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MULTIVALUED PICARD OPERATORS

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Abstract. The aim of this paper is to realize a systematic study of the theory of multivalued Picard operators. Some open questions are also presented.

1. Introduction

Let (X,d) be a metric space and $T : X \to P(X)$ be a multivalued operator. The symbol $F_T := \{x \in X|\ x \in T(x)\}$ denotes the fixed point set of T, while $(SF)_T := \{x \in X|\ \{x\} = T(x)\}$ is the strict fixed point set of T. We also denote by H_d the Pompeiu-Hausdorff functional generated by d.

By definition, $T : X \to P(X)$ is called a multivalued Picard operator (see [12], [15]) if and only if:

(i) $(SF)_T = F_T = \{x^*\}$;

(ii) $T^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

The aim of this paper is to realize a systematic study of the theory of multivalued Picard operators. Several new results and some open questions are presented. We will focus our attention on the (strict) fixed point problem for a ψ-multivalued Picard operator from the following perspectives:

- data dependence of the (strict) fixed point set;
- well-posedness of the (strict) fixed point problem;
- Ulam-Hyers stability of the (strict) fixed point problem.

2. Preliminaries

We recall first the notations and concepts used in this paper. Let X be a nonempty set. Then we denote

$$\mathcal{P}(X) := \{Y|\ Y \text{ is a subset of } X\},\ P(X) := \{Y \in \mathcal{P}(X)|\ Y \text{ is non-empty}\}.$$

Let (X,d) be a metric space. We introduce the following notations:

$$P_b(X) := \{Y \in P(X)|\ Y \text{ is bounded }\},\ P_{cl}(X) := \{Y \in P(X)|\ Y \text{ is closed}\},$$

$$P_{cp}(X) := \{Y \in P(X)|\ Y \text{ is compact}\},\ P_{b,cl}(X) := P_b(X) \cap P_{cl}(X).$$

The following (generalized) functionals are used throughout the paper.

The gap functional

$$D_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ D_d(A,B) := \inf\{d(a,b)|\ a \in A,\ b \in B\}.$$

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The δ generalized functional

$$\delta_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ \delta_d(A,B) := \sup\{d(a,b)|\ a \in A,\ b \in B\}.$$

In particular, $\delta(A) := \delta(A,A)$.

The excess generalized functional

$$\rho_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ \rho_d(A,B) := \sup\{D_d(a,B)|\ a \in A\}.$$

The Hausdorff-Pompeiu generalized functional

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H_d(A,B) := \max\{\rho_d(A,B), \rho_d(B,A)\}.$$

If no confusion is possible, we will avoid the subscript d from the above notations.

If $T : X \to P(X)$ is a multivalued operator, then by

$$Graph(T) := \{(x,y) \in X \times X\ :\ y \in T(x)\}$$

we denote the graphic of the multivalued operator T and by

$$I(T) := \{Y \subset X | T(Y) \subset Y\},$$

the set of all invariant subsets of T. A selection for T is an operator $t : X \to X$ with the property $t(x) \in T(x)$ for each $x \in X$.

We also denote by $T^0 := 1_X,\ T^1 := T, \ldots, T^{n+1} = T \circ T^n,\ n \in \mathbb{N}$ the iterate operators of T. A sequence of successive approximations of T starting from $x \in X$ is a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in X with $x_0 = x,\ x_{n+1} \in T(x_n)$, for $n \in \mathbb{N}$. In the same framework, the operator $\hat{T} : P(X) \to P(X)$, defined by

$$\hat{T}(Y) := \bigcup_{x\in Y} T(x),\ \text{for } Y \in P(X)$$

is called the fractal operator generated by T.

If (X,d) is a metric space, then a multivalued operator $T : X \to P(X)$ is called upper semicontinuous (briefly u.s.c.) on X if and only if $T^+(V) := \{x \in X |\ T(x) \subset V\}$ is open, for each open set $V \subset X$ and it is said to be lower semicontinuous (briefly l.s.c.) on X if and only if $T^-(W) := \{x \in X |\ T(x) \cap W \neq \emptyset\}$ is open, for each open set $W \subset X$. If T is u.s.c. and l.s.c. on X then it is called continuous on X.

