Journal of Nonlinear and Convex Analysis
Volume 13, Number 1, 2012, 31–48

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

UNICAST DECENTRALIZED ALGORITHM FOR SOLVING CENTRALIZED OPTIMIZATION PROBLEMS IN NETWORK RESOURCE ALLOCATION

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Abstract. Decentralized resource allocation algorithms are desired for large-scale and complex networks that lack centralized operators and where each user cannot use other users' private information such as their utility functions. In this paper, we devise a unicast decentralized algorithm that treats network resource allocation as a variational inequality problem and enables each user in a network to decide its own optimal resource allocation by using only its own private information and the transmitted information from the neighbor user. We present a convergence analysis of the algorithm and an example application.

1. Introduction

Networks have finite resources and support a finite number of users. Resource allocation methods for power [13, 24, 27, 36], channel [1, 17, 19, 20, 31], bandwidth [15, 18, 23, 25, 29], and storage allocations [22] are needed for making networks stable and reliable. When users have to compete for the resource, a control mechanism is needed to allocate their shares and to avoid network congestion. Each user should be able to use the limited resources without interfering with other users. We can define the utility of user i ($i \in I := \{1, 2, \ldots, K\}$, where I stands for the set of users participating in the network) as a function of the allocations to other users and model the utility function, $\mathcal{U}^{(i)}$, of user i as a function from $\mathbb{R}^K$ into $\mathbb{R}$. The domain of $\mathcal{U}^{(i)}$ becomes a more than K-dimensional Euclidean space. For example, reference [17] presented the utility function for a multi-carrier system of which the domain is $\mathbb{R}^{KN}$, where $N > 0$ is the total number of subcarriers. To deal with a network in which users share more than one network resource, we shall assume that the domain of $\mathcal{U}^{(i)}$ is a real Hilbert space H. The feasible set, $C^{(i)}$, of user i is a subset of H, and the set, $C := \bigcap_{i \in I} C^{(i)}$, is called the feasible region for allocating the resource. Since users will have different processor speeds and storage capacities, they will have different utility functions and feasible sets as well. We shall formulate the network resource allocation problem as follows.

Problem 1.1 (Network resource allocation problem)**.** Suppose that $\mathcal{U}^{(i)} \colon H \to \mathbb{R}$ ($i \in I$) is concave and continuously Fréchet differentiable and that $C^{(i)}$ ($\subset H$)

2010 *Mathematics Subject Classification.* 47J25, 65K10, 91A10, 91A40.
Key words and phrases. Monotone operator, resolvent, centralized optimization problem, variational inequality problem, network resource allocation, decentralized algorithm.
This work was supported by the Japan Society for the Promotion of Science through a Grant-in-Aid for Young Scientists (B) (21760062) and by the Japan Society for the Promotion of Science through a Grant-in-Aid for Young Scientists (B) (23760077).

$(i \in I)$ is nonempty, closed, and convex. Our objective is to

$$\text{maximize } \sum_{i \in I} \mathcal{U}^{(i)}(x) \text{ subject to } x \in C := \bigcap_{i \in I} C^{(i)}.$$

Problem 1.1 can be represented as a *variational inequality problem* [5, Subchapter 8.3], [21, Chapter III], [35, Chapters 54-57]:

Problem 1.2 (Variational inequality problem).

$$\text{Find } x^* \in C := \bigcap_{i \in I} C^{(i)} \text{ such that } \left\langle y - x^*, \sum_{i \in I} A^{(i)}(x^*) \right\rangle \geq 0 \text{ for all } y \in C,$$

where $\langle \cdot, \cdot \rangle$ is the inner product of H, $A^{(i)} := -\nabla \mathcal{U}^{(i)}$ $(i \in I)$, and $\nabla \mathcal{U}^{(i)} \colon H \to H$ $(i \in I)$ is the gradient of $\mathcal{U}^{(i)}$.

Many iterative algorithms for solving Problem 1.2 have been presented. For instance, the projection algorithm, $x_{n+1} := P_C(x_n - \lambda A(x_n))$ $(n \in \mathbb{N})$, with an adequate parameter $\lambda > 0$ strongly converges to a unique solution of Problem 1.2 when $A := \sum_{i \in I} A^{(i)}$ is strongly monotone and Lipschitz continuous and the metric projection, P_C, (see Subsection 2.2) onto C $(\subset H)$ can be calculated explicitly [11], [35, Subsection 46.6]. The projection algorithm for the case that A is monotone was presented in [7]. This method requires the mean, $z_n := \sum_{k=1}^n \lambda_k x_k / \sum_{k=1}^n \lambda_k$, of $x_{n+1} := P_C(x_n - \lambda_n A(x_n))$ $((\lambda_n)_{n \in \mathbb{N}} \subset (0,1))$. The sequence, $(z_n)_{n \in \mathbb{N}}$, converges weakly to a solution of Problem 1.2. References [8, 12, 16, 33] presented iterative algorithms that work when A is strongly monotone and Lipschitz continuous and C $(\subset H)$ is a fixed point set of a nonexpansive mapping. These algorithms can be used when P_C cannot be calculated explicitly, and they converge strongly to the solution of Problem 1.2 under standard assumptions [8, 12, 16, 33]. The method in [8] was applied to signal recovery, and the method in [33] was applied to beamforming [28]. The methods in [16, 33] were applied to network bandwidth allocation problems [15]. Iterative algorithms [8, 12, 16, 33] for solving Problem 1.2 with a fixed point constrained set are referred to as *fixed point optimization algorithms*.

The projection algorithms presented in [7, 11] would not be a good choice for solving Problem 1.2 because it is not easy to compute the projection onto the set, $C := \bigcap_{i \in I} C^{(i)}$, even when $C^{(i)}$ $(\subset H)$ $(i \in I)$ in Problem 1.1 is simple in the sense that $P_{C^{(i)}}$ can be explicitly calculated. Instead, we could try to use the fixed point optimization algorithm in [33]. Here, we define a mapping, $T \colon H \to H$, by $T(x) := \prod_{i \in I} P_{C^{(i)}}(x)$ for all $x \in H$. T satisfies the nonexpansivity condition, $\|T(x) - T(y)\| \leq \|x - y\|$ $(x, y \in H)$, and the fixed point set, $\mathrm{Fix}(T) := \{x \in H \colon T(x) = x\}$, has the same elements as C [33, Proposition 4.2]. We reach a solution by iterating $x_{n+1} := T(x_n) - \lambda_n A(T(x_n))$ $(n \in \mathbb{N})$, where $(\lambda_n)_{n \in \mathbb{N}} \subset (0,1)$. The problem is that this method requires a centralized resource allocation; i.e., an operator who knows all the explicit forms of $A^{(i)}$s and $C^{(i)}$s. A centralized operator (Fig.1) manages all the resource allocations in the network by executing a *centralized allocation algorithm* (e.g., fixed point optimization algorithm) to solve a *centralized optimization problem* (Problem 1.2). For example, the base station acts as the centralized operator determining the power control for the uplink or downlink

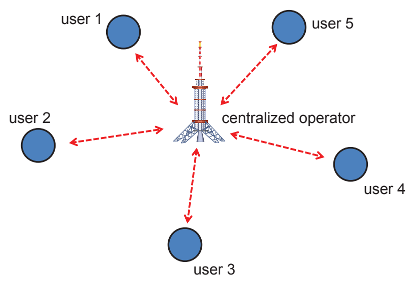

FIGURE 1. A network with a centralized operator

in a code-division multiple-access (CDMA) data network. It executes the allocation algorithm and transmits the calculated powers to all users in the network.

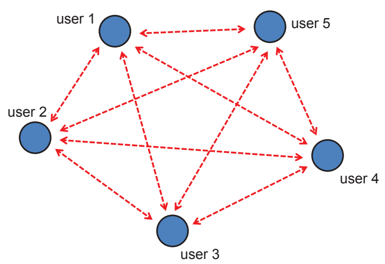

FIGURE 2. A network without a centralized operator in which each user communicates with other users directly and executes a broadcast decentralized algorithm.

