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SEPARATION BY LINEAR INTERPOLATION FAMILIES

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Abstract. By standard separation theorems, if a convex function majorizes a concave one, then there exists an affine function between them. Moreover, a characterization of the existence of an affine separation between two arbitrary functions is also known. Motivated by these results, this note presents a necessary and sufficient condition for the existence of separation by members of a given linear interpolation family. The proof is based on the classical Helly theorem and reflects the geometric feature of the problem. In a particular case, the existence of a generalized convex (or concave) separation is also characterized.

1. Introduction

According to a classical result, if g is a convex and f is a concave function defined on a real interval with $f \le g$, then there exists an affine function ω fulfilling the separation inequalities $f \le \omega \le g$. Of course, the existence of an affine separation does not imply that one of the functions in the separation is convex and the other one is concave. Surprisingly, the characterization for two functions possessing an affine separation was given by Baron, Matkowski and Nikodem [3]. An independent approach is due to Nikodem and Wąsowicz [15]. (In fact, the first paper gives more general results, presenting a convex separation.)

Applying the selection theorems by Behrend and Nikodem [4] and by Balaj and Nikodem [1], analogous results were obtained for polynomial separation by Wąsowicz [20] and by Balaj and Wąsowicz [2]. The separation problem was also studied by Nikodem and Páles [14] when the underlying interpolation family is a two parameter Beckenbach family (which is not supposed to have a linear structure).

Motivated by these results, the aim of this note is to present necessary and sufficient conditions for the existence of separation by members of linear interpolation families via a new insight. Contrary to the previous approaches, we emphasize the geometric feature of the problem. The proof of the main result does not require preliminary selection methods. It needs only the following version of the classical Helly Theorem (for details, consult [12] or [18]): If a collection of convex compact sets of an n-dimensional space is given such that the intersection of every $(n+1)$-member subcollection is nonempty, then the intersection of the entire collection is also nonempty. In fact, it suffices to require that the members are convex, closed, and there exists a finite subcollection with compact intersection.

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In what follows, we recall the terminology about Chebyshev systems and the induced generalized convexity.

Definition 1.1. Let H be a subset of the reals of at least n elements, and $\omega_1, \dots, \omega_n : H \to \mathbb{R}$ be given functions. We say that $\boldsymbol{\omega} := (\omega_1, \dots, \omega_n)$ is a (positive) *Chebyshev system* over H, if, for all elements $x_1 < \dots < x_n$ of H, the following inequality holds:

$$\begin{vmatrix} \boldsymbol{\omega}(x_1) & \dots & \boldsymbol{\omega}(x_n) \end{vmatrix} := \begin{vmatrix} \omega_1(x_1) & \dots & \omega_1(x_n) \\ \vdots & \ddots & \vdots \\ \omega_n(x_1) & \dots & \omega_n(x_n) \end{vmatrix} > 0.$$

Let $\boldsymbol{\omega}$ be a fixed positive Chebyshev system on H. The linear hull of the components of $\boldsymbol{\omega}$ is called a *Haar space* or a *linear interpolation family* on H and is denoted by $\Omega_n(H)$. It is immediate to see, that if n points of the product $H \times \mathbb{R}$ are given with pairwise distinct first coordinates, then there exists exactly one member of $\Omega_n(H)$ passing trough the given points. Exactly this fact is reflected in the terminology "linear interpolation family".

Keeping the notations and assumptions above, the definition of classical convexity can be extended to the case when a linear interpolation family is given.

Definition 1.2. Let H be a subset of the reals of at least n elements, and let ω be a positive Chebyshev system on H. A function $f : H \to \mathbb{R}$ is said to be *generalized convex with respect to* $\boldsymbol{\omega}$ (or briefly: $\boldsymbol{\omega}$*-convex*) if, for all elements $x_0 \le \dots \le x_n$ of H, the inequality

$$\begin{vmatrix} \boldsymbol{\omega}(x_1) & \dots & \boldsymbol{\omega}(x_n) \\ f(x_0) & \dots & f(x_n) \end{vmatrix} := \begin{vmatrix} \omega_1(x_0) & \dots & \omega_1(x_n) \\ \vdots & \ddots & \vdots \\ \omega_n(x_0) & \dots & \omega_n(x_n) \\ f(x_0) & \dots & f(x_n) \end{vmatrix} \ge 0$$

holds true.