Lemma 2.1 (see e.g. [1], [3], [10]). *If (X,d) is a metric space and $T : X \to P_{cp}(X)$, then the following conclusions hold:*

(a) *if T is upper semicontinuous, then $T(Y) \in P_{cp}(X)$, for every $Y \in P_{cp}(X)$;*

(b) *the continuity of T implies the continuity of $\hat{T} : P_{cp}(X) \to P_{cp}(X)$;*

(c) *If T is a multivalued α-contraction (i.e., $\alpha \in [0,1[$ and $H_d(T(x),T(y)) \le \alpha d(x,y)$, for each $x, y \in X$), then the operator $\hat{T} : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$ is a (singlevalued) α-contraction.*

Definition 2.2 ([23]). Let (X,d) be a metric space. Then, $T : X \to P(X)$ is called a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

i) $x_0 = x,\ x_1 = y$;

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

The sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying (i) and (ii) from the above definition is also called a sequence of successive approximations of T starting from $(x,y) \in Graph(T)$.

Definition 2.3 ([12], [15]). Let (X,d) be a metric space and $T : X \to P(X)$ be a MWP operator. Then we define the multivalued operator $T^\infty : Graph(T) \to P(F_T)$ by the formula $T^\infty(x,y) = \{\ z \in F_T \mid$ there exists a sequence of successive approximations of T starting from (x,y) that converges to $z\ \}$.

Definition 2.4 ([12], [15]). Let (X,d) be a metric space and $T : X \to P(X)$ a MWP operator. Then T is said to be a ψ-multivalued weakly Picard operator (briefly ψ-MWP operator) if and only if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 and satisfies $\psi(0) = 0$ and there exists a selection t^∞ of T^∞ such that

$$d(x, t^\infty(x,y)) \le \psi(d(x,y)), \text{ for all } (x,y) \in Graph(T).$$

In particular, if ψ has a linear representation, i.e., there exists $c > 0$ such that $\psi(t) = ct$ for all $t \in \mathbb{R}_+$, then T is called a c-multivalued weakly Picard operator.

Definition 2.5 (see [18]). Let (X,d) be a metric space. An operator $f : X \to X$ is Picard operator if and only if:
(i) $F_f = \{x^*\}$;
(ii) $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

For basic notions and results on the theory of weakly Picard and Picard operators see [12], [11], [15], [18], [24]. For related results concerning metric spaces, operators on metric spaces and fixed points see [2], [7], [8], [26].

3. Multivalued ψ-Picard operators

Let (X,d) be a metric space. Recall that $T : X \to P(X)$ is called a multivalued Picard operator if:
(i) $(SF)_T = F_T = \{x^*\}$;
(ii) $T^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

We will give now some examples of multivalued Picard operators. Let us mention here that the theoretical support of these examples comes from several research directions in metric fixed point theory.

A first direction was given by the following result of S. Reich in [17].

Theorem 3.1 (Reich's Theorem). *Let (X,d) be a complete metric space and $T : X \to P_b(X)$ be a multivalued δ-contraction of Reich type with coefficients α,β,γ, i.e., there exist $\alpha,\beta,\gamma \in \mathbb{R}_+$ with $\alpha+\beta+\gamma < 1$ such that*
$\delta(T(x),T(y)) \le \alpha d(x,y) + \beta\delta(x,T(x)) + \gamma\delta(y,T(y))$, *for all* $x,y \in X$.
Then, $(SF)_T = F_T = \{x^*\}$.

Proof. Let $q > 1$ and let $x_0 \in X$ be arbitrary. Then there exists $x_1 \in T(x_0)$ such that $\delta(x_0, T(x_0)) \le q \cdot d(x_0,x_1)$. Thus, we have

$$\delta(x_1,T(x_1)) \le \delta(T(x_0),T(x_1)) \le \alpha d(x_0,x_1) + \beta\delta(x_0,T(x_0)) + \gamma\delta(x_1,T(x_1))$$
$$\le \alpha d(x_0,x_1) + \beta q d(x_0,x_1) + \gamma\delta(x_1,T(x_1)).$$

Hence, we get $\delta(x_1, T(x_1)) \leq \frac{\alpha+\beta q}{1-\gamma} d(x_0, x_1)$. By this approach we can construct a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T, such that

$$d(x_n, x_{n+1}) \leq \delta(x_n, T(x_n)) \leq \left(\frac{\alpha+\beta q}{1-\gamma}\right)^n d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

