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OPTIMALITY CONDITION FOR VECTOR-VALUED DC PROGRAMMING PROBLEMS

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ABSTRACT. In this paper, we introduce a general concept of the difference of two cone convex functions (DC function) in a different way from what Blanquero and Carrizosa [1] did. Also we show a certain optimality condition for dc programming by using some properties of DC functions.

1. Introduction

Horst, Pardalos and Thoai [4] introduced several mathematical methods in global optimization including nonconvex quadratic programming, general concave minimization, network optimization, Lipschitz and dc programming. Many multiextremal problems can be transformed into equivalent problems where each objective function is the sum of a convex and a concave function, that is, the difference of two convex functions (dc function). Such optimization problems are called dc programming, and there are several proposed algorithms for solving them, such as outer approximation and branch-and-bound methods. On the other hand, for a vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$ which consists of dc function components and a gauge $\gamma : \mathbb{R}^m \to \mathbb{R}$ associated with the unit ball in $\mathbb{R}^n$, Blanquero and Carrizosa [1] showed that the composition $\gamma \circ f : \mathbb{R}^n \to \mathbb{R}$ is also a dc function, where $\mathbb{R}$ is the set of real numbers. In general, there are many concepts of convexity for vector-valued functions with respect to a convex cone in the range of function; see [5]. Such convexities are referred to as cone-convexity, and its typical one is defined by the convexity of the epigraph of function; see [7].

In this paper, we study vector-valued DC programming problems with multiobjective which is represented as the difference of two cone-convex functions (DC function). The main purpose of the paper is to give a certain optimality condition for the problem above. This paper is organized as follows. In Section 2, we introduce some properties of real-valued dc functions and verify that similar properties hold for vector-valued DC functions. In Section 3, we explain the idea of Blanquero and Carrizosa [1] which the composition $\gamma \circ f : \mathbb{R}^n \to \mathbb{R}$ is also a dc function. Inspired by [1], we show that the composite function of a DC function and Gerstewitz's (Tammer's) sublinear scalarizing function [3] is also a dc function. In Section 4, we show a certain optimality condition for dc programming by using some properties of DC functions.

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2. Preliminaries

Throughout the paper, let X and Y be two real vector spaces and let C be a convex cone in Y which induces the following ordering $\leq_C$: for $x, y \in X$,

$$x \leq_C y \quad \Leftrightarrow \quad y - x \in C.$$

For a subset A of Y with $0 \in A$, A is said to be absorbing if for every $x \in Y$, there exists $\delta > 0$ such that $tx \in A$ for all $t \in [0, \delta]$. Moreover, when Y is a topological vector space we denote the topological interior and topological closure of $A \subset Y$ by $\operatorname{int} A$ and $\operatorname{cl} A$, respectively. We start with the following definitions.

Definition 2.1. Let E be a convex subset of X. A real-valued function $f : E \to \mathbb{R}$ is said to be a dc function on E if there exist two convex real-valued functions g and h on E such that

$$f(x) = g(x) - h(x), \ \forall x \in E. \tag{2.1}$$

When $E = X$, f is said to be a "dc function" simply. The representation (2.1) is called a dc decomposition of f on E. When $E = X$, (2.1) is said to be a dc decomposition.

Definition 2.2. A global optimization problem is called a dc programming problem or a dc programming if it has the form:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \in E, \\ & f_i(x) \leq 0, \ i = 1, \ldots, m, \end{array} \tag{2.2}$$

where E is a closed convex subset of X, and all functions $f_i : X \to \mathbb{R}, i = 0, 1, \ldots, m$, are dc functions.

The class of dc functions contains convex functions, concave functions, and more general ones like neither convex nor concave function. Moreover, it is invariant under many operations frequently encountered in optimization.

Lemma 2.3 (Horst, Pardalos and Thoai [4])**.** *Let X be a vector space. Let $f, f_1, \ldots,$ $f_m : X \to \mathbb{R}$ be real valued dc functions. Then the following functions are also dc functions*:

(i) $x \mapsto \displaystyle\sum_{i=1}^{m} \lambda_i f_i(x)$ *for any* $\{\lambda_i\}_{i=1}^m \subset \mathbb{R}$;

(ii) $x \mapsto \displaystyle\max_{i=1,\ldots,m} f_i(x)$ *and* $x \mapsto \displaystyle\min_{i=1,\ldots,m} f_i(x)$;

(iii) $x \mapsto |f(x)|$, $x \mapsto \max\{0, f(x)\}$ *and* $x \mapsto \min\{0, f(x)\}$;

(iv) $x \mapsto \displaystyle\prod_{i=1}^{m} f_i(x)$.