If the reversed inequality is valid, then f is termed $\boldsymbol{\omega}$*-concave*. The members of $\Omega_n(H)$ can be considered as *generalized affine functions* that is, functions that are simultaneously $\boldsymbol{\omega}$-convex and $\boldsymbol{\omega}$-concave. Substituting $\boldsymbol{\omega} := (1, \mathrm{id})$, the definition above reduces to the notion of classical convexity. Moreover, it involves also the case of polynomial convexity introduced by Hopf [9] and Popoviciu [16]. For an exhaustive summary of the theory of Chebyshev systems and the convexity induced by Chebyshev systems, we refer to the book [10] by Karlin and Studden. Note also, that if the set H has exactly n elements, then each function given on H belongs to $\Omega_n(H)$ and hence all functions are generalized affine on H.

2. The main results

In the sequel we assume that H is a subset of $\mathbb{R}$ of at least n elements and $\boldsymbol{\omega}$ is a positive Chebyshev system on H. We shall frequently apply the following simple but useful observation.

Lemma 2.1. *For every system of fixed points $x_0 \leq \cdots \leq x_n$ of H, the function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by*

$$F(u_0, u_1, \dots, u_n) := \begin{vmatrix} \boldsymbol{\omega}(x_0) & \boldsymbol{\omega}(x_1) & \dots & \boldsymbol{\omega}(x_n) \\ u_0 & u_1 & \dots & u_n \end{vmatrix}$$

is monotone nondecreasing (resp. nonincreasing) in its $(n-k)$th variable if k is even (resp. odd).

Proof. The statement follows immediately from the fact that the coefficient of the term u_k in the determinant expansion of F is nonnegative (resp. nonpositive) according to the positivity of the system $\boldsymbol{\omega}$. □

We start with three propositions that are crucial tools in proving the main separation theorem. The first proposition is trivial, therefore its proof is omitted.

Proposition 2.2. *Let $f, g : H \to \mathbb{R}$. If there exists an $\boldsymbol{\omega}$-affine function $\omega : H \to \mathbb{R}$ such that $f \leq \omega \leq g$, then there exists an $\boldsymbol{\omega}$-concave function $\varphi : H \to \mathbb{R}$ and an $\boldsymbol{\omega}$-convex function $\psi : H \to \mathbb{R}$ such that $f \leq \varphi \leq g$ and $f \leq \psi \leq g$.*

The next propositions give necessary conditions for the existence of generalized concave and convex separation, respectively. Their proof are quite similar, therefore we deal only with the concave case.

Proposition 2.3. *Let $f, g : H \to \mathbb{R}$. If there exists an $\boldsymbol{\omega}$-concave function $\varphi : H \to \mathbb{R}$ such that $f \leq \varphi \leq g$, then, for all elements $x_0 \leq \cdots \leq x_n$ of H, the inequality*

$$\begin{vmatrix} \dots & \boldsymbol{\omega}(x_{n-3}) & \boldsymbol{\omega}(x_{n-2}) & \boldsymbol{\omega}(x_{n-1}) & \boldsymbol{\omega}(x_n) \\ \dots & g(x_{n-3}) & f(x_{n-2}) & g(x_{n-1}) & f(x_n) \end{vmatrix} \leq 0 \tag{2.1}$$

holds true.

Proof. To prove (2.1), fix elements $x_0 \leq \cdots \leq x_n$ of H. Then, using the monotonicity properties of the function F defined in Lemma 2.1, the inequalities $f \leq \varphi \leq g$, and the $\boldsymbol{\omega}$-concavity of φ, we get

$$\begin{aligned} F\big(\dots, g(x_{n-3}), f(x_{n-2}), g(x_{n-1}), f(x_n)\big) &\leq F\big(\dots, \varphi(x_{n-3}), \varphi(x_{n-2}), \varphi(x_{n-1}), \varphi(x_n)\big) \\ &\leq 0, \end{aligned}$$