Choosing $q > 1$ with $q\beta < 1 - \alpha - \gamma$ we obtain $\frac{\alpha+\beta q}{1-\gamma} < 1$. Hence $(x_n)_{n\in\mathbb{N}}$ ia a Cauchy sequence in the complete metric space (X, d). Let us denote by $x^* \in X$ its limit. We show that x^* is a strict fixed point for T, i.e., $T(x^*) = \{x^*\}$. Indeed, since

$$\delta(x^*, T(x^*)) \leq d(x^*, x_{n+1}) + D(x_{n+1}, T(x_n)) + \delta(T(x_n), T(x^*)) \leq$$
$$d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta\delta(x_n, T(x_n)) + \gamma\delta(x^*, T(x^*)) \leq$$
$$d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta\left(\frac{\alpha+\beta q}{1-\gamma}\right)^n \cdot d(x_0, x_1) + \gamma\delta(x^*, T(x^*)),$$

we have, for each $n \in \mathbb{N}$, that

$$\delta(x^*, T(x^*)) \leq \frac{1}{1-\gamma}(d(x^*, x_{n+1}) + \alpha d(x_n, x^*) + \beta\left(\frac{\alpha+\beta q}{1-\gamma}\right)^n \cdot d(x_0, x_1)).$$

As $n \to \infty$, we obtain that $\delta(x^*, T(x^*)) = 0$ and thus $T(x^*) = \{x^*\}$.

We will show now that $F_T = (SF)_T$. For this purpose it is enough to show that $F_T \subset (SF)_T$. Let $x \in F_T$ be arbitrarily chosen. Then, from the contraction type condition, by putting $y = x \in T(x)$, we get

$$\delta(T(x)) \leq (\beta+\gamma) \cdot \delta(x, T(x)) \leq (\beta+\gamma) \cdot \delta(T(x)).$$

If, we suppose, by contradiction, that $card(T(x)) > 1$, then $\delta(T(x)) > 0$ and, by above relation, we obtain $1 \leq \beta + \gamma$, a contradiction. Thus, we have proved that $\delta(T(x)) = 0$ and so $\{x\} = T(x)$.

For the uniqueness of the strict fixed point, let us consider $y \in (SF)_T$, with $y \neq x^*$. Then

$$d(x^*, y) = \delta(T(x^*), T(y)) \leq \alpha d(x^*, y) + \beta\delta(x^*, T(x^*)) + \gamma\delta(y, T(y)) = \alpha d(x^*, y),$$

which let us the contradiction $\alpha \geq 1$. The proof is now complete. □

Remark 3.2. The original proof of this result is based on the idea of constructing a singlevalued Ćirić-Reich-Rus selection of the operator T, see [17]. See also I.A. Rus [21].

Adding a new condition on the coefficients we obtain an example of multivalued Picard operator.

Corollary 3.3. *Let (X, d) be a complete metric space and $T : X \to P_b(X)$ be a multivalued δ-contraction of Reich type with coefficients α, β, γ. Additionally suppose that $\alpha + 2\beta < 1$. Then T is a multivalued Picard operator.*

Proof. By Theorem 3.1 we know that $(SF)_T = F_T = \{x^*\}$. We have to prove that $T^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$. We successively have:

$\delta(T(x), x^*) = \delta(T(x), T(x^*)) \leq \alpha d(x, x^*) + \beta\delta(x, T(x)) + \gamma\delta(x^*, T(x^*)) =$

$\alpha d(x,x^*) + \beta\delta(x,T(x)) \leq \alpha d(x,x^*) + \beta(d(x,x^*) + \delta(x^*,T(x)))$. Thus

$$\delta(T(x),x^*) \leq \frac{\alpha+\beta}{1-\beta} d(x,x^*), \text{ for all } x \in X.$$

Then

$$\delta(T^2(x),x^*) = \sup_{y\in T(x)} \delta(T(y),x^*) \leq \sup_{y\in T(x)} \left(\frac{\alpha+\beta}{1-\beta}\right) d(y,x^*) \leq \left(\frac{\alpha+\beta}{1-\beta}\right)^2 d(x,x^*).$$

By induction, we get that

$$\delta(T^n(x),x^*) \leq \left(\frac{\alpha+\beta}{1-\beta}\right)^n d(x,x^*) \to 0 \text{ as } n \to +\infty, \text{ for each } x \in X.$$

The proof is now complete. □

For the next result we need the following two notions, see [21] for details. A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$. As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is continuous in 0. A comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ having the property that $t - \varphi(t) \to +\infty$, as $t \to +\infty$ is said to be a strict comparison function.

A general result for multivalued operators satisfying a nonlinear contraction type condition was proved by I.A. Rus in [21], see Theorem 8.4.3, page 85.