We give two definitions for vector-valued functions.

Definition 2.4. Let X and Y be two real vector spaces. Let $f : X \to Y$. Then

$$\operatorname{epi}(f) := \big\{(x, y) \in X \times Y \ \big| \ f(x) \leq_C y\big\}$$

is called the epigraph of f.

Definition 2.5 (Luc [7]). Let $f : X \to Y$. We say that f is a C-convex function on X if for every $x_1,\ x_2 \in X$ and $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_C \lambda f(x_1) + (1-\lambda)f(x_2).$$

Equivalently, it means that the epi(f) is convex.

Now, we introduce a general concept of the difference of two cone-convex functions (DC function).

Definition 2.6. Let X and Y be two real vector spaces. $f : X \to Y$ is said to be a DC function if there exist two C-convex functions p and q such that

$$f(x) = p(x) - q(x),\ \forall x \in X.$$

By Proposition 6.7 in [7], we can prove the following result easily.

Proposition 2.7. *If $f_1, \ldots, f_m : X \to Y$ are DC functions, then for any $\lambda_1, \ldots,$ $\lambda_m \in \mathbb{R}$,*

$$x \mapsto \sum_{i=1}^{m} \lambda_i f_i(x)$$

is also a DC function.

3. Scaralization for DC functions

In this section, we introduce the composition of a gauge with a dc function by using the way of Blanquero and Carrizosa [1, Prop.1.1]. We assume that $Y = \mathbb{R}^n$ with the usual inner product $\langle \cdot, \cdot \rangle$.

Definition 3.1. Let $S \subset \mathbb{R}^n$ be a set with $0 \in S$. The gauge of S is the function $\gamma_S : \mathbb{R}^n \mapsto (-\infty, +\infty]$ defined by

$$\gamma_S(x) := \inf\big\{t > 0 \;\big|\; x \in tS\big\},\ \forall\, x \in \mathbb{R}^n.$$

When S is absorbing, γ_S is called the Minkowski functional associated with S.

By Theorem 14.5 in [8], we can prove the following lemma.

Lemma 3.2. *Let S be a closed convex absorbing set in Y. Then the gauge γ_S can be written as*

$$\gamma_S(x) = \max\big\{\langle u', x\rangle \;\big|\; u' \in S^\circ\big\},\ \forall x \in \mathbb{R}^n,$$

where S° is the polor set of S, i.e., $S^\circ := \big\{u \in \mathbb{R}^n \;\big|\; \langle u, x\rangle \leq 1,\ \forall x \in S\big\}$.

For a special DC function f with respect to $\mathbb{R}^n_+ = \big\{x = (x_1, x_2, \ldots, x_n) \;\big|\; x_i \geq 0,\ i = 1, 2, \ldots, n\big\}$, we have the following proposition.

Proposition 3.3 (Blanquero and Carrizosa [1]). *Let $\Omega \subset \mathbb{R}^n$ be a convex set, and let $\gamma_B : \mathbb{R}^n \to \mathbb{R}$ be the gauge function of the closed unit ball $B \subset \mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$, and let each component f_i be a real-valued dc function with known dc decomposition*

$$f_i = f_i^+ - f_i^-,$$

where f_i^+ and f_i^- are two $\mathbb{R}^n_+$-convex functions. For any $i = 1, \ldots, n$, let

$$M_i := \max\{\gamma_B(e_i), \gamma_B(-e_i)\},$$

where e_i is the i-th unit vector of $\mathbb{R}^n$. Then $\gamma_B \circ f : \Omega \to \mathbb{R}$ is a dc function and a dc decomposition of $\gamma_B \circ f$ is

$$\gamma_B \circ f = g - h,$$

where $g = \gamma_B \circ f + \sum_{i=1}^{n} M_i(f_i^+ - f_i^-)$ *and* $h = \gamma_B \circ f + \sum_{i=1}^{n} M_i(f_i^+ + f_i^-)$.