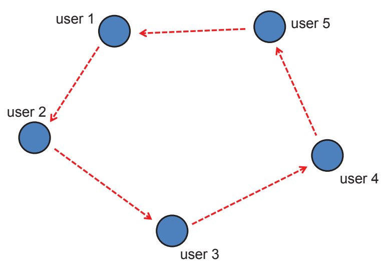

FIGURE 3. A network without a centralized operator in which each user communicates only with its neighbors and executes a unicast decentralized algorithm.

However, pure peer-to-peer networks, sensor networks, mesh networks, and ad hoc networks (FIGS.2 and 3) that do not have centralized operators and that can change size at any time require a different kind of allocation algorithm. *Decentralized* allocation enables individual users to adjust their own resource allocations without using other users' utility functions and feasible sets. For example, reference [14] presents a broadcast type of decentralized algorithm which can be implemented through cooperation, whereby all users communicate with each other (see FIG.2). Given that all users initially have a point, $x_0 \in H$ and a step size, $(\alpha_n)_{n\in\mathbb{N}} \subset (0,1]$, the algorithm in [14] works as follows:

Algorithm 1.3 (Broadcast decentralized algorithm [14])**.**

Step 0. Set $n := 0$.

Step 1. Given $x_n \in H$, user i computes $x_{n+1}^{(i)} \in H$ as $x_{n+1}^{(i)} = P_{C^{(i)}}(x_n - \alpha_{n+1}A^{(i)}(x_{n+1}^{(i)}))$ $(i \in I)$[1]. Each point, $x_{n+1}^{(i)}$ $(i \in I)$, is broadcast to all users in the system.

Step 2. The users compute $x_{n+1} \in H$ by calculating

$$x_{n+1} := \frac{1}{K}\sum_{i\in I} x_{n+1}^{(i)}.$$

User i computes the following mean:

$$z_{n+1}^{(i)} := \frac{1}{\sum_{k=1}^{n+1}\alpha_k}\sum_{k=1}^{n+1}\alpha_k x_k^{(i)}.$$

Put $n := n+1$, and go to Step 1.

The sequence, $(z_{n+1}^{(i)})$ $(i \in I)$, converges weakly to the solution to Problem 1.2 under the standard assumptions [14, Theorem 3.1]. Algorithm 1.3 can be used when each user can directly communicate with other users, as shown in FIG.2. However, Algorithm 1.3 encounters a problem when it is used to solve to Problem 1.2 when each user directly communicates with only neighbor users, as shown in FIG.3. When user i transmits the value it computed, $x_n^{(i)}$, for each $n \in \mathbb{N}$ to all users, its value must be transmitted via other users, which implies that all users assume the burden of transmitting all the $x_n^{(i)}$s $(i = 1,2,\dots,K)$. Hence, Step 1 in Algorithm 1.3 would be inefficient to implement in the case of FIG.3. Moreover, it would not be easy to compute x_n $(n \in \mathbb{N})$ in Step 2 because each user cannot directly get all other users' computed values, $x_n^{(i)}$s $(i = 1,2,\dots,K)$.

This paper presents a *unicast type of decentralized allocation algorithm*, which can be applied to the case that each user directly communicates with only one of its neighbor users. The algorithm is based on the ergodic algorithm in [7] and the ideas of the *resolvents* [2, 4, 9] of bifunctions and monotone operators, which are used to solve important problems in image processing [10] and network flows [2]. Our algorithm enables each user to determine its own optimal resource allocation without using other users' private information such as their utility functions and feasible sets. We also prove that the algorithm weakly converges to the solution to the centralized optimization problem. Although there are many decentralized algorithms being used in the network field, few of them have been proven mathematically to converge to the desired solutions. The analyses presented in the literature tend to rely on computational simulations. Although certain methods exist to solve Problem 1.1 in specific networks and Euclidean spaces [27, 36], the literature does not seem to have any unicast type of decentralized algorithm for solving a centralized variational

[1] $x_{n+1}^{(i)}$ is referred to as a *resolvent* of $\alpha_{n+1}A^{(i)}$ at x_n. For more details, see Subsection 2.3.

inequality problem in a real Hilbert space and applying it to a general network resource allocation problem. The proposed algorithm can be modified to work in large-scale and complex networks that have the properties of incompleteness and asymmetry.

This paper is organized as follows. Section 2 briefly gives mathematical preliminaries. Section 3 describes the centralized optimization problem and the decentralized allocation algorithm for solving it. It also presents a convergence analysis on the algorithm. Section 4 applies the algorithm to a network resource allocation problem. Section 5 concludes the paper.

2. Mathematical preliminaries

2.1. **Variational Inequality Problems for Monotone Operators.** Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$, and let $\mathbb{N}$ be the set of zero and all positive integers; that is, $\mathbb{N} := \{0, 1, 2, \ldots\}$. A set-valued operator, $A \colon H \to 2^H$, is said to be *monotone* [34, Definition 32.2 (c)] if, for all $(x, u), (y, v) \in \mathrm{G}(A) := \{(z, w) \in H \times H \colon w \in A(z)\}$, $\langle x - y, u - v \rangle \geq 0$. $A \colon H \to H$ is referred to as a *strictly monotone* operator [34, Definition 25.2 (ii)] if, for all $x, y \in H$ with $x \neq y$, $\langle x - y, A(x) - A(y) \rangle > 0$. A monotone operator, $A \colon H \to 2^H$, is said to be *maximal* [34, Definition 32.2 (b), (d)] if $\mathrm{G}(A)$ is not properly contained in $\mathrm{G}(B)$ of any monotone operator $B \colon H \to 2^H$. $A \colon H \to H$ is said to be *hemicontinuous* [32, p.204], [34, Definition 27.14] if, for any $x, y \in H$, a mapping, $g \colon [0, 1] \to H$, defined by $g(t) := A(tx + (1 - t)y)$ $(t \in [0, 1])$ is continuous, where H has a weak topology. Any single-valued, monotone, hemicontinuous operator satisfies the maximality condition [34, Proposition 32.7].

The variational inequality problem [21, Chapter III], [5, Subchapter 8.3], [35, Chapters 54-57] for a monotone operator, $A \colon H \to H$, over a nonempty, closed convex set, D $(\subset H)$, is to

$$\text{find } x^* \in \mathrm{VI}(D, A) := \{x^* \in D \colon \langle y - x^*, A(x^*) \rangle \geq 0 \text{ for all } y \in D\}.$$

The following theorem characterizes the solution set of the variational inequality problem and proves the existence of a point in the set:

Proposition 2.1. *Let D $(\subset H)$ be a nonempty, closed convex set, and let $A \colon H \to H$ be monotone and hemicontinuous. Then,*

(i) [32, Lemma 7.1.7] $\mathrm{VI}(D, A) = \{x^* \in D \colon \langle y - x^*, A(y) \rangle \geq 0$ *for all* $y \in D\}$;

(ii) [5, Theorem 8.3.6], [32, Theorem 7.1.8] $\mathrm{VI}(D, A) \neq \emptyset$ *if D is compact;*

(iii) *there exists a unique point in* $\mathrm{VI}(D, A)$ *if $A \colon H \to H$ is strictly monotone and if* $\mathrm{VI}(D, A) \neq \emptyset$.