which verifies (2.1). □

Proposition 2.4. *Let $f, g : H \to \mathbb{R}$. If there exists an $\boldsymbol{\omega}$-convex function $\psi : H \to \mathbb{R}$ such that $f \leq \psi \leq g$, then, for all elements $x_0 \leq \cdots \leq x_n$ of H, the inequality*

$$\begin{vmatrix} \dots & \boldsymbol{\omega}(x_{n-3}) & \boldsymbol{\omega}(x_{n-2}) & \boldsymbol{\omega}(x_{n-1}) & \boldsymbol{\omega}(x_n) \\ \dots & f(x_{n-3}) & g(x_{n-2}) & f(x_{n-1}) & g(x_n) \end{vmatrix} \geq 0 \tag{2.2}$$

holds true.

Our first main result provides a characterization for the existence of a generalized affine functions separating two given ones in terms of determinant inequalities in the previous two propositions. In the proof, a system of convex sets is constructed such that the intersection of each n-member subsystem turns out to be compact. This property is verified via the characterization of compact sets in finite dimension spaces and the equivalence of the norms of finite dimensional spaces. Finally, the classical Helly theorem is applied.

Theorem 2.5. *Let $f, g : H \to \mathbb{R}$. Then, the following statements are equivalent:*

(i) *There exists an element ω of $\Omega_n(H)$ such that $f \le \omega \le g$;*
(ii) *there exists an $\boldsymbol{\omega}$-concave function $\varphi : H \to \mathbb{R}$ and an $\boldsymbol{\omega}$-convex function $\psi : H \to \mathbb{R}$ satisfying the inequalities $f \le \varphi \le g$ and $f \le \psi \le g$;*
(iii) *the inequalities (2.1) and (2.2) are fulfilled for all elements $x_0 \le \cdots \le x_n$ of H.*

Proof. Without loss of generality we may assume, that H contains at least $(n+1)$ elements (otherwise, every real valued function on H is $\boldsymbol{\omega}$-affine). Moreover, in view of the previous propositions, it is sufficient to prove the implication $(iii) \Rightarrow (i)$.

Let $x \in H$ be arbitrary. Then there exists $0 \le k \le n-1$ and elements $x_0, \dots, x_n \in H$ such that $x_i < x_{i+1}$ if $i \in \{0, \dots, n\} \setminus \{k\}$ and $x = x_k = x_{k+1}$. Substituting these values into (2.1), and expanding the determinant by its last row, and applying the positivity of $\boldsymbol{\omega}$, the inequality $f(x) \le g(x)$ follows. This proves $f \le g$ on H.

For a fixed element x of H, consider the sets $K(x) \subset \Omega_n(H)$ defined by the formula
$$K(x) := \{\omega \in \Omega_n(H) \,|\, f(x) \le \omega(x) \le g(x)\}.$$
Clearly, $K(x)$ is a convex, closed subset of $\Omega_n(H)$. Moreover, the intersection $K(x_1) \cap \ldots \cap K(x_n)$ is compact for any system of pairwise distinct points $x_1, \dots, x_n$ of H. To verify this latter statement, it suffices to check that this intersection is bounded in a specific norm since it is a subset of the finite dimensional space $\Omega_n(H)$, and the norms of finite dimensional spaces are equivalent. Define, for $\omega \in \Omega_n(H)$,
$$\|\omega\| := \max\{|\omega(x_1)|, \dots, |\omega(x_n)|\}.$$
Then, $\|\cdot\|$ is a nonnegative valued, positive homogeneous, subadditive functional on $\Omega_n(H)$. Assume that $\|\omega\| = 0$ for some $\omega \in \Omega_n(H)$. Then $\omega(x_k) = 0$ for all indices $k \in \{1, \dots, n\}$, which implies that ω must be the zero element of $\Omega_n(H)$. Therefore, $\|\cdot\|$ is a norm indeed. With respect to this norm, the intersection $K(x_1) \cap \ldots \cap K(x_n)$ is obviously bounded.