Theorem 3.4 (Rus' Theorem)**.** *Let (X,d) be a complete metric space, $T : X \to P_b(X)$ be a multivalued operator and $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ be a mapping. Suppose:*

(i) *$r, s \in \mathbb{R}^5_+, r \leq s$ implies that $\varphi(r) \leq \varphi(s)$;*
(ii) *there exists $p > 1$ such that the mapping $\Phi_p : \mathbb{R}_+ \to \mathbb{R}_+$ given by $t \longmapsto \varphi(t, pt, pt, t, t)$ is a strict comparison function;*
(iii) *$\delta(T(x),T(y)) \leq \varphi(d(x,y), \delta(x,T(x)), \delta(y,T(y)), \delta(x,T(y)), \delta(y,T(x)))$, for all $x, y \in X$.*

Then, $(SF)_T = F_T = \{x^*\}$.

The above result also assures the first assumption from the definition of a multivalued Picard operator. In order to obtain the second one too, we have to impose, as before, another condition on the operator T. A result in this direction is the following theorem.

Theorem 3.5. *Let (X,d) be a complete metric space, $T : X \to P_b(X)$ be a multivalued operator satisfying all the assumptions from Theorem 3.4. If, additionally, there exists a comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$r_0, r_1 \in \mathbb{R}_+ \text{ with } r_1 \leq \varphi(r_0, r_0 + r_1, 0, r_0, r_1) \text{ implies that } r_1 \leq \psi(r_0),$$

then T is a multivalued Picard operator.

Proof. By Theorem 3.4 we have that $(SF)_T = F_T = \{x^*\}$. We have to prove that $T^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$. We successively have:

$$\begin{aligned}
\delta(T(x),x^*) &= \delta(T(x),T(x^*)) \\
&\leq \varphi(d(x,x^*), \delta(x,T(x)), \delta(x^*,T(x^*)), \delta(x,T(x^*)), \delta(x^*,T(x))) \\
&= \varphi(d(x,x^*), \delta(x,T(x)), 0, d(x,x^*), \delta(x^*,T(x))) \\
&\leq \varphi(d(x,x^*), d(x,x^*) + \delta(x^*,T(x)), 0, d(x,x^*), \delta(x^*,T(x))).
\end{aligned}$$

Thus, by the additional hypothesis we get that

$$\delta(T(x), x^*) \leq \psi(d(x, x^*)).$$

Next, $\delta(T^2(x), x^*) = \sup_{y \in T(x)} \delta(T(y), x^*) \leq \sup_{y \in T(x)} \psi(d(y, x^*)) \leq \psi^2(d(x, x^*))$.
By induction, for each $x \in X$, we get that

$$\delta(T^n(x), x^*) \leq \psi^n(d(x, x^*)) \to 0 \text{ as } n \to +\infty.$$

The proof is complete. □

A second direction concerning multivalued Picard operators is given by the following result of I.A. Rus, see [21], Theorem 8.5.1, page 87.

Theorem 3.6 (Rus' Theorem)**.** *Let (X, d) be a complete metric space and let $T : X \to P_{b,cl}(X)$ be a multivalued α-contraction such that $(SF)_T \neq \emptyset$. Then $F_T = (SF)_T = \{x^*\}$.*

We get now another example of multivalued Picard operator. For the sake of completeness we recall here the proof of the whole theorem.

Theorem 3.7. *Let (X, d) be a complete metric space and let $T : X \to P_{cp}(X)$ be a multivalued α-contraction with $(SF)_T \neq \emptyset$. Then, we have:*
(i) $F_T = (SF)_T = \{x^*\}$;
(ii) $T^n(x) \overset{H_d}{\to} \{x^*\}$ *as* $n \to +\infty$*, for each* $x \in X$*,*
i.e., T is a multivalued Picard operator.

Proof. (i) (see [21]). Let $x^* \in (SF)_T$. Notice first that $(SF)_T = \{x^*\}$. Indeed, if $y \in (SF)_T$ with $y \neq x^*$, then $d(x^*, y) = H(T(x^*), T(y)) \leq \alpha d(x^*, y)$. Thus, since $\alpha < 1$, we immediately get that $y = x^*$.

Suppose now that $y \in F_T$. Then,

$$d(x^*, y) = D(T(x^*), y) \leq H(T(x^*), T(y)) \leq \alpha d(x^*, y).$$

Thus, again we have that $y = x^*$. Hence $F_T \subset (SF)_T$. Since $(SF)_T \subset F_T$, we get that $(SF)_T = F_T$.

(ii) Let $x \in X$ be arbitrarily chosen. Then, using Lemma 2.1 (c), we have

$$H(T^n(x), x^*) = H(T^n(x), T^n(x^*)) \leq \alpha H(T^{n-1}(x), T^{n-1}(x^*)) \leq \cdots \leq \alpha^n d(x, x^*).$$

Thus, $T^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to +\infty$. □

Remark 3.8. A similar result for the case of multivalued φ-contractions (i.e., multivalued operators $T : X \to P_{cl}(X)$ satisfying, with a strict comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, the following assumption

$$H_d(T(x), T(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in X)$$

can be established (see also A. Sîntămărian [25]).

