F : X \rightrightarrows Y$ at $(\bar{x},\bar{y}) \in$ *gph*(F) in the direction $(\bar{u},\bar{v}) \in X \times Y$ is the set-valued map $D^2_\ell F(\bar{x},\bar{y},\bar{u},\bar{v})$ such that $y \in D^2_\ell F(\bar{x},\bar{y},\bar{u},\bar{v})(x)$, if and only if for every $(t_n) \subset P$ and for every $(x_n) \subset X$ with $t_n \downarrow 0$ and $x_n \to x$ there are a sequence $(y_n) \subset Y$ with $y_n \to y$ and an integer $m \in N$ such that $\bar{y} + t_n\bar{v} + t_n^2 y_n \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n)$ for every $n \geq m$.

Remark 4.14. The above notion is inspired by the lower Dini derivative introduced by Penot [22] and used to study sensitivity analysis. In fact, if $(\bar{u},\bar{v}) = (0,0)$, then the above notion recovers the lower Dini derivative of Penot [22]. The second-order analogue, given above, was used in [16] to give some necessary optimality conditions in set-valued optimization.

We now formulate the notion of second-order semidifferentiable maps.

Definition 4.15. A set-valued map $F : X \rightrightarrows Y$ is called second-order semidifferentiable at $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ in direction $(\bar{u},\bar{v}) \in X \times Y$, if and only if the second order lower Dini derivative $D^2_\ell F(\bar{x},\bar{y},\bar{u},\bar{v})$ and the second-order contingent derivative $D^2F(\bar{x},\bar{y},\bar{u},\bar{v})$ coincide.

Remark 4.16. The mapping F defined in Example 3.17 is second-order semidifferentiable at $(1,(1,1))$ in the direction $(v_1+v_2,(v_1,v_2))$.

The following result makes good use of the above notions.

Theorem 4.17. *If G is second-order semidifferentiable at $(\bar{x}, \bar{y})$ in direction $(\bar{u}, \bar{v})$, then (4.12) holds.*

Proof. Let $y \in D^2R(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Then there exist sequences $(t_n) \subset \mathbb{P}$, $(x_n, y_n) \subset X \times Y$ such that $t_n \downarrow 0$, $(x_n, y_n) \to (x, y)$ and

$$a_n := \bar{y} + t_n\bar{v} + t_n^2 y_n \in R(\bar{x} + t_n\bar{u} + t_n^2 x_n). \tag{4.13}$$

Assume that $y \notin \mathrm{WMin}_C D^2G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Then there exists $\tilde{y} \in D^2G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) = D^2_\ell G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$ such that $y - \tilde{y} \in \mathrm{int}(C)$. Since $t_n \downarrow 0$ and $x_n \to x$, there exist $n_0 \in \mathbb{N}$ and $\tilde{y}_n \to \tilde{y}$ such that

$$b_n := \bar{y} + t_n\bar{v} + t_n^2\tilde{y}_n \in G(\bar{x} + t_n\bar{u} + t_n^2 x_n) \quad \text{for } n > n_0.$$

Since $y - \tilde{y} \in \mathrm{int}(C)$, we have $y_n - \tilde{y_n} \in \mathrm{int}(C)$ for sufficiently large n. Moreover, because C is a cone, we obtain that

$$a_n - b_n = (\bar{y} + t_n\bar{v} + t_n^2 y_n) - (\bar{y} + t_n\bar{v} + t_n^2\tilde{y}_n) = t_n^2(y_n - \tilde{y_n}) \in \mathrm{int}(C).$$

This however is a contradiction to (4.13). The proof is complete. □

We will give another proof of (4.12) under Aubin's property defined as follows.

Definition 4.18. Let B_Y be the unit ball of the space Y. The map $F : X \rightrightarrows Y$ is said to have the Aubin property around $(u, v) \in \mathrm{gph}(F)$, if there are a constant $L \geq 0$ and neighborhoods U of u and V of v so that

$$F(x_1) \cap V \subset F(x_2) + L \parallel x_1 - x_2 \parallel B_Y \quad \text{for all } x_1,\, x_2 \in U \cap \mathrm{dom}(F).$$

This concept is due to J. P. Aubin. For several useful features of this notion, see [2], [21], [23].

The following result is an analogue of Theorem 4.17.

Theorem 4.19. *If G has the Aubin property around $(\bar{x}, \bar{y})$ and Y is finite dimensional, then (4.12) holds.*

Proof. The proof is similar to that of Theorem 4.17 (See also the proof of Theorem 4.22). □

For our next result we consider the following parameter dependent vector-optimization problem

$$\mathrm{Min}_C f(w, x) \quad \text{subject to } w \in H(x), \tag{4.14}$$

where H is a given set-valued map, and confine our discussion to the case when the set-valued map G is given by

$$G(x) := \{y \in Y |\; y = f(w, x) \text{ for some } w \in H(x)\}, \tag{4.15}$$

where X is the parameter space, $f : W \times X \to Y$ is a single-valued map, and $H : X \rightrightarrows W$ is a set-valued map.