Proof. (iii) Let $x_1^*, x_2^* \in \mathrm{VI}(D, A)$. Since $x_1^*, x_2^* \in D$, $\langle x_1^* - x_2^*, A(x_1^*) \rangle \leq 0$ and $\langle x_2^* - x_1^*, A(x_2^*) \rangle \leq 0$. Assume that $x_1^* \neq x_2^*$. Then, the strict monotonicity condition of A implies that $0 < \langle x_1^* - x_2^*, A(x_1^*) - A(x_2^*) \rangle = \langle x_1^* - x_2^*, A(x_1^*) \rangle + \langle x_2^* - x_1^*, A(x_2^*) \rangle \leq 0$, which is a contradiction. Therefore, $x_1^* = x_2^*$; that is, the uniqueness of the point in $\mathrm{VI}(D, A)$ is guaranteed. □

2.2. **Metric Projections onto Closed Convex Sets.** Let D $(\subset H)$ be nonempty, closed, and convex. A mapping that assigns every point, $x \in H$, to its unique nearest point in D is called a *metric projection* [3, Facts 1.5], [30, Equation (2.3-13)], [32, p.56] onto D and is denoted by P_D; that is, $P_D(x) \in D$ and $\|x - P_D(x)\| = \inf_{y \in D} \|x - y\|$. The metric projection, P_D, satisfies the following conditions:

Proposition 2.2.
(i) [3, Facts 1.5 (ii)], [32, Lemma 3.1.3] *Let $x \in H$. Then, $\bar{x} = P_D(x)$ if and only if $\bar{x} \in D$ and $\langle \bar{x} - x, y - \bar{x} \rangle \geq 0$ for all $y \in D$.*
(ii) *The fixed point set of P_D is coincident with D; that is,* $\mathrm{Fix}(P_D) := \{x \in H \colon P_D(x) = x\} = D$.
(iii) [3, Facts 1.5 (i)], [30, Theorem 2.4-1 (ii)], [32, Proof (i) of Theorem 3.1.4] *P_D satisfies the firm nonexpansivity condition; that is, $\|P_D(x) - P_D(y)\|^2 \leq \langle x - y, P_D(x) - P_D(y) \rangle$ for all $x, y \in H$.*

By using Proposition 2.2 (iii) and the equality, $\langle x, y \rangle = (1/2)\{\|x\|^2 + \|y\|^2 - \|x - y\|^2\}$ $(x, y \in H)$, we can show that, for all $x, y \in H$,

$$\begin{aligned}\|P_D(x) - P_D(y)\|^2 &\leq \langle x - y, P_D(x) - P_D(y) \rangle \\ &= \frac{1}{2}\left\{\|x - y\|^2 + \|P_D(x) - P_D(y)\|^2 - \|(x - y) - (P_D(x) - P_D(y))\|^2\right\},\end{aligned}$$

and hence,

$$\|P_D(x) - P_D(y)\|^2 \leq \|x - y\|^2 - \|(x - y) - (P_D(x) - P_D(y))\|^2 \ (x, y \in H). \tag{2.1}$$

Note that Inequality (2.1) is used in the proof of Lemma 3.7.

2.3. **Resolvents of Bifunctions.** Suppose that D $(\subset H)$ is a simple, closed convex set and that a bifunction, $F \colon D \times D \to \mathbb{R}$, satisfies the following standard conditions [9, Condition 1.1]:

(I) $F(x, x) = 0$ for all $x \in D$;
(II) $F(x, y) + F(y, x) \leq 0$ for all $x, y \in D$;
(III) for every $x \in D$, $F(x, \cdot) \colon D \to \mathbb{R}$ is lower semi-continuous and convex;
(IV) $\limsup_{t \to 0+} F((1 - t)x + tz, y) \leq F(x, y)$ for all $x, y, z \in D$.

The *resolvent*, $J_F \colon H \to 2^H$, of $F \colon D \times D \to \mathbb{R}$ is defined as follows [4, Corollary 1], [9, Definition 2.11] for all $x \in H$:

$$J_F(x) := \{z \in D \colon F(z, y) + \langle z - x, y - z \rangle \geq 0 \text{ for all } y \in D\}.$$

Given a monotone, hemicontinuous operator $A \colon H \to H$, which satisfies the maximality condition [34, Proposition 32.7], the mapping, $F_A \colon D \times D \to \mathbb{R}$, defined by

$$F_A(x, y) := \langle y - x, A(x) \rangle \ (x, y \in D) \tag{2.2}$$

will have the properties listed below:

Proposition 2.3. *Let $F_A \colon D \times D \to \mathbb{R}$ be the function defined by Equation* (2.2) *and let $x \in H$. Then, F_A has the following properties:*
(i) [9, Lemma 2.15 (i)] *F_A satisfies Conditions* (I)-(IV)*;*
(ii) [4, Corollay 1], [9, Lemma 2.12 (i)] *$J_{F_A}(x) \neq \emptyset$;*
(iii) [9, Lemma 2.12 (ii)] *J_{F_A} is single-valued;*
(iv) [9, Lemma 2.15 (i)] *$\bar{x} := J_{F_A}(x)$ if and only if $\langle y - \bar{x}, \bar{x} - (x - A(\bar{x})) \rangle \geq 0$ for all $y \in D$.*

We can use Proposition 2.2 (i) and Proposition 2.3 (iv) to represent $\bar{x} := J_{F_{rA}}(x)$ $(r > 0)$ as

$$\bar{x} = P_D(x - rA(\bar{x})). \tag{2.3}$$

Reference [10, Subsection 2.6] contains examples of the resolvents of monotone operators that can be explicitly calculated. When A is the gradient, ∇f, of a convex, Fréchet differentiable function, $f\colon H \to \mathbb{R}$, we can represent $\bar{x}$ in Equation (2.3) as

$$\{\bar{x}\} = \operatorname*{Argmin}_{y \in D} \left[f(y) + \frac{1}{2r}\|y - x\|^2 \right]. \tag{2.4}$$

Since $\bar{x} \in D$ in Equation (2.4) is a minimizer of the convex function, $g(\cdot) := f(\cdot) + \|\cdot - x\|^2/(2r)$, over D $(\subset H)$, $\bar{x} \in D$ satisfies Equation (2.4) if and only if, for all $y \in D$, $0 \leq \langle y - \bar{x}, \nabla g(\bar{x})\rangle = \langle y - \bar{x}, \nabla f(\bar{x}) + (\bar{x} - x)/r\rangle$. This is equivalent to $0 \leq \langle y - \bar{x}, \bar{x} - (x - r\nabla f(\bar{x}))\rangle$ $(y \in D)$. Proposition 2.2 (i) guarantees that $\bar{x} \in D$ satisfies Equation (2.3) when $A := \nabla f$.

3. Unicast decentralized algorithm for solving variational inequality problem

3.1. System Model, Assumptions, and Problem Formulation.

Let $I := \{1, 2, \ldots, K\}$ be the set of users who must compete for the network resource. We need to make the following assumptions about the network structure:

Assumption 3.1.

(A1) Each user's objective operator, $A^{(i)}\colon H \to H$ $(i \in I)$, is strictly monotone and hemicontinuous. The explicit form of $A^{(i)}$ is its own private information; that is, other users cannot know the explicit form of $A^{(i)}$.

(A2) Each user's feasible set, $C^{(i)}$ $(i \in I)$, is a nonempty, bounded, closed convex subset of H. The explicit form of $C^{(i)}$ is its own private information.

We will assume the network has the following properties:

Assumption 3.2.

(A3) A set, $C := \bigcap_{i \in I} C^{(i)}$, is nonempty.

(A4) The network topology is a closed structure such as a ring. The communication direction is constant and each user can communicate with one neighbor user directly. Let user 1 be a user who initially executes an algorithm, and let user i $(i \in I)$ be a user who can communicate with user $(i+1)$, where user $(K+1)$ stands for user 1 (Fig.4 describes the case of $K = 5$).

Let us consider the following variational inequality problem with information on the whole network (see also Section 1):

Problem 3.3 (Centralized variational inequality problem)**.** Under Assumptions 3.1 and 3.2,

$$\text{find } x^* \in \mathrm{VI}\left(C, \sum_{i \in I} A^{(i)}\right) := \left\{ x^* \in C\colon \left\langle y - x^*, \sum_{i \in I} A^{(i)}(x^*) \right\rangle \geq 0 \text{ for all } y \in C \right\},$$

where

(A5) $\mathrm{VI}(C, \sum_{i \in I} A^{(i)}) \neq \emptyset$ is assumed.

Assumption (A1) implies that the operator, $\sum_{i\in I} A^{(i)}\colon H \to H$, is strictly monotone and hemicontinuous. Moreover, Assumptions (A2) and (A3) imply that $C := \bigcap_{i\in I} C^{(i)} (\subset H)$ is nonempty, closed, and convex. Hence, Assumption (A5) and Proposition 2.1 (iii) guarantee the uniqueness and existence of the solution of Problem 3.3.

3.2. **Unicast Decentralized Algorithm and Convergence Analysis.** We assume that all users have the following common information before they execute the algorithm.

Assumption 3.4.

(A6) The step size, $(\alpha_n)_{n\in\mathbb{N}} \subset (0,1]$, initially satisfies the following conditions[2]:

(C1) $\alpha_{n+1} \leq \alpha_n$ for all $n \in \mathbb{N}$,

(C2) $\lim_{n\to\infty} \alpha_n = 0$,

(C3) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The algorithm for solving Problem 3.3 under Assumptions (A1)-(A6) is as follows.