For general DC functions, we take a similar approach to the idea in Proposition 3.3, i.e., the scalarizing method by using the composition of a sublinear function with a DC function. For this purpose, we consider a polyhedral convex cone as C in the rest of the paper; that is,

$$C := \{z \in \mathbb{R}^n \mid \langle c_i, z \rangle \geq 0, \forall i = 1, \ldots, m\},$$

where $c_i \neq \theta$ for each $i = 1, \ldots, m$.

Lemma 3.4. *Let X be a real vector space. For a C-convex function $p : X \to \mathbb{R}^n$, and $i \in \{1, \ldots, m\}$, define $g^{(i)} : X \to \mathbb{R}$ by*

$$g^{(i)}(x) := \langle c_i, p(x) \rangle, \ \forall x \in X.$$

Then $g^{(i)}$ is a convex function for each $i = 1, \ldots, m$.

Proof. Take $x_1, x_2 \in X$, $\lambda \in [0, 1]$. Then we have that

$$\begin{aligned} &\lambda g^{(i)}(x_1) + (1-\lambda) g^{(i)}(x_2) - g^{(i)}\big(\lambda x_1 + (1-\lambda)x_2\big) \\ &= \big\langle c_i, \lambda p(x_1) + (1-\lambda)p(x_2) - p\big(\lambda x_1 + (1-\lambda)x_2\big)\big\rangle. \end{aligned}$$

Since the function p is a C-convex function, we have that

$$\lambda p(x_1) + (1-\lambda)p(x_2) - p\big(\lambda x_1 + (1-\lambda)x_2\big) \in C.$$

For each $i = 1, \ldots, m$, we have that

$$\big\langle c_i, \lambda p(x_1) + (1-\lambda)p(x_2) - p\big(\lambda x_1 + (1-\lambda)x_2\big)\big\rangle \geq 0.$$

Thus

$$g^{(i)}\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \lambda g^{(i)}(x_1) + (1-\lambda) g^{(i)}(x_2).$$

This implies that $g^{(i)}$ is a convex function for each $i = 1, \ldots, m$. □

Assume that there exists $k \in \operatorname{int} C$, and define $c^i(k) \in \mathbb{R}^n$ as follows:

$$c^i(k) := \frac{1}{\langle c_i, k \rangle} c_i, \ i = 1, \ldots, m.$$

As in the proof of Lemma 3.4, we have from $\langle c_i, k \rangle > 0$ that the mapping $x \mapsto \langle c^i(k), p(x) \rangle$ is convex for any C-convex function $p : X \to \mathbb{R}^n$. The following proposition is used in the proof of Theorem 4.4.

Proposition 3.5. *Consider the following sublinear function*

$$\varphi_k(y) := \inf\{t \in \mathbb{R} \mid y \in tk - C\}, \ \forall y \in \mathbb{R}^n$$

which was defined by Gerstewitz (Tammer) in [3]*. Then*

$$\varphi_k(y) = \max_{i=1,\ldots,m} \langle c^i(k), y \rangle, \ \forall y \in \mathbb{R}^n.$$

Proof. For any $y \in \mathbb{R}^n$ and $\varepsilon > 0$, by the definition of φ_k there exists $t_\varepsilon \in \mathbb{R}$ such that $t_\varepsilon k - y \in C$ and $t_\varepsilon < \varphi_k(y) + \varepsilon$. We obtain from $t_\varepsilon k - y \in C$ that $\langle c_i, t_\varepsilon k - y \rangle \geq 0$ for all $i = 1, 2, \ldots, m$. Then we have that

$$t_\varepsilon \langle c_i, k \rangle \geq \langle c_i, y \rangle$$

and

$$t_\varepsilon \geq \frac{1}{\langle c_i, k \rangle} \langle c_i, y \rangle = \langle c^i(k), y \rangle.$$

Thus we have that

$$t_\varepsilon \geq \max_{i=1,\ldots,m} \langle c^i(k), y \rangle.$$

From $t_\varepsilon < \varphi_k(y) + \varepsilon$, we have that

$$\varphi_k(y) + \varepsilon > t_\varepsilon \geq \max_{i=1,\ldots,m} \langle c^i(k), y \rangle.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\varphi_k(y) \geq \max_{i=1,\ldots,m} \langle c^i(k), y \rangle.$$