The next goal is to verify that each $(n+1)$-member subsystem of $\{K(x) \,|\, x \in H\}$ has a nonempty intersection. Let $x_0 < \cdots < x_n$ be arbitrary elements of H, and consider the sets Φ, Ψ given by
$$\Phi := \{\dots, \big(x_{n-3}, g(x_{n-3})\big), \big(x_2, f(x_{n-2})\big), \big(x_{n-1}, g(x_{n-1})\big)\},$$
$$\Psi := \{\dots, \big(x_{n-3}, f(x_{n-3})\big), \big(x_2, g(x_{n-2})\big), \big(x_{n-1}, f(x_{n-1})\big)\}.$$
Denote by φ the unique element of $\Omega_n(H)$ that interpolates the points of Φ; similarly, let ψ be the unique element determined by Ψ. By their construction, the functions φ and ψ belong to $K(x_0) \cap \ldots \cap K(x_{n-1})$. Assume now, that $f(x_n) > \varphi(x_n)$ holds. Applying the interpolation properties of φ and the monotonicity of F defined in Lemma 2.1,
$$\begin{aligned} F\big(\dots, f(x_{n-2}), g(x_{n-1}), f(x_n)\big) &= F\big(\dots, \varphi(x_{n-2}), \varphi(x_{n-1}), f(x_n)\big) \\ &> F\big(\dots, \varphi(x_{n-2}), \varphi(x_{n-1}), \varphi(x_n)\big) = 0 \end{aligned}$$
follows, which contradicts inequality (2.1). That is, necessarily $f(x_n) \le \varphi(x_n)$ is valid. Due to inequality (2.2), a similar argument provides that $g(x_n) \ge \psi(x_n)$ also holds. If either $\varphi(x_n)$ or $\psi(x_n)$ belongs to the interval $[f(x_n), g(x_n)]$, then φ or ψ

is an element of the intersection $K(x_0) \cap \ldots \cap K(x_n)$, respectively. Assume that neither $\varphi(x_n)$, nor $\psi(x_n)$ belongs to $[f(x_n), g(x_n)]$. Then, according to the previous observation, $[f(x_n), g(x_n)] \subset [\psi(x_n), \varphi(x_n)]$; hence, with a suitable number $\lambda \in]0,1[$,

$$f(x_n) \le \lambda\psi(x_n) + (1-\lambda)\varphi(x_n) \le g(x_n).$$

With this λ, set $\omega := \lambda\psi + (1-\lambda)\varphi$. It can immediately be seen that $\omega \in K(x_0) \cap \ldots \cap K(x_n)$, proving the nonemptiness.

In view of the above established properties, by Helly's theorem, it follows that the intersection of the family $\{K(x) \,|\, x \in H\}$ is nonempty, which is equivalent to the first assertion (i) of the theorem. $\square$

Let us mention that a similar construction to the sets $K(x)$ and the norm $\|\cdot\|$ of the proof appears in [4]. In fact, the same norm was considered by Tornheim [19] when the underlying structure is a *nonlinear* interpolation family. According to Tornheim's results, the norm $\|\cdot\|$ is also equivalent to the supremum norm in the more general setting.

When the underlying Chebyshev system is two dimensional, the reversed statements of Proposition 2.1 and Proposition 2.2 also remain true (see below). A two-dimensional Chebyshev system $\boldsymbol{\omega} = (\omega_1, \omega_2)$ is said to be *regular*, if it is given on an open interval I such that ω_1 is positive and the function ω_2/ω_1 is strictly monotone increasing. In this setting, there is a tight connection between the standard and (ω_1, ω_2)-convexity. Namely, the convexity of a function χ is equivalent to the (ω_1, ω_2)-convexity of the function $\omega_1 \cdot \chi \circ (\omega_2/\omega_1)$ on a properly chosen domain. For precise details, consult [5] or [6]. (In fact, similar statement holds under more general circumstances, namely, when the underlying two parameter interpolation family is not necessarily a linear one [14].)