Algorithm 3.5 (Unicast decentralized algorithm)**.**

Step 0. User 1 sets $x_0 \in H$ arbitrarily.

Step 1. User 1 computes $x_0^{(1)} \in H$ by $x_0^{(1)} = P_{C^{(1)}}(x_0 - \alpha_0 A^{(1)}(x_0^{(1)}))$ and transmits this point to user 2. User 2 computes $x_1^{(2)} \in H$ by $x_1^{(2)} = P_{C^{(2)}}(x_0^{(1)} - \alpha_1 A^{(2)}(x_1^{(2)}))$ and transmits this point to user 3. User i ($i = 3, 4, \ldots, K$) computes $x_1^{(i)} \in H$ by $x_1^{(i)} = P_{C^{(i)}}(x_1^{(i-1)} - \alpha_1 A^{(i)}(x_1^{(i)}))$ and transmits this point to user $(i+1)$, where user $(K+1)$ means user 1.

Step 2. Given $x_n^{(K)} \in H$, user 1 computes $x_n^{(1)} \in H$ by

$$x_n^{(1)} = P_{C^{(1)}}\left(x_n^{(K)} - \alpha_n A^{(1)}\left(x_n^{(1)}\right)\right) \tag{3.1}$$

and transmits this point to user 2. User 2 computes $x_{n+1}^{(2)} \in H$ by

$$x_{n+1}^{(2)} = P_{C^{(2)}}\left(x_n^{(1)} - \alpha_{n+1} A^{(2)}\left(x_{n+1}^{(2)}\right)\right) \tag{3.2}$$

and transmits this point to user 3. User i ($i = 3, 4, \ldots, K$) computes $x_{n+1}^{(i)} \in H$ by

$$x_{n+1}^{(i)} = P_{C^{(i)}}\left(x_{n+1}^{(i-1)} - \alpha_{n+1} A^{(i)}\left(x_{n+1}^{(i)}\right)\right) \tag{3.3}$$

and transmits this point to user $(i+1)$, where user $(K+1)$ means user 1. User 1 then computes $z_n^{(1)} \in H$ by calculating

$$z_n^{(1)} := \frac{1}{\sum_{k=1}^{n} \alpha_k} \sum_{k=1}^{n} \alpha_k x_k^{(1)}. \tag{3.4}$$

User i ($i = 2, 3, \ldots, K$) computes $z_{n+1}^{(i)} \in H$ by calculating

$$z_{n+1}^{(i)} := \frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{n+1} \alpha_k x_k^{(i)}. \tag{3.5}$$

[2]Examples of $(\alpha_n)_{n\in\mathbb{N}}$ satisfying Conditions (C1)-(C3) are $\alpha_n := 1/(n+1)^{\rho}$ ($\rho \in (0,1]$).

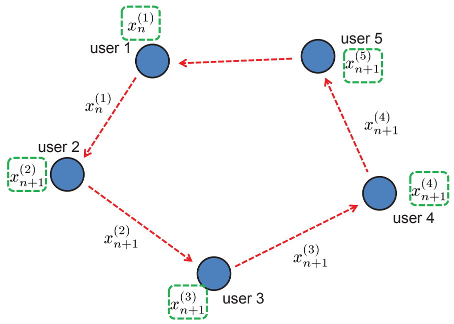

FIGURE 4. Algorithm 3.5 when $K = 5$ (The values in the dotted frame are the values obtained by Algorithm 3.5. Each user transmits its own computed value to its neighbor.)

Put $n := n + 1$, and go to Step 2.

Figure 4 illustrates the concept of Algorithm 3.5 when $K = 5$. The uniqueness and existence of $x_{n+1}^{(i)}$ $(i \in I, n \in \mathbb{N})$ in Equations (3.1), (3.2), and (3.3) are guaranteed by Equation (2.3) and Proposition 2.3 (ii) and (iii). The possibility of calculating $x_{n+1}^{(i)}$ $(i \in I, n \in \mathbb{N})$ depends on the forms of $A^{(i)}$ and $C^{(i)}$. Some important examples of the resolvents of monotone operators are given in [10, Subsection 2.6]. $z_{n+1}^{(i)}$ $(i \in I, n \in \mathbb{N})$ in Equations (3.4) and (3.5) is defined by the mean of $(x_k^{(i)})_{k=1}^{n+1}$. This idea is based on the ergodic algorithm [7] for solving the variational inequality problem. From $z_{n+1}^{(i)} := (1/\sum_{k=1}^{n+1} \alpha_k) \sum_{k=1}^{n+1} \alpha_k x_k^{(i)} = (1/\sum_{k=1}^{n+1} \alpha_k)\{\sum_{k=1}^{n} \alpha_k z_n^{(i)} + \alpha_{n+1} x_{n+1}^{(i)}\}$, user i can compute $z_{n+1}^{(i)} \in H$ by using not all $x_k^{(i)}$s $(k = 1, 2, \ldots, n+1)$ but $z_n^{(i)}$ and $x_{n+1}^{(i)}$.

The following theorem constitutes the convergence analysis of Algorithm 3.5.

Theorem 3.6. *The sequence,* $(z_{n+1}^{(i)})_{n \in \mathbb{N}}$ $(i \in I)$, *in Algorithm 3.5 converges weakly to a unique solution to Problem 3.3.*

3.3. **Lemmas and Proof of Theorem 3.6.** We first prove the following lemma:

Lemma 3.7. *Let* $(z_{n+1}^{(i)})_{n \in \mathbb{N}}$ *and* $(x_{n+1}^{(i)})_{n \in \mathbb{N}}$ $(i \in I)$ *be sequences generated by Algorithm 3.5. Then,*

(i) $(x_{n+1}^{(i)})_{n \in \mathbb{N}}$ *and* $(z_{n+1}^{(i)})_{n \in \mathbb{N}}$ $(i \in I)$ *are bounded;*
(ii) *for all* $n \in \mathbb{N}$ *and for all* $y \in C$,

$$-\frac{\left\|x_0^{(1)} - y\right\|^2}{\sum_{k=0}^{n} \alpha_{k+1}} \leq 2\sum_{i \in I} \left\langle y - z_{n+1}^{(i)}, A^{(i)}(y) \right\rangle - \frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2$$

$$-\sum_{i=3}^{K} \frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \left\| x_{k+1}^{(i)} - x_{k+1}^{(i-1)} \right\|^2;$$

(iii) *for each* $i \in I \backslash \{2\}$,

$$\left(\frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \left\| x_{k+1}^{(i)} - x_{k+1}^{(i-1)} \right\|^2 \right)_{n \in \mathbb{N}} \textit{ is bounded,}$$

where $x_{n+1}^{(0)} := x_{n+1}^{(K)}$ $(n \in \mathbb{N})$.

Proof. (i) Equations (3.1), (3.2), and (3.3) and the boundedness of $C^{(i)}$ $(i \in I)$ guarantee that $(x_{n+1}^{(i)})_{n \in \mathbb{N}} \subset C^{(i)}$ $(i \in I)$ is bounded. Moreover, the convexity of $C^{(i)}$ $(i \in I)$, and Equations (3.4) and (3.5) ensure that $(z_{n+1}^{(i)})_{n \in \mathbb{N}} \subset C^{(i)}$ $(i \in I)$. Accordingly, the boundedness of $C^{(i)}$ $(i \in I)$ implies that $(z_{n+1}^{(i)})_{n \in \mathbb{N}}$ is bounded.