Conversely, fix $y \in \mathbb{R}^n$ and let $\hat{t} = \max_{i=1,\ldots,m} \langle c^i(k), y \rangle$. For any $i = 1, \ldots, m$, we have that

$$\hat{t} \geq \langle c^i(k), y \rangle = \frac{1}{\langle c_i, k \rangle} \langle c_i, y \rangle$$

and hence

$$\langle c_i, \hat{t}k - y \rangle \geq 0.$$

Then

$$\hat{t}k - y \in C.$$

From the definition of φ_k we get

$$\varphi_k(y) \leq \hat{t} = \max_{i=1,\ldots,m} \langle c^i(k), y \rangle.$$

This completes the proof. □

Using Proposition 3.5, and Lemmas 3.4 and 2.3, we have the following proposition.

Proposition 3.6. *Let X be a real vector space. Let $f : X \to \mathbb{R}^n$ be a DC function which a DC decomposition of f is as follows:*

$$f(x) = p(x) - q(x), \ \forall x \in X,$$

where p and q are C-convex functions. Then

$$(\varphi_k \circ f)(x) = \max_{i=1,\ldots,m} \langle c^i(k), f(x) \rangle, \ \forall x \in X.$$

Furthermore $\varphi_k \circ f$ is a dc function.

Proof. We have from Proposition 3.5 that for any $x \in X$,

$$\begin{aligned}(\varphi_k \circ f)(x) &= \varphi_k(f(x)) \\ &= \max_{i=1,\dots,m} \langle c^i(k), f(x)\rangle \\ &= \max_{i=1,\dots,m} \{\langle c^i(k), p(x)\rangle - \langle c^i(k), q(x)\rangle\}.\end{aligned}$$

By Lemma 3.4, we obtain that $\langle c^i(k), p(x)\rangle$ and $\langle c^i(k), q(x)\rangle$ are convex functions. Then $\langle c^i(k), p(x)\rangle - \langle c^i(k), q(x)\rangle$ is a dc function. By Lemma 2.3, we have $\varphi_k \circ f$ is also a dc function. □

4. Optimality condition

In this section, we show a certain optimality condition for dc programming by using properties of DC functions.

Let E be a closed convex subset of $\mathbb{R}^n$. An important dc optimization problem is the following:

$$\omega_0 := \inf\{g(x) - h(x) \mid x \in E\}, \tag{4.1}$$

where g and h are two convex functions on $\mathbb{R}^n$.

Problem (4.1) is solvable if there exist $x^* \in E$ and $\omega_0 \in \mathbb{R}^n$ such that

$$\omega_0 := g(x^*) - h(x^*) = \inf\{g(x) - h(x) \mid x \in E\}.$$

The following result gives an optimality condition for Problem (4.1).

Theorem 4.1 (Horst, Pardalos and Thoai [4]). *If Problem (4.1) is solvable then a point $x^* \in E$ is an optimal solution to it if and only if there exists $t^* \in \mathbb{R}$ such that*

$$\inf\{-h(x) + t \mid x \in E,\ t \in \mathbb{R},\ g(x) - t \le g(x^*) - t^*\} = 0.$$

Next, we give the concept of infimal point with respect to $\operatorname{int} C$, which is regarded as an approximately efficient point.

Definition 4.2 (Tanaka [6]). Let A be a nonempty subset of $\mathbb{R}^n$. A point $w^* \in \mathbb{R}^n$ is said to be an infimal point of A with respect to $\operatorname{int} C$ if it satisfies the following two conditions:

(i) $w^* \in \operatorname{cl} A$;
(ii) for any $a \in A$ with $a \neq w^*$, $w^* - a \notin \operatorname{int} C$.

The set of all infimal points of A with respect to $\operatorname{int} C$ is denoted by $\operatorname{Inf} A$.

Before proving our main theorems, we recall that Gerstewitz's (Tammer's) sublinear scalarizing function φ_k in Proposition 3.5 satisfies the following convenient properties.

Lemma 4.3 (Göpfert, Tammer, Riahi and Zălinescu [2]). *Let Y be a real vector space, C be a proper, closed and convex cone in Y and $k \in \operatorname{int} C$. Then φ_k is a continuous function such that for every $\lambda \in \mathbb{R}$,*

$$\{y \in \mathbb{R}^n \mid \varphi_k(y) < \lambda\} = \lambda k - \operatorname{int} C.$$

Now, we prove our main theorems.