Theorem 2.6. *Let I be an open interval, $\boldsymbol{\omega} = (\omega_1, \omega_2)$ be a two-dimensional regular Chebyshev system on I, and $f, g : I \to \mathbb{R}$. Then, there exists an $\boldsymbol{\omega}$-concave function $\varphi : I \to \mathbb{R}$ fulfilling the inequalities $f \le \varphi \le g$ if and only if, for all elements $x_0 \le x_1 \le x_2$ of I, the following inequality holds:*

$$\begin{vmatrix} \boldsymbol{\omega}(x_0) & \boldsymbol{\omega}(x_1) & \boldsymbol{\omega}(x_2) \\ f(x_0) & g(x_1) & f(x_2) \end{vmatrix} \le 0.$$

Analogously, there exists an $\boldsymbol{\omega}$-convex function $\psi : I \to \mathbb{R}$ satisfying $f \le \psi \le g$ if and only if, for all elements $x_0 \le x_1 \le x_2$ of I, the inequality

$$\begin{vmatrix} \boldsymbol{\omega}(x_0) & \boldsymbol{\omega}(x_1) & \boldsymbol{\omega}(x_2) \\ g(x_0) & f(x_1) & g(x_2) \end{vmatrix} \ge 0$$

is valid.

Proof. In view of Proposition 2.1 and Proposition 2.2 the necessity of the above condition is obvious. We shall prove the sufficiency only in the convex case.

Set $J := (\omega_2/\omega_1)(I)$. Since ω_2/ω_1 is a continuous, strictly monotone increasing function, J is an open interval. Consider the identities

$$\begin{vmatrix} \omega_1(x_0) & \omega_1(x_1) & \omega_1(x_2) \\ \omega_2(x_0) & \omega_2(x_1) & \omega_2(x_2) \\ g(x_0) & f(x_1) & g(x_2) \end{vmatrix} =$$

$$= \omega_1(x_0)\omega_1(x_1)\omega_1(x_2)\begin{vmatrix} 1 & 1 & 1 \\ (\omega_2/\omega_1)(x_0) & (\omega_2/\omega_1)(x_1) & (\omega_2/\omega_1)(x_2) \\ (g/\omega_1)(x_0) & (f/\omega_1)(x_1) & (g/\omega_1)(x_2) \end{vmatrix}$$

$$= \omega_1(x_0)\omega_1(x_1)\omega_1(x_2)\begin{vmatrix} 1 & 1 & 1 \\ u_0 & u_1 & u_2 \\ G(u_0) & F(u_1) & G(u_2) \end{vmatrix},$$

where

$$u_0 := (\omega_2/\omega_1)(x_0), \quad u_1 := (\omega_2/\omega_1)(x_1), \quad u_2 := (\omega_2/\omega_1)(x_2),$$

and

$$F := \frac{f}{\omega_1} \circ \Big(\frac{\omega_2}{\omega_1}\Big)^{-1}, \qquad G := \frac{g}{\omega_1} \circ \Big(\frac{\omega_2}{\omega_1}\Big)^{-1}.$$

The positivity of ω_1 forces that the first and the last term have the same sign. In particular, if the first one is nonnegative, then so is the last one. Hence, according to the convex separation theorem of [3], there exists a convex function $\chi : J \to \mathbb{R}$ satisfying $F \leq \chi \leq G$. Define the function $h : I \to \mathbb{R}$ by

$$\psi := \omega_1 \cdot \chi \circ \Big(\frac{\omega_2}{\omega_1}\Big).$$

The inequalities $F \leq \chi \leq G$ and the monotone increasing property of ω_2/ω_1 implies $f \leq \psi \leq g$; on the other hand, applying the connection between the standard and the $\boldsymbol{\omega}$-convexity (consult [6]), the $\boldsymbol{\omega}$-convexity of ψ follows. $\square$

3. Applications

Not claiming completeness, we list up here some consequences of the main result Theorem 2.5. The first one states that a generalized concave function majorized by a generalized convex one can be separated by a generalized affine function. In particular, if an ordinary convex function majorizes an ordinary concave function, then there exists an ordinary affine separation between them. Moreover, the role of concavity and convexity can also be interchanged.

In the subsequent results, condition (ii) of Theorem 2.5 trivially holds, and hence (i), that is, the existence of an affine separation follows.