(ii) Inequality (2.1) and Equation (3.1) guarantee that, for all $y \in C := \bigcap_{j \in I} C^{(j)} = \bigcap_{j \in I} \mathrm{Fix}(P_{C^{(j)}}) \subset \mathrm{Fix}(P_{C^{(1)}})$ and for all $k \in \mathbb{N}$,

$$\begin{aligned}
\left\| x_{k+1}^{(1)} - y \right\|^2 &= \left\| P_{C^{(1)}} \left(x_{k+1}^{(K)} - \alpha_{k+1} A^{(1)} \left(x_{k+1}^{(1)} \right) \right) - P_{C^{(1)}}(y) \right\|^2 \\
&\leq \left\| x_{k+1}^{(K)} - \alpha_{k+1} A^{(1)} \left(x_{k+1}^{(1)} \right) - y \right\|^2 \\
&\quad - \left\| \left(x_{k+1}^{(K)} - \alpha_{k+1} A^{(1)} \left(x_{k+1}^{(1)} \right) - y \right) - \left(x_{k+1}^{(1)} - y \right) \right\|^2 \\
&= \left\| \left(x_{k+1}^{(K)} - y \right) - \alpha_{k+1} A^{(1)} \left(x_{k+1}^{(1)} \right) \right\|^2 \\
&\quad - \left\| \left(x_{k+1}^{(K)} - x_{k+1}^{(1)} \right) - \alpha_{k+1} A^{(1)} \left(x_{k+1}^{(1)} \right) \right\|^2 \\
&= \left\| x_{k+1}^{(K)} - y \right\|^2 - 2\alpha_{k+1} \left\langle x_{k+1}^{(K)} - y, A^{(1)} \left(x_{k+1}^{(1)} \right) \right\rangle \\
&\quad - \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2 - 2\alpha_{k+1} \left\langle x_{k+1}^{(1)} - x_{k+1}^{(K)}, A^{(1)} \left(x_{k+1}^{(1)} \right) \right\rangle \\
&= \left\| x_{k+1}^{(K)} - y \right\|^2 + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(1)}, A^{(1)} \left(x_{k+1}^{(1)} \right) \right\rangle - \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2.
\end{aligned}$$

Assumption (A1) implies that $\langle y - x_{k+1}^{(1)}, A^{(1)}(y) \rangle \geq \langle y - x_{k+1}^{(1)}, A^{(1)}(x_{k+1}^{(1)}) \rangle$. Therefore, for all $y \in C$ and for all $k \in \mathbb{N}$, we find that

(3.6)

$$\left\| x_{k+1}^{(1)} - y \right\|^2 \leq \left\| x_{k+1}^{(K)} - y \right\|^2 + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(1)}, A^{(1)}(y) \right\rangle - \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2.$$

Inequality (2.1) and Equation (3.2) also imply that, for all $y \in C := \bigcap_{j \in I} C^{(j)} \subset \mathrm{Fix}(P_{C^{(2)}})$ and for all $k \in \mathbb{N}$,

$$\begin{aligned}
\left\| x_{k+1}^{(2)} - y \right\|^2 &= \left\| P_{C^{(2)}} \left(x_k^{(1)} - \alpha_{k+1} A^{(2)} \left(x_{k+1}^{(2)} \right) \right) - P_{C^{(2)}}(y) \right\|^2 \\
&\leq \left\| \left(x_k^{(1)} - y \right) - \alpha_{k+1} A^{(2)} \left(x_{k+1}^{(2)} \right) \right\|^2 - \left\| \left(x_k^{(1)} - x_{k+1}^{(2)} \right) - \alpha_{k+1} A^{(2)} \left(x_{k+1}^{(2)} \right) \right\|^2 \\
&= \left\| x_k^{(1)} - y \right\|^2 - 2\alpha_{k+1} \left\langle x_k^{(1)} - y, A^{(2)} \left(x_{k+1}^{(2)} \right) \right\rangle
\end{aligned}$$

$$- \left\| x_k^{(1)} - x_{k+1}^{(2)} \right\|^2 - 2\alpha_{k+1} \left\langle x_{k+1}^{(2)} - x_k^{(1)}, A^{(2)} \left(x_{k+1}^{(2)} \right) \right\rangle$$
$$= \left\| x_k^{(1)} - y \right\|^2 + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(2)}, A^{(2)} \left(x_{k+1}^{(2)} \right) \right\rangle - \left\| x_k^{(1)} - x_{k+1}^{(2)} \right\|^2.$$

Together with the monotonicity of $A^{(2)}$, this means

$$\left\| x_{k+1}^{(2)} - y \right\|^2 \leq \left\| x_k^{(1)} - y \right\|^2 + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(2)}, A^{(2)}(y) \right\rangle - \left\| x_k^{(1)} - x_{k+1}^{(2)} \right\|^2. \tag{3.7}$$

For each $i \in I \backslash \{1, 2\}$, Inequality (2.1) and Equation (3.3) guarantee that, for all $y \in C \subset \mathrm{Fix}(P_{C^{(i)}})$ and for all $k \in \mathbb{N}$,

$$\begin{aligned}
\left\| x_{k+1}^{(i)} - y \right\|^2 &= \left\| P_{C^{(i)}} \left(x_{k+1}^{(i-1)} - \alpha_{k+1} A^{(i)} \left(x_{k+1}^{(i)} \right) \right) - P_{C^{(i)}}(y) \right\|^2 \\
&\leq \left\| \left(x_{k+1}^{(i-1)} - y \right) - \alpha_{k+1} A^{(i)} \left(x_{k+1}^{(i)} \right) \right\|^2 \\
&\quad - \left\| \left(x_{k+1}^{(i-1)} - x_{k+1}^{(i)} \right) - \alpha_{k+1} A^{(i)} \left(x_{k+1}^{(i)} \right) \right\|^2 \\
&= \left\| x_{k+1}^{(i-1)} - y \right\|^2 - 2\alpha_{k+1} \left\langle x_{k+1}^{(i-1)} - y, A^{(i)} \left(x_{k+1}^{(i)} \right) \right\rangle \\
&\quad - \left\| x_{k+1}^{(i-1)} - x_{k+1}^{(i)} \right\|^2 - 2\alpha_{k+1} \left\langle x_{k+1}^{(i)} - x_{k+1}^{(i-1)}, A^{(i)} \left(x_{k+1}^{(i)} \right) \right\rangle \\
&= \left\| x_{k+1}^{(i-1)} - y \right\|^2 + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(i)}, A^{(i)} \left(x_{k+1}^{(i)} \right) \right\rangle - \left\| x_{k+1}^{(i-1)} - x_{k+1}^{(i)} \right\|^2.
\end{aligned}$$

Hence, the monotonicity of $A^{(i)}$ implies that

$$\left\| x_{k+1}^{(i)} - y \right\|^2 \leq \left\| x_{k+1}^{(i-1)} - y \right\|^2 + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(i)}, A^{(i)}(y) \right\rangle - \left\| x_{k+1}^{(i-1)} - x_{k+1}^{(i)} \right\|^2. \tag{3.8}$$

Therefore, Inequalities (3.6), (3.7), and (3.8) ensure that, for all $y \in C$ and for all $k \in \mathbb{N}$,

$$\begin{aligned}
\left\| x_{k+1}^{(1)} - y \right\|^2 &\leq \left\| x_{k+1}^{(K)} - y \right\|^2 + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(1)}, A^{(1)}(y) \right\rangle - \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2 \\
&\leq \left\{ \left\| x_{k+1}^{(K-1)} - y \right\|^2 + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(K)}, A^{(K)}(y) \right\rangle - \left\| x_{k+1}^{(K-1)} - x_{k+1}^{(K)} \right\|^2 \right\} \\
&\quad + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(1)}, A^{(1)}(y) \right\rangle - \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2 \\
&\leq \left\| x_{k+1}^{(2)} - y \right\|^2 + 2\alpha_{k+1} \sum_{i=3}^{K} \left\langle y - x_{k+1}^{(i)}, A^{(i)}(y) \right\rangle - \sum_{i=3}^{K} \left\| x_{k+1}^{(i-1)} - x_{k+1}^{(i)} \right\|^2 \\
&\quad + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(1)}, A^{(1)}(y) \right\rangle - \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2 \\
&\leq \left\{ \left\| x_k^{(1)} - y \right\|^2 + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(2)}, A^{(2)}(y) \right\rangle - \left\| x_k^{(1)} - x_{k+1}^{(2)} \right\|^2 \right\} \\
&\quad + 2\alpha_{k+1} \sum_{i=3}^{K} \left\langle y - x_{k+1}^{(i)}, A^{(i)}(y) \right\rangle - \sum_{i=3}^{K} \left\| x_{k+1}^{(i-1)} - x_{k+1}^{(i)} \right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ 2\alpha_{k+1} \left\langle y - x_{k+1}^{(1)}, A^{(1)}(y) \right\rangle - \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2 \\
&\leq \left\| x_k^{(1)} - y \right\|^2 + 2\alpha_{k+1} \sum_{i \in I} \left\langle y - x_{k+1}^{(i)}, A^{(i)}(y) \right\rangle - \sum_{i=3}^{K} \left\| x_{k+1}^{(i-1)} - x_{k+1}^{(i)} \right\|^2 \\
&\quad - \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2 .
\end{aligned}
$$