Theorem 4.4. *Let A be a nonempty subset of $\mathbb{R}^n$, $w^* \in \mathrm{cl}A$ and $k \in \mathrm{int}\, C$. Then*

$$w^* \in \mathrm{Inf}\, A \Leftrightarrow \inf\{\varphi_k(y) \mid y \in A - w^*\} = 0.$$

Proof. Let $\beta := \inf\{\varphi_k(y) \mid y \in A - w^*\}$. At first, we assume that $w^* \in \mathrm{Inf}\, A$. Since $w^* \in \mathrm{cl}\, A$, there is $\{a_n\}_{n=1}^{+\infty} \subset A$ such that $a_n \to w^*$ as $n \to +\infty$. Then we have from the continuity of φ_k (Lemma 4.3) and Proposition 3.5 that

$$\begin{aligned}\beta &= \inf\{\varphi_k(y) \mid y \in A - w^*\}\\ &\leq \inf\{\varphi_k(a_n - w^*) \mid n \in \mathbb{N}\}\\ &\leq \lim_{n\to+\infty} \varphi_k(a_n - w^*)\\ &= \varphi_k(0) = 0.\end{aligned}$$

Therefore $\beta \leq 0$. We shall show $\beta = 0$. If $\beta < 0$, then from the definition of β there is $a_0 \in A$ such that $\varphi_k(a_0 - w^*) < 0$. Hence by Lemma 4.3, $a_0 - w^* \in -\mathrm{int}\, C$. This is a contradiction to the infimality of w^*, i.e., (ii) in Definition 4.2. Thus $\beta = 0$.

Conversely, we assume that $\beta = 0$. If $w^* \notin \mathrm{Inf}A$, then there exists $a \in A$ with $a \neq w^*$ such that $a - w^* \in -\mathrm{int}\, C$. Since

$$0 \cdot k - \mathrm{int}\, C = -\mathrm{int}\, C \ni a - w^*,$$

we have from Lemma 4.3 that $\varphi_k(a - w^*) < 0$. On the other hands, from $\beta = 0$ we have that $\varphi_k(a - w^*) \geq 0$ for all $a \in A$. This is a contradiction. Then we obtain that $w^* \in \mathrm{Inf}A$. □

Let $f : X \to \mathbb{R}^n$ be a DC function. Let us consider the following problem:

$$w \in \mathrm{Inf}\{f(x) \mid x \in X\}. \tag{4.2}$$

Problem (4) is called solvable if there exist $x^* \in X$ and $w^* \in \mathbb{R}^n$ such that

$$w^* \in \mathrm{Inf}\{f(x) \mid x \in X\} \quad \text{and} \quad f(x^*) = w^*.$$

Theorem 4.5. *If Problem (4) is solvable then there exists $t^* \in \mathbb{R}$ such that*

$$\inf\{-h_{w^*}(x) + t \mid x \in X,\ t \in \mathbb{R},\ g_{w^*}(x) - t \leq g_{w^*}(x^*) - t^*\} = 0,$$

where g_{w^*} and h_{w^*} are two convex functions as decomposition of $\varphi_k(f(x) - w^*)$.

Proof. Since f is a DC function, it follows from Proposition 3.6 that $\varphi_k(f(x) - w^*)$ is a dc function. By Theorem 4.1,

$$w^* \in \mathrm{Inf}\{f(x) \mid x \in X\} \Leftrightarrow \inf\{\varphi_k(f(x) - w^*) \mid x \in X\} = 0.$$

Hence

$$\begin{aligned}0 &= \inf\{\varphi_k(f(x) - w^*) \mid x \in X\}\\ &= \inf\{g_{w^*}(x) - h_{w^*}(x) \mid x \in X\}.\end{aligned}$$

From Theorem 4.1, there exists $t^* \in \mathbb{R}^n$ such that

$$\inf\{-h_{w^*}(x) + t \mid x \in X,\ t \in \mathbb{R},\ g_{w^*}(x) - t \leq g_{w^*}(x^*) - t^*\} = 0.$$

This completes the proof. □

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