For this situation, we have the following result (see [26]).

Theorem 4.20. *Assume that X and Y are finite dimensional. Assume that f is continuous and C-convex. If H is convex and G is C-dominated by Q near $\bar{x}$, then*

$$\mathrm{Min}_C D^2G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset D^2Q(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x).$$

Proof. Assume that $y \in \mathrm{Min}_C D^2G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ which implies that $y \in D^2G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$ and hence there are sequences $(t_n) \subset \mathbb{P}$ and $(x_n, y_n) \subset X \times Y$ such that $t_n \downarrow 0$, $(x_n, y_n) \to (x, y)$, and

$$\bar{y} + t_n\bar{v} + t_n^2 y_n \in G(\bar{x} + t_n\bar{u} + t_n^2 x_n).$$

Since G is C-dominated by Q near $\bar{x}$, for sufficiently large n, we have

$$\bar{y} + t_n\bar{v} + t_n^2 y_n \in Q(\bar{x} + t_n\bar{u} + t_n^2 x_n) + C,$$

Consequently, we ensure the existence of a sequence $(\tilde{y}_n) \subset Y$ such that

$$\begin{aligned} \bar{y} + t_n\bar{v} + t_n^2\tilde{y}_n &\in Q(\bar{x} + t_n\bar{u} + t_n^2 x_n) \\ y_n - \tilde{y}_n &\in C. \end{aligned} \tag{4.16}$$

Assume that $(\tilde{y}_n)$ is bounded. Since Y is finite dimensional, we may assume that $\tilde{y}_n \to \tilde{y}$ which implies

$$\tilde{y} \in D^2Q(\bar{x},\bar{y},\bar{u},\bar{v})(x) \subset D^2G(\bar{x},\bar{y},\bar{u},\bar{v})(x).$$

In view of (4.16), we have $y - \tilde{y} \in C$, and because $y \in \mathrm{Min}_C D^2G(\bar{x},\bar{y},\bar{u},\bar{v})(x)$, we must have $y = \tilde{y}$. This implies that $y \in D^2Q(\bar{x},\bar{y},\bar{u},\bar{v})(x)$.

Therefore, to complete the proof it suffices to show that $(\tilde{y}_n)$ remains bounded. If possible, assume that this is not the case and $\|\tilde{y}_n\| \to \infty$. In view of (4.15), there are sequences (w_n) and $(\tilde{w}_n)$ such that

$$\begin{aligned} \bar{w} + t_n\bar{u} + t_n^2 w_n &\in H(\bar{x} + t_n\bar{u} + t_n^2 x_n) \\ \bar{y} + t_n\bar{v} + t_n^2 y_n &= f(\bar{w} + t_n\bar{u} + t_n^2 w_n, \bar{x} + t_n\bar{u} + t_n^2 x_n) \\ \bar{w} + t_n\bar{u} + t_n^2\tilde{w}_n &\in H(\bar{x} + t_n\bar{u} + t_n^2 x_n) \\ \bar{y} + t_n\bar{v} + t_n^2\tilde{y}_n &= f(\bar{w} + t_n\bar{u} + t_n^2\tilde{w}_n, \bar{x} + t_n\bar{u} + t_n^2 x_n). \end{aligned}$$

Due to the convexity of H, for some $0 \le \alpha \le 1$, we have

$$\begin{aligned} \alpha(\bar{w} + t_n\bar{u} + t_n^2 w_n) + (1-\alpha)(\bar{w} + t_n\bar{u} + t_n^2\tilde{w}_n) &= \bar{w} + t_n\bar{u} + t_n^2(\alpha w_n + (1-\alpha)\tilde{w}_n) \\ &\in H(\bar{x} + t_n\bar{u} + t_n^2 x_n). \end{aligned}$$

Moreover, due to the assumption that f is C-convex, we obtain

$$\begin{aligned} \bar{y} + t_n\bar{v} + t_n^2 y_n(\alpha) &:= f(\bar{w} + t_n\bar{u} + t_n^2(\alpha w_n + (1-\alpha)\tilde{w}_n), \bar{x} + t_n\bar{u} + t_n^2 x_n) \\ &\le_C \bar{y} + t_n\bar{v} + t_n^2(\alpha y_n + (1-\alpha)\tilde{y}_n) \\ &\le_C \bar{y} + t_n\bar{v} + t_n^2 y_n, \end{aligned}$$

where we used the fact that

$$(\bar{y} + t_n\bar{v} + t_n^2 y_n) - (\bar{y} + t_n\bar{v} + t_n^2(\alpha y_n + (1-\alpha)\tilde{y}_n)) = t_n^2(1-\alpha)(y_n - \tilde{y}_n) \in C.$$

Since f is continuous, we have

$$\begin{aligned} \bar{y} + t_n\bar{v} + t_n^2 y_n(\alpha) &\to \bar{y} + t_n\bar{v} + t_n^2\tilde{y}_n \quad \text{as } \alpha \to 0 \\ \bar{y} + t_n\bar{v} + t_n^2 y_n(\alpha) &\to \bar{y} + t_n\bar{v} + t_n^2 y_n \quad \text{as } \alpha \to 1. \end{aligned}$$