The above presented results give rise to the following definition.

Definition 3.9. Let (X, d) be a metric space. Then, by definition, a multivalued operator $T : X \to P(X)$ satisfies Rus' alternative if

$$\text{either } (SF)_T = \emptyset \text{ or } F_T = (SF)_T = \{x^*\}.$$

Using this definition, we have the following results.

Theorem 3.10. *Let (X,d) be a complete metric space and let $T : X \to P_{cl}(X)$ be a strong Caristi multivalued operator, i.e., there exists a lower semicontinuous function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each $(x,y) \in Graph(T)$, we have*

$$\eta(y) + d(x,y) \leq \eta(x).$$

Suppose that T satisfies Rus' alternative and it has closed graph.
Then, $F_T = (SF)_T = \{x^*\}$.

Proof. By Aubin-Siegel's Theorem (see [4]), we have that $(SF)_T \neq \emptyset$. Now, by Rus's alternative we obtain the conclusion. □

Theorem 3.11. *Let (X,d) be a complete metric space and let $T : X \to P_b(X)$ for which there exists $a \in [0,1[$ such that, for each $(x,y) \in Graph(T)$, we have*

$$\delta(T(x), T(y)) \leq a \max\{\delta(x, T(x)), \delta(y, T(y)), \frac{1}{2} D(x, T(y))\}.$$

Suppose that T satisfies Rus' alternative and it has closed graph.
Then, $F_T = (SF)_T = \{x^*\}$.

Proof. By Theorem 2.1 in [16] we have that $F_T = (SF)_T \neq \emptyset$. Next, by Rus's alternative we get the conclusion. □

A third direction concerning the theory of multivalued Picard operators is related to the following theorem given by Tarafdar-Yuan in [27], (see also Yuan [28], Theorem 9.3.14. on page 559).

Let X be a topological space. By definition, $T : X \to P_{cl}(X)$ is called a topological contraction (Tarafdar-Yuan [27], see also [28]) if:

a) T is u.s.c.
b) $Y \in P_{cl}(X)$ with $T(Y) = Y \;\Rightarrow\; Y = \{x^*\}$.

Theorem 3.12 (Tarafdar-Yuan's Theorem)**.** *Let X be a Hausdorff compact topological space and $T : X \to P_{cl}(X)$ be a topological contraction. Then T has a unique strict fixed point $x^* \in X$ and $\bigcap_{n \geq 0} T^n(X) = \{x^*\}$.*

From the above theorem we obtain another example of multivalued Picard operator.

Theorem 3.13. *Let (X,d) be a compact metric space and $T : X \to P_{cl}(X)$ be a l.s.c. topological contraction. Then T is a multivalued Picard operator.*

Proof. Notice first that, from Tarafdar-Yuan's Theorem, there exists a unique $x^* \in X$ such that $T(x^*) = \{x^*\} = \bigcap_{n \geq 0} T^n(X)$. Thus $(SF)_T = \{x^*\}$. Let $x \in F_T$ be arbitrary chosen. Then $x \in T(x) \subset T^2(x) \subset \cdots \subset T^n(x) \subset \cdots$. So, $x \in T^n(x) \subset T^n(X)$, for each $n \in \mathbb{N}$. Hence $x = x^*$ and so $F_T = (SF)_T = \{x^*\}$. Consider now the fractal operator $\hat{T}$ generated by T, i.e., the operator $\hat{T} : P_{cp}(X) \to P_{cp}(X)$, defined by

$$\hat{T}(Y) := \bigcup_{x\in Y} T(x), \text{ for } Y \in P_{cp}(X).$$

Since T is continuous it follows that $\hat{T}$ is a H_d-continuous operator from $P_{cp}(X)$ to itself. Moreover, $\hat{T}$ is a (singlevalued) self topological contraction on $P_{cp}(X)$, since $\hat{T}(Y) = Y$ implies $T(Y) = Y$ and so $Y = \{x^*\}$. Thus, by Corollary 9.3.16. in [28] (page 559), we have that $(\hat{T}^n(Z))_{n\in\mathbb{N}}$ converges to $\{x^*\}$ as $n \to \infty$, for each $Z \in P_{cp}(X)$. Hence, $T^n(x) \stackrel{H}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$. □

The above theorem give rise to the following definition.