Summing this inequation from $k = 0$ to $k = m$ ($m \in \mathbb{N}$) gives, for all $y \in C$,

$$
\begin{aligned}
\left\| x_{m+1}^{(1)} - y \right\|^2 &\leq \left\| x_0^{(1)} - y \right\|^2 + 2 \sum_{k=0}^{m} \alpha_{k+1} \sum_{i \in I} \left\langle y - x_{k+1}^{(i)}, A^{(i)}(y) \right\rangle \\
&\quad - \sum_{k=0}^{m} \sum_{i=3}^{K} \left\| x_{k+1}^{(i)} - x_{k+1}^{(i-1)} \right\|^2 - \sum_{k=0}^{m} \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2 ,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
- \left\| x_0^{(1)} - y \right\|^2 &\leq 2 \sum_{i \in I} \left\langle \sum_{k=0}^{m} \alpha_{k+1} y - \sum_{k=0}^{m} \alpha_{k+1} x_{k+1}^{(i)}, A^{(i)}(y) \right\rangle \\
&\quad - \sum_{k=0}^{m} \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2 - \sum_{i=3}^{K} \sum_{k=0}^{m} \left\| x_{k+1}^{(i)} - x_{k+1}^{(i-1)} \right\|^2 .
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
-\frac{\left\| x_0^{(1)} - y \right\|^2}{\sum_{k=0}^{m} \alpha_{k+1}} &\leq 2 \sum_{i \in I} \left\langle y - z_{m+1}^{(i)}, A^{(i)}(y) \right\rangle - \frac{1}{\sum_{k=0}^{m} \alpha_{k+1}} \sum_{k=0}^{m} \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2 \\
&\quad - \sum_{i=3}^{K} \frac{1}{\sum_{k=0}^{m} \alpha_{k+1}} \sum_{k=0}^{m} \left\| x_{k+1}^{(i)} - x_{k+1}^{(i-1)} \right\|^2 .
\end{aligned}
$$

(iii) Condition (C3) and the boundedness of $(z_{n+1}^{(i)})_{n \in \mathbb{N}}$ ($i \in I$) guarantee that, for all $y \in C$, the sequences, $(\|x_0^{(1)} - y\| / \sum_{k=0}^{n} \alpha_{k+1})_{n \in \mathbb{N}}$ and $(\sum_{i \in I} \langle y - z_{n+1}^{(i)}, A^{(i)}(y) \rangle)_{n \in \mathbb{N}}$, are bounded. From Lemma 3.7 (ii), we find that, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
&\sum_{i=3}^{K} \frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \left\| x_{k+1}^{(i)} - x_{k+1}^{(i-1)} \right\|^2 \leq \frac{\left\| x_0^{(1)} - y \right\|^2}{\sum_{k=0}^{n} \alpha_{k+1}} + 2 \sum_{i \in I} \left\langle y - z_{n+1}^{(i)}, A^{(i)}(y) \right\rangle , \\
&\frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \left\| x_{k+1}^{(K)} - x_{k+1}^{(1)} \right\|^2 \leq \frac{\left\| x_0^{(1)} - y \right\|^2}{\sum_{k=0}^{n} \alpha_{k+1}} + 2 \sum_{i \in I} \left\langle y - z_{n+1}^{(i)}, A^{(i)}(y) \right\rangle .
\end{aligned}
$$

Therefore, $(\sum_{k=0}^{n} \|x_{k+1}^{(i)} - x_{k+1}^{(i-1)}\|^2 / \sum_{k=0}^{n} \alpha_{k+1})_{n \in \mathbb{N}}$ ($i \in I \backslash \{1, 2\}$) and $(\sum_{k=0}^{n} \|x_{k+1}^{(K)} - x_{k+1}^{(1)}\|^2 / \sum_{k=0}^{n} \alpha_{k+1})_{n \in \mathbb{N}}$ are bounded. □

Lemma 3.7 leads directly to the next lemma:

Lemma 3.8. *Let $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ be the sequence generated by Algorithm 3.5. Then,*

(i) *for each $i \in I$, $\lim_{n\to\infty} \|z_{n+1}^{(i+1)} - z_{n+1}^{(i)}\| = 0$, where $z_{n+1}^{(K+1)} := z_{n+1}^{(1)}$ $(n \in \mathbb{N})$;*

(ii) *$\lim_{n\to\infty} \|z_{n+1}^{(i)} - z_{n+1}^{(j)}\| = 0$ for all $i, j \in I$.*

Proof. (i) Choose any $i \in I\backslash\{1\}$. From Lemma 3.7 (i) and (iii), there exist M_1, $M_2 > 0$ such that, for all $n \in \mathbb{N}$, $\|x_{n+1}^{(i+1)} - x_{n+1}^{(i)}\|^2 \leq M_1$ and $\sum_{k=0}^{n} \|x_{k+1}^{(i+1)} - x_{k+1}^{(i)}\|^2 / \sum_{k=0}^{n} \alpha_{k+1} \leq M_2$, where $x_{n+1}^{(K+1)} := x_{n+1}^{(1)}$ $(n \in \mathbb{N})$. Choose $\varepsilon > 0$ arbitrarily. Condition (C2) guarantees that $m_1 \in \mathbb{N}$ exists such that $\alpha_n \leq \varepsilon$ for all $n \geq m_1$. Moreover, Condition (C3) guarantees that $m_2 := m_2(m_1) \in \mathbb{N}$ exists such that

$$\frac{1}{\sum_{k=1}^{m_2} \alpha_k} \sum_{k=1}^{m_1} \alpha_k \left\| x_k^{(i+1)} - x_k^{(i)} \right\|^2 \leq \frac{M_1}{\sum_{k=1}^{m_2} \alpha_k} \sum_{k=1}^{m_1} \alpha_k \leq \varepsilon,$$

which implies that, for all $n \geq m_2$,

$$\frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{m_1} \alpha_k \left\| x_k^{(i+1)} - x_k^{(i)} \right\|^2 \leq \varepsilon.$$

Accordingly, Equations (3.4) and (3.5), the convexity of $\|\cdot\|^2$, and Condition (C1) imply that, for all $n \geq n_0 := \max\{m_1, m_2\}$,

$$\begin{aligned}
\left\| z_{n+1}^{(i+1)} - z_{n+1}^{(i)} \right\|^2 &= \left\| \frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{n+1} \alpha_k \left(x_k^{(i+1)} - x_k^{(i)} \right) \right\|^2 \\
&\leq \frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{n+1} \alpha_k \left\| x_k^{(i+1)} - x_k^{(i)} \right\|^2 \\
&= \frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{m_1} \alpha_k \left\| x_k^{(i+1)} - x_k^{(i)} \right\|^2 \\
&\quad + \frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=m_1+1}^{n+1} \alpha_k \left\| x_k^{(i+1)} - x_k^{(i)} \right\|^2 \\
&\leq \varepsilon + \frac{\alpha_{m_1+1}}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=m_1+1}^{n+1} \left\| x_k^{(i+1)} - x_k^{(i)} \right\|^2 \\
&\leq \varepsilon + \frac{\varepsilon}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{n+1} \left\| x_k^{(i+1)} - x_k^{(i)} \right\|^2 \\
&\leq (1 + M_2)\varepsilon,
\end{aligned}$$

which in turn implies that $\lim_{n\to\infty} \|z_{n+1}^{(i+1)} - z_{n+1}^{(i)}\| = 0$ for all $i \in I\backslash\{1\}$, where $z_{n+1}^{(K+1)} := z_{n+1}^{(1)}$ $(n \in \mathbb{N})$. On the other hand, the triangle inequality ensures that, for all $n \in \mathbb{N}$, $\|z_{n+1}^{(1)} - z_{n+1}^{(2)}\| \leq \|z_{n+1}^{(1)} - z_{n+1}^{(K)}\| + \sum_{l=2}^{K-1} \|z_{n+1}^{(l+1)} - z_{n+1}^{(l)}\|$. Hence, we find that $\lim_{n\to\infty} \|z_{n+1}^{(1)} - z_{n+1}^{(2)}\| = 0$. Therefore, $\lim_{n\to\infty} \|z_{n+1}^{(i+1)} - z_{n+1}^{(i)}\| = 0$ for all $i \in I$.