Since $\|\tilde{y}_n\| \to \infty$ and $y_n \to y$, by taking α_n appropriately close to 1, we have

$$\epsilon t_n^2 \le \|\bar{y} + t_n\bar{v} + t_n^2 y_n - (\bar{y} + t_n\bar{v} + t_n^2 y_n(\alpha_n))\| \le t_n^2,$$

where $\epsilon \in (0,1)$. Setting $y_n(\alpha_n) = z_n$ we see that $\epsilon \leq \|y_n - z_n\| \leq 1$. Because $y_n \to y$, we deduce that (z_n) is bounded. Assume that $z_n \to z$, which implies $z \in D^2G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Since $\epsilon \leq \|y_n - z_n\|$, we have $\epsilon \leq \|y - z\|$, and hence $y \neq z$.

From the inequality $\bar{y} + t_n\bar{v} + t_n^2 z_n \leq_C \bar{y} + t_n\bar{v} + t_n^2 y_n$, we infer that $y_n - z_n \in C$, which implies $y - z \in C$. This, however, is a contradiction to the assumption that $y \in \operatorname{Min}_C D^2G(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Therefore, $(\tilde{y}_n)$ has to be a bounded sequence. The proof is complete. □

Example 4.21. As an illustration of Theorem 4.20, let $X = W = (0, +\infty)$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and define $H : X \rightrightarrows W$ by $H(x) = [x, +\infty)$ and $f : W \times X \to Y$ by $f(w, x) = (w, 1/w + x^2)$. In this example, the hypotheses of Theorem 4.20 are satisfied for any $\bar{x} > 0$. One can also see directly that the conclusion of Theorem 4.20 holds, since for each $x > 0$, we have

$$Q(x) = G(x) = \{(w, 1/w + x^2) \mid w \geq x\}.$$

To see what DG and D^2G look like in this example, consider the case where $\bar{x} = 1$ and $\bar{y} = (2, 3/2)$. Then

$$DG(1, (2, 3/2))(x) = \{(y, 2x - y/4) \mid y \in \mathbb{R}\}$$

and

$$D^2G(1, (2, 3/2), v_1, (v_2, 2v_1 - v_2/4))(x) = \{(y, 2x - y/4 + {v_1}^2 + {v_2}^2/8) \mid y \in \mathbb{R}\}.$$

The main focus of this work is on the second-order sensitivity analysis, and consequently issues related to second-order stability analysis have been ignored. In the following we briefly touch upon the subject by proving an upper-continuity result for the second-order contingent derivative.

Theorem 4.22. *Let $F : X \rightrightarrows Y$ be set-valued, let $(\bar{x}, \bar{y}) \in \operatorname{gph}(F)$, and let $(\bar{u}, \bar{v}) \in X \times Y$. Assume that the set-valued map $F : X \rightrightarrows Y$ possesses the Aubin property and Y is finite dimensional. Then $D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$ is upper-semicontinuous.*

Proof. Let $x_0 \in \operatorname{dom}(D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$. To show that the map is upper-semicontinuous at x_0, we prove that for every $\epsilon > 0$, there exists a neighborhood $\mathcal{N}(x_0)$ of x_0 such that

$$D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) \subset D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x_0) + \epsilon B_Y \quad \text{for all } x \in \mathcal{N}(x_0).$$

For an arbitrary $\epsilon > 0$, we define $\mathcal{N}(x_0) = x_0 + \epsilon^{-1} B_Y$. Let $x \in \mathcal{N}(x_0)$ and let $y \in D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x)$. Hence there exists sequences $(t_n) \subset \mathbb{P}$, $(x_n, y_n) \subset X \times Y$ such that $t_n \downarrow 0$, $(x_n, y_n) \to (x, y)$ and $\bar{y} + t_n\bar{v} + t_n^2 y_n \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n)$. Let $z_n \in X$ be a sequence such that $z_n \to x_0$. Therefore, there exist $n_0 \in \mathbb{N}$, and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\bar{y} + t_n\bar{v} + t_n^2 y_n \in F(\bar{x} + t_n\bar{u} + t_n^2 x_n) \cap V \subset F(\bar{x} + t_n\bar{u} + t_n^2 z_n) + L t_n^2 \|x_n - z_n\| B_Y.$$

Therefore,

$$\bar{y} + t_n\bar{v} + t_n^2 w_n \in F(\bar{x} + t_n\bar{u} + t_n^2 z_n),$$

where $w_n = y_n + L b_n \|x_n - z_n\|$ for some $b_n \in B_Y$. Since Y is finite-dimensional, we may assume that $b_n \to b \in B_Y$, and consequently, $w_n \to y + Lb\|x - x_0\|$, confirming that $y \in D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) + \epsilon B_Y$. This establishes that $D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ is upper-semicontinuous. □

5. Conclusion

In this paper we have presented a wide variety of basic results from which it should be possible to build a more comprehensive second-order theory of sensitivity analysis in set-valued optimization. We hope in future work to develop such a theory further.

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