Definition 3.14. Let (X, d) be a metric space. Then, by definition, a multivalued operator $T : X \to P(X)$ satisfies Tarafdar-Yuan's alternative if

$$\text{either } F_{\hat{T}} = \emptyset \text{ or } F_{\hat{T}} = \{x^*\},$$

where $\hat{T}$ is the fractal operator generated by T, i.e., $\hat{T} : P(X) \to P(X)$ is given by $\hat{T}(Y) := \bigcup\limits_{x\in Y} T(x)$, for $Y \in P(X)$.

From the above results, we get the following interesting open question: find sufficient conditions for the existence of a set $Y \subset X$ such that $T(Y) = Y$ and $F_T \subset Y$. Notice that, from Tarafdar-Yuan's Theorem the core $\bigcap\limits_{n\geq 0} T^n(X)$ of the operator T is an example of such a set. For other examples and related rsults see Theorem 3.1 (vii) in A. Petruşel, I.A. Rus [15].

Another example comes from Martelli's Theorem, see [9].

Theorem 3.15 (Martelli's Theorem)**.** *Let X be a compact topological space and $T : X \to P_{cl}(X)$ be an u.s.c. multivalued operator. Then, there exists $Y \in P_{cl}(X)$ such that $T(Y) = Y$.*

By combining Tarafdar-Yuan's Theorem and Martelli's Theorem, we get the following result.

Theorem 3.16. *Let X be a Hausdorff compact topological space and $T : X \to P_{cl}(X)$ be an u.s.c. multivalued operator which satisfies Tarafdar-Yuan's alternative. Then T has a unique strict fixed point $x^* \in X$.*

Proof. By Martelli's Theorem there exists $Y \in P_{cl}(X)$ such that $T(Y) = Y$. By Tarafdar-Yuan's alternative, we obtain that $F_{\hat{T}} = \{x^*\}$, i.e., there exists a unique $x^* \in X$ such that $T(x^*) = \{x^*\}$. The proof is complete. □

Another result of this type was given by I.A. Rus in [22].

Theorem 3.17. *Let X be a complete metric space and $T : X \to P_b(X)$ be a multivalued (δ, φ)-contraction, i.e.,*

$$\delta(T(Y)) \leq \varphi(\delta(Y)), \textit{ for all } Y \in I(T),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function. If $T(X)$ is bounded, then $F_T = (SF)_T = \{x^*\}$.

It is an open problem to give, for the two above mentioned results, conditions which guarantee that $T^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$, i.e., to obtain other examples of multivalued Picard operators.

Other examples of Picard operators are given in what follows.

Theorem 3.18. *Let (X, d) be a complete metric space and $f_1, ..., f_m : X \to X$ be α_i-contractions, such that $F_{f_i} = \{x^*\}$ for each $i \in \{1, 2, ..., m\}$. Consider the multivalued operator $T : X \to P_{cp}(X)$ defined by*

$$T(x) = \{f_1(x), f_2(x), \cdots f_m(x)\}.$$

Then, T is a multivalued Picard operator.

Proof. Notice first that $x^* \in (SF)_T$. Moreover, $F_T = (SF)_T = \{x^*\}$. Indeed, if $y \in X$ is another fixed point of T with $y \neq x^*$, then $y \in T(y)$ implies there exists $j \in \{1, 2, \cdots, m\}$ such that $y = f_j(y)$, which is a contradiction with our assumptions. Since any self contraction on a complete metric space is a Picard operator, we immediately obtain that $T^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$. □

More generally, we have:

Theorem 3.19. *Let (X, d) be a metric space and $f_1, ..., f_m : X \to X$ be Picard operators, such that $F_{f_i} = \{x^*\}$ for each $i \in \{1, 2, ..., m\}$. Consider the multivalued operator $T : X \to P_{cp}(X)$ defined by*

$$T(x) = \{f_1(x), f_2(x), \cdots f_m(x)\}.$$

Then, T is a multivalued Picard operator.

Theorem 3.20. *Let (X, d) be a metric space and $T_1, ..., T_m : X \to P_{cp}(X)$ be multivalued Picard operators, such that $F_{T_i} = (SF)_{T_i} = \{x^*\}$ for each $i \in \{1, 2, ..., m\}$. Consider the multivalued operator $T : X \to P_{cp}(X)$ defined by*

$$T(x) = \bigcup_{i=1}^{m} T_i(x).$$

Then, T is a multivalued Picard operator.