(ii) Choose an $i, j \in I$. If $j > i$, the triangle inequality and Lemma 3.8 (i) ensure that $\lim_{n\to\infty} \|z_{n+1}^{(i)} - z_{n+1}^{(j)}\| \leq \lim_{n\to\infty} \sum_{l=i}^{j-1} \|z_{n+1}^{(l)} - z_{n+1}^{(l+1)}\| = 0$. Similarly, if $j \leq i$, we find that $\lim_{n\to\infty} \|z_{n+1}^{(i)} - z_{n+1}^{(j)}\| = 0$. Hence, $\lim_{n\to\infty} \|z_{n+1}^{(i)} - z_{n+1}^{(j)}\| = 0$ holds for all $i, j \in I$. □

The next lemma follows from Lemmas 3.7 and 3.8:

Lemma 3.9. *For all $i \in I$, $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ generated by Algorithm 3.5 has a subsequence converging weakly to a point in* $\mathrm{VI}(C, \sum_{j\in I} A^{(j)})$.

Proof. Choose an $i \in I$. From the boundedness of $(z_n^{(i)})_{n\in\mathbb{N}\setminus\{0\}}$ in Lemma 3.7 (i), we can prove that a subsequence, $(z_{n_l}^{(i)})_{l\in\mathbb{N}}$, of $(z_n^{(i)})_{n\in\mathbb{N}\setminus\{0\}}$ and a point, $z_*^{(i)} \in H$, exist such that $(z_{n_l}^{(i)})_{l\in\mathbb{N}}$ converges weakly to $z_*^{(i)}$. We shall show that $z_*^{(i)} \in C$. The closedness and convexity of $C^{(i)} (\subset H)$ and $(z_{n_l}^{(i)})_{l\in\mathbb{N}} \subset C^{(i)}$ guarantee that $z_*^{(i)} \in C^{(i)}$. Choose $j \in I\setminus\{i\}$ arbitrarily. Then, from Lemma 3.8 (ii), we find that $\lim_{l\to\infty} \|z_{n_l}^{(j)} - z_{n_l}^{(i)}\| = 0$, which means that $(z_{n_l}^{(j)})_{l\in\mathbb{N}}$ converges weakly to $z_*^{(i)} \in C^{(i)}$. Moreover, the closedness and convexity of $C^{(j)}$ and $(z_{n_l}^{(j)})_{l\in\mathbb{N}} \subset C^{(j)}$ guarantee that $z_*^{(i)} \in C^{(j)}$. Therefore, $z_*^{(i)} \in C^{(i)} \cap \bigcap_{j\in I\setminus\{i\}} C^{(j)} =: C$.

Next, we shall show that $z_*^{(i)} \in \mathrm{VI}(C, \sum_{j\in I} A^{(j)})$. Lemma 3.7 (ii) ensures that, for all $y \in C$ and for all $l \in \mathbb{N}$,

$$-\frac{\left\|x_0^{(1)} - y\right\|^2}{\sum_{k=0}^{n_l-1} \alpha_{k+1}} \leq 2 \sum_{j\in I} \left\langle y - z_{n_l}^{(j)}, A^{(j)}(y) \right\rangle.$$

The weak convergence of $(z_{n_l}^{(j)})_{l\in\mathbb{N}}$ $(j \in I)$ to $z_*^{(i)} \in C$ and Condition (C3) guarantee that, for all $y \in C$,

$$0 \leq 2 \sum_{j\in I} \left\langle y - z_*^{(i)}, A^{(j)}(y) \right\rangle = 2 \left\langle y - z_*^{(i)}, \sum_{j\in I} A^{(j)}(y) \right\rangle.$$

From the monotonicity and hemicontinuity of $\sum_{i\in I} A^{(i)}$ (Assumption (A1)) and Proposition 2.1 (i), we find that

$$0 \leq \left\langle y - z_*^{(i)}, \sum_{j\in I} A^{(j)} \left(z_*^{(i)}\right) \right\rangle \text{ for all } y \in C,$$

which implies that $z_*^{(i)} \in \mathrm{VI}(C, \sum_{j\in I} A^{(j)})$. This completes the proof. □

We can prove Theorem 3.6 by using Lemma 3.9:

Proof of Theorem 3.6. Choose an $i \in I$. As shown in Subsection 3.1, the uniqueness and existence of the point, x^*, in $\mathrm{VI}(C, \sum_{j\in I} A^{(j)})$ are guaranteed. Therefore, from Lemma 3.9, $x^* = z_*^{(i)}$, where $z_*^{(i)} \in \mathrm{VI}(C, \sum_{j\in I} A^{(j)})$ is as in the proof of Lemma 3.9. Moreover, Lemma 3.9 ensures that, for each $i \in I$, there exists a subsequence of $(z_n^{(i)})_{n\in\mathbb{N}\setminus\{0\}}$ such that it converges weakly to the unique point,

$x^* \in \mathrm{VI}(C, \sum_{j\in I} A^{(j)})$. Therefore, we conclude that $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ converges weakly to the solution to Problem 3.3 for all $i \in I$. □

4. Application to network resource allocation problem

We shall apply Algorithm 3.5 to the following network resource allocation:

Problem 4.1. Suppose that Assumptions (A1)-(A6) are satisfied, where the utility function, $\mathcal{U}^{(i)}\colon H \to \mathbb{R}$, of user i is strictly concave and continuously Fréchet differentiable and $A^{(i)} := -\nabla \mathcal{U}^{(i)}$ $(i \in I)$. Then,

$$\text{find } x^* \in C \text{ such that } \sum_{i\in I} \mathcal{U}^{(i)}(x^*) \geq \sum_{i\in I} \mathcal{U}^{(i)}(y) \text{ for all } y \in C.$$

By using Algorithm 3.5 and the relation between Equations (2.3) and (2.4), we can devise the following unicast decentralized resource allocation algorithm:

Algorithm 4.2 (Unicast decentralized resource allocation algorithm)**.**

Step 0. User 1 sets $x_0 \in H$ arbitrarily.

Step 1. User 1 computes $x_0^{(1)} \in H$ by $\{x_0^{(1)}\} := \mathrm{Argmax}_{x\in C^{(1)}}[\mathcal{U}^{(1)}(x) - \|x - x_0\|^2/(2\alpha_0)]$ and transmits this point to user 2. User 2 computes $x_1^{(2)} \in H$ by $\{x_1^{(2)}\} := \mathrm{Argmax}_{x\in C^{(2)}}[\mathcal{U}^{(2)}(x) - \|x - x_0^{(1)}\|^2/(2\alpha_1)]$ and transmits this point to user 3. User i $(i = 3, 4, \ldots, K)$ computes $x_1^{(i)} \in H$ by $\{x_1^{(i)}\} := \mathrm{Argmax}_{x\in C^{(i)}}[\mathcal{U}^{(i)}(x) - \|x - x_1^{(i-1)}\|^2/(2\alpha_1)]$ and transmits this point to user $(i + 1)$, where user $(K + 1)$ means user 1.