Corollary 3.1. *If $f, g : H \to \mathbb{R}$ are such that $f \leq g$, and f is $\boldsymbol{\omega}$-concave, g is $\boldsymbol{\omega}$-convex, then there exists a $\boldsymbol{\omega}$-affine function $\omega \in \Omega_n(H)$ with $f \leq \omega \leq g$.*

Corollary 3.2. *If $f, g : H \to \mathbb{R}$ are such that $f \leq g$, and f is $\boldsymbol{\omega}$-convex, g is $\boldsymbol{\omega}$-concave, then there exists a $\boldsymbol{\omega}$-affine function $\omega \in \Omega_n(H)$ with $f \leq \omega \leq g$.*

In the standard setting (i.e., when $\boldsymbol{\omega} = (1, \mathrm{id})$), the main result reduces to a characterization of functions having affine separation [15]. To prove this, one should substitute $x_0 = x$ and $x_2 = y$; then, x_1 can be expressed as a convex combination of x and y. The expansion of the determinants involved gives the inequalities of the corollary.

Corollary 3.3. *Let I be an interval and $f, g : I \to \mathbb{R}$. Then, the following conditions are equivalent:*

(i) *There exists an affine function $\omega : I \to \mathbb{R}$ such that $f \leq \omega \leq g$;*

(ii) *there exist a concave function* $\varphi : I \to \mathbb{R}$ *and a convex function* $\psi : I \to \mathbb{R}$ *satisfying* $f \le \varphi \le g$ *and* $f \le \psi \le g$;

(iii) *for all* $\lambda \in [0,1]$ *and* $x, y \in I$,

$$\lambda f(x) + (1-\lambda) f(y) \le g\big(\lambda x + (1-\lambda) y\big),$$
$$\lambda g(x) + (1-\lambda) g(y) \ge f\big(\lambda x + (1-\lambda) y\big).$$

At last, we consider the case when the underlying interpolation family is generated by polynomials (up to a fixed degree). For technical convenience, first we need the next concepts: if points $x_0 \le \cdots \le x_n$ are fixed elements of an interval I, then denote the Vandermonde determinants built on the system $\{x_0, \ldots, x_n\} \setminus \{x_k\}$ by $V_k(x_0, \ldots, x_n)$. Further, denote the sets $(2\mathbb{Z}) \cap [0,n]$ and $(2\mathbb{Z}+1) \cap [0,n]$ by $N_0(n)$ and $N_1(n)$, respectively. Then, we can formulate the following corollary, which is equivalent to the main result of the paper [20]. The last assertion of the corollary can be derived directly from that of Theorem 2.5, expanding the determinants therein with respect to the last rows, and then arranging the inequalities obtained.

Corollary 3.4. *Let* I *be an interval and* $f, g : I \to \mathbb{R}$. *Then, the following conditions are equivalent:*

(i) *There exists a polynomial* ω *of degree at most* $(n-1)$ *such that* $f \le \omega \le g$;

(ii) *there exists an* n*-concave function* $\varphi : H \to \mathbb{R}$ *and an* n*-convex function* $\psi : H \to \mathbb{R}$ *satisfying the inequalities* $f \le \varphi \le g$ *and* $f \le \psi \le g$;

(iii) *for all elements* $x_0 \le \cdots \le x_n$ *of* I, *the following inequalities hold:*

$$\sum_{k \in N_0(n)} f(x_{n-k}) V_{n-k}(x_0, \ldots, x_n) \le \sum_{k \in N_1(n)} g(x_{n-k}) V_{n-k}(x_0, \ldots, x_n),$$
$$\sum_{k \in N_0(n)} g(x_{n-k}) V_{n-k}(x_0, \ldots, x_n) \le \sum_{k \in N_1(n)} f(x_{n-k}) V_{n-k}(x_0, \ldots, x_n).$$

Let us finish this note with some historical remarks and open problems. As the works [7] and [13] point out, Helly obtained his result in 1913, but, he did not published it until 1923 [8]. By this time, two alternative proofs had appeared: the first one was due to Radon [17] from 1921, while the second one was obtained by Kőnig [11] in 1923.

In view of Theorem 2.6 (and also in its corresponding cited result of [14]), an evident question arises, whether the reversed statements of Proposition 2.1 and Proposition 2.2 also remain true or not (under some reasonable restrictions) as in the two-dimensional case. Till now, neither an affirmative nor negative answer has been obtained, and this problem may be the subject of further research.

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