We will present now the concept of multivalued ψ-Picard operators.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0) = 0$. By definition (see [13], [14]), a multivalued Picard operator $T : X \to P(X)$ is said to be:

(1) a multivalued (ψ, H_d)-Picard operator if and only if

$$d(x, x^*) \leq \psi(H_d(x, T(x))), \quad \text{for all } x \in X.$$

(2) a multivalued (ψ, D_d)-Picard operator if and only if

$$d(x, x^*) \leq \psi(D_d(x, T(x))), \quad \text{for all } x \in X.$$

Remark 3.21. Any multivalued (ψ, D_d)-Picard operator is a multivalued (ψ, H_d)-Picard operator.

Theorem 3.22. *Let (X, d) be a complete metric space and let $T : X \to P_{cp}(X)$ be a multivalued α-contraction, such that $(SF)_T \neq \emptyset$. Then T is a multivalued (ψ, D)-Picard operator with $\psi(t) := \frac{1}{1-\alpha}t$, $t \in \mathbb{R}_+$.*

Proof. By Theorem 3.7 we have that T is a multivalued Picard operator with $F_T = (SF)_T = \{x^*\}$. We also have:

$$d(x, x^*) = D(x, T(x^*)) \leq D(x, T(x)) + H(T(x), T(x^*)) \leq D(x, T(x)) + \alpha d(x, x^*).$$

Hence, we get that

$$d(x, x^*) \leq \frac{1}{1-\alpha} D(x, T(x)), \text{ for all } x \in X.$$

Theorem 3.23. *Let (X, d) be a complete metric space and let $T : X \to P_{cl}(X)$ be a multivalued ψ-weakly Picard operator, such that $F_T = (SF)_T = \{x^*\}$ and $T^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to +\infty$, for each $x \in X$. Then T is a multivalued (ψ, H)-Picard operator.*

Proof. Notice first that T is a multivalued Picard operator and $t^\infty(x, y) = x^*$, for each $(x, y) \in Graph(T)$. Since T is a multivalued ψ-weakly Picard operator, we also have that $d(x, x^*) \leq \psi(d(x, y))$, for each $x \in X$ and $y \in T(x)$. Hence, $d(x, x^*) \leq \psi(H(x, T(x)))$, for all $x \in X$. □

Theorem 3.24. *Let (X, d) be a complete metric space and let $T : X \to P_b(X)$ be a multivalued δ-contraction of Reich type with coefficients α, β, γ and $\alpha + 2\beta < 1$. Then T is a multivalued (ψ, H)-Picard operator with $\psi(t) := \frac{1+\beta}{1-\alpha}t$, $t \in \mathbb{R}_+$.*

Proof. By Theorem 3.1 we have that T is a multivalued Picard operator with $F_T = (SF)_T = \{x^*\}$. For $x \in X$, we also have:
$d(x, x^*) \leq \delta(x, T(x)) + \delta(T(x), x^*) = \delta(x, T(x)) + \delta(T(x), T(x^*)) \leq \delta(x, T(x)) + \alpha d(x, x^*) + \beta\delta(x, T(x)) + \gamma\delta(x^*, T(x^*)) = (1 + \beta)\delta(x, T(x)) + \alpha d(x, x^*)$. Thus,

$$d(x, x^*) \leq \frac{1+\beta}{1-\alpha} H(x, T(x)), \text{ for all } x \in X.$$

□

4. Data dependence of the (strict) fixed point set

We will present now some data dependence results for multivalued Picard operators.

The following result was proved in A. Petruşel, I.A. Rus [15].

Theorem 4.1. *Let (X, d) be a complete metric space and $T, S : X \to P_{cl}(X)$ be two multivalued operators. Suppose:*

(i) *T is a multivalued α-contraction;*
(ii) *$(SF)_T \neq \emptyset$;*
(iii) *$F_S \neq \emptyset$;*
(iv) *there exists $\eta > 0$ such that $H(S(x), T(x)) \leq \eta$, for each $x \in X$.*

Then

$$H(F_S, F_T) \leq \frac{\eta}{1-\alpha}.$$

Proof. Let $x^* \in (SF)_T$. Then, it is easy to observe (see the proof of Theorem 3.7) that $F_T = (SF)_T = \{x^*\}$. Let $y \in F_S$. Then

$$d(y,x^*) \le H(S(y),x^*) \le H(S(y),T(y)) + H(T(y),x^*) \le \eta + \alpha d(y,x^*).$$

Thus, $d(y,x^*) \le \frac{\eta}{1-\alpha}$. Since

$$H(F_S,F_T) = \sup_{y\in F_S} d(y,x^*) \le \frac{\eta}{1-\alpha},$$

we get the desired conclusion. □

As an immediate consequence of the above result, we have the following.