Step 2. Given $x_n^{(K)} \in H$, user 1 computes $x_n^{(1)} \in H$ by

$$\left\{x_n^{(1)}\right\} := \operatorname*{Argmax}_{x\in C^{(1)}} \left[\mathcal{U}^{(1)}(x) - \frac{1}{2\alpha_n}\left\|x - x_n^{(K)}\right\|^2\right]$$

and transmits this point to user 2. User 2 computes $x_{n+1}^{(2)} \in H$ by

$$\left\{x_{n+1}^{(2)}\right\} := \operatorname*{Argmax}_{x\in C^{(2)}} \left[\mathcal{U}^{(2)}(x) - \frac{1}{2\alpha_{n+1}}\left\|x - x_n^{(1)}\right\|^2\right]$$

and transmits this point to user 3. User i $(i = 3, 4, \ldots, K)$ computes $x_{n+1}^{(i)} \in H$ by

$$\left\{x_{n+1}^{(i)}\right\} := \operatorname*{Argmax}_{x\in C^{(i)}} \left[\mathcal{U}^{(i)}(x) - \frac{1}{2\alpha_{n+1}}\left\|x - x_{n+1}^{(i-1)}\right\|^2\right]$$

and transmits this point to user $(i + 1)$, where user $(K + 1)$ means user 1. User 1 then computes $z_n^{(1)} \in H$ by calculating

$$z_n^{(1)} := \frac{1}{\sum_{k=1}^n \alpha_k} \sum_{k=1}^n \alpha_k x_k^{(1)}.$$

User i $(i = 2, 3, \ldots, K)$ computes $z_{n+1}^{(i)} \in H$ by calculating

$$z_{n+1}^{(i)} := \frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{n+1} \alpha_k x_k^{(i)}.$$

Put $n := n + 1$, and go to Step 2.

The point, $x_{n+1}^{(i)}$ $(i \in I, n \in \mathbb{N})$, in Step 2 is the unique maximizer of a strictly concave function over a closed convex set. If $H = \mathbb{R}^L$ $(L \geq 1)$, this problem can be solved by using convex optimization techniques such as projection methods [11], interior-point methods [6, Chapter III, 10 and 11], [26, Chapters 15-19], and fixed point optimization algorithms [8, 12, 16, 33]. Some important examples in which $x_{n+1}^{(i)}$ in Step 2 can be solved explicitly are given in [10, Subsection 2.6].

From Theorem 3.6, we reach the following conclusion:

Corollary 4.3. *The sequence, $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$, in Algorithm 4.2 converges weakly to the unique solution to Problem 4.1.*

5. Conclusion

This paper presented a unicast decentralized algorithm for solving the centralized variational inequality problem associated with the network resource allocation problem. The proposed algorithm enables each user to adjust its optimal allocation by using only its own private information and the transmitted information from the neighbor user. Moreover, this paper presented a convergence analysis of the algorithm. The analysis ensures that the algorithm converges weakly to the solution to the centralized variational inequality problem.

Acknowledgments

I am sincerely grateful to Professor Wataru Takahashi of Keio University for helping me improve the manuscript.

References

[1] R. Agrawal and V. Subramanian, *Optimality of certain channel aware scheduling policies*, Proc. 40th Annual Allerton Conf. Comm., Control, Comput., Monticello, IL, 2002, pp. 1532–1541.

[2] H. Attouch, L. M. Briceño-Arias, and P. L. Combettes, *A parallel splitting method for coupled monotone inclusions*, SIAM J. Control Optim. **48** (2010), 3246–3270.

[3] H. H. Bauschke and J. M. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Review **38** (1996), 367–426.

[4] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[5] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples*, Springer, New York, 2000.

[6] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

[7] R. E. Bruck, Jr., *On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space*, J. Math. Anal. Appl. **61** (1977), 159–164.

[8] P. L. Combettes, *A block-iterative surrogate constraint splitting method for quadratic signal recovery*, IEEE Trans. Signal Process. **51** (2003), 1771–1782.

[9] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, Journal of Nonlinear and Convex Analysis **6** (2005), 117–136.

[10] P. L. Combettes and V. R. Wajs, *Signal recovery by proximal forward-backward splitting*, Multiscale Model. Simul. **4** (2005), 1168–1200.

[11] A. A. Goldstein, *Convex programming in Hilbert space*, Bull. Amer. Math. Soc. **70** (1964), 709–710.

[12] H. Iiduka, *Three-term conjugate gradient method for the convex optimization problem over the fixed point set of a nonexpansive mapping*, Applied Mathematics and Computation **217** (2011), 6315–6327.

[13] H. Iiduka, *Fixed point optimization algorithm and its application to power control in CDMA data networks*, Mathematical Programming, to appear (DOI: 10.1007/s10107-010-0427-x).
[14] H. Iiduka, *Decentralized algorithm for centralized variational inequalities in network resource allocation*, Journal of Optimization Theory and Applications **151** (2011), 525–540.
[15] H. Iiduka and M. Uchida, *Fixed point optimization algorithms for network bandwidth allocation problems with compoundable constraints*, IEEE Communications Letters **15** (2011), 596–598.
[16] H. Iiduka and I. Yamada, *A use of conjugate gradient direction for the convex optimization problem over the fixed point set of a nonexpansive mapping*, SIAM J. Optim. **19** (2009), 1881–1893.
[17] M. Kaneko, P. Popovski and J. Dahl, *Proportional fairness in multi-carrier system: upper bound and approximation algorithms*, IEEE Commun. Lett. **10** (2006), 462–464.
[18] F. P. Kelly, *Charging and rate control for elastic traffic*, European Transactions on Telecommunications **8** (1997), 33–37.
[19] F. P. Kelly, A. K. Maulloo and D. K. H. Tan, *Rate control in communication networks: shadow prices, proportional fairness and stability*, J. of the Operational Research Society **49** (1998), 237–252.
[20] H. Kim and Y. Han, *A proportional fair scheduling for multicarrier transmission systems*, IEEE Commun. Lett. **9** (2005), 210–212.
[21] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
[22] P. Maillé and L. Toka, *Managing a peer-to-peer data storage system in a selfish society*, IEEE Journal on Selected Areas in Communications **26** (2008), 1295–1301.
[23] L. Massoulié and J. Roberts, *Bandwidth sharing: Objectives and algorithms*, in Proceedings of the 18th Annual IEEE Conference on Computer Communications (INFOCOM' 99), New York, NY, USA, March 1999, pp. 1395-1403.
[24] F. Meshkati, H. V. Poor, S. C. Schwartz, and N. B. Mandayam, *An energy-efficient approach to power control and receiver design in wireless data networks*, IEEE Trans. Commun. vol. 53, no. 11, pp. 1885-1894, Nov. 2005.
[25] A. Nedić and A. Ozdaglar, *Cooperative distributed multi-agent optimization*, in Convex Optimization in Signal Processing and Communications, D. P. Palomar, Y. C. Eldar (eds.), Cambridge University Press, Cambridge, 2010, pp. 340–386.
[26] J. Nocedal and S. J. Wright, *Numerical Optimization*, Springer Ser. Oper. Res., Springer, New York, 1999.
[27] S. Sharma and D. Teneketzis, *An externalities-based decentralized optimal power allocation algorithm for wireless networks*, IEEE/ACM Trans. Networking **17** (2009), 1819–1831.
[28] K. Slavakis and I. Yamada, *Robust wideband beamforming by the hybrid steepest descent method*, IEEE Trans. Signal Process **55** (2007), 4511–4522.
[29] R. Srikant, *Mathematics of Internet Congestion Control*, Birkhauser, 2004.
[30] H. Stark and Y. Yang, *Vector Space Projections: A Numerical Approach to Signal and Image Processing, Neural Nets, and Optics*, John Wiley & Sons Inc, New York, 1998.
[31] A. L. Stolyar, *On the asymptotic optimality of the gradient scheduling algorithm for multiuser throughput allocation*, Oper. Res. **53** (2005), 12–25.
[32] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.
[33] I. Yamada, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, in Inherently Parallel Algorithms for Feasibility and Optimization and Their Applications, D. Butnariu, Y. Censor and S. Reich (Eds.), Elsevier, New York, 2001, pp. 473–504.
[34] E. Zeidler, *Nonlinear Functional Analysis ans Its Applications II/B. Nonlinear Monotone Operators*, Springer, New York, 1985.
[35] E. Zeidler, *Nonlinear Functional Analysis ans Its Applications III. Variational Methods and Optimization*, Springer, New York, 1985.
[36] C. Zhang, J. Kurose, Y. Liu, D. Towsley, and M. Zink, *A distributed algorithm for joint sensing and routing in wireless networks with non-steerable directional antennas*, in Proceedings of the

14th IEEE International Conference on Network Protocols 2006 (ICNP ’06), Santa Barbara, CA, Nov. 2006, pp. 218–227.

Manuscript received June 1, 2011
revised January 3, 2012

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