Theorem 4.2. *Let (X,d) be a complete metric space and let $T : X \to P_{cl}(X)$ be a multivalued α-contraction with $(SF)_T \neq \emptyset$. For each $n \in \mathbb{N}$, let $T_n : X \to P_{cl}(X)$, $n \in \mathbb{N}$ be a sequence of multivalued operators such that $F_{T_n} \neq \emptyset$ for each $n \in \mathbb{N}$ and $T_n(x) \overset{H_d}{\to} T(x)$ as $n \to +\infty$, uniformly with respect to $x \in X$.*

Then,

$$F_{T_n} \overset{H_d}{\to} F_T \text{ as } n \to +\infty.$$

More generally, we have the following abstract result.

Theorem 4.3. *Let (X,d) be a metric space and $T,S : X \to P(X)$ be two multivalued operators. Suppose:*

(i) *T is a multivalued (ψ, D_d)-Picard operator;*
(ii) *$S(x)$ is closed for each $x \in X$ and $F_S \neq \emptyset$;*
(iv) *there exists $\eta > 0$ such that $H(S(x),T(x)) \le \eta$, for each $x \in X$.*

Then

$$H(F_S,F_T) \le \psi(\eta).$$

Proof. Let $x^* \in X$ be the unique fixed (and also strict) point for T. Let $y \in F_S$. Then we have:
$d(y,x^*) \le \psi(D_d(y,T(y))) \le \psi(D_d(y,S(y)) + H(S(y),T(y))) = \psi(H(S(y),T(y)))$
$\le \psi(\eta)$. Thus,

$$H(F_S,F_T) = \sup_{y\in F_S} d(y,x^*) \le \psi(\eta).$$

□

Remark 4.4. In particular, if $T : X \to P_{cp}(X)$ is a multivalued α-contraction with $(SF)_T \neq \emptyset$, then (via Theorem 3.22) Theorem 4.3 reduces to Theorem 4.1.

As a consequence of the above theorem, we have the following.

Theorem 4.5. *Let (X,d) be a complete metric space and let $T : X \to P_{cl}(X)$ be a multivalued (ψ, D_d)-Picard operator. For each $n \in \mathbb{N}$, let $T_n : X \to P_{cl}(X)$ be a sequence of multivalued operators such that $F_{T_n} \neq \emptyset$ for each $n \in \mathbb{N}$ and $T_n(x) \overset{H_d}{\to} T(x)$ as $n \to +\infty$, uniformly with respect to $x \in X$.*

Then,

$$F_{T_n} \overset{H_d}{\to} F_T \text{ as } n \to +\infty.$$

Proof. Let $\epsilon > 0$. Then, there exists $n_\epsilon \in \mathbb{N}$, such that

$$H(T_n(x), T(x)) < \epsilon, \text{ for all } n \geq n_\epsilon \text{ and each } x \in X.$$

Then, by Theorem 4.3, we get that

$$H(F_{T_n}, F_T) \leq \psi(\epsilon), \text{ for all } n \geq n_\epsilon.$$

This completes the proof. □

Another data dependence result is the following.

Theorem 4.6. *Let (X, d) be a metric space and $T, S : X \to P(X)$ be two multivalued operators. Suppose:*

(i) *T is a multivalued (ψ, H_d)-Picard operator;*

(ii) *$(SF)_S \neq \emptyset$;*

(iv) *there exists $\eta > 0$ such that $H(S(x), T(x)) \leq \eta$, for each $x \in X$.*

Then

$$H((SF)_S, (SF)_T) \leq \psi(\eta).$$

Proof. Let $x^* \in X$ be the unique fixed (and also strict) point for T. Let $y \in (SF)_S$. Then we have $d(y, x^*) \leq \psi(H_d(y, T(y))) = \psi(H_d(S(y), T(y))) \leq \psi(\eta)$. Thus,

$$H((SF)_S, (SF)_T) = \sup_{y \in (SF)_S} d(y, x^*) \leq \psi(\eta).$$

□

5. Well-posedness of the (strict) fixed point problem

We will devote this section to the study of the well-posedness of (strict) fixed point problems. Firstly, we recall some definitions of a well-posed (strict) fixed point problem.

Definition 5.1. Let (X, d) be a metric space, $Y \in P(X)$ and $T : Y \to P_{cl}(X)$ be a multivalued operator. The fixed point problem is well-posed for T with respect to D_d if:

(a_1) $F_T = \{x^*\}$

(b_1) If $(x_n)_{n\in\mathbb{N}} \subset Y$ and $D_d(x_n, T(x_n)) \to 0$ as $n \to +\infty$, then $x_n \to x^*$ as $n \to +\infty$.

Definition 5.2. Let (X, d) be a metric space, $Y \in P(X)$ and $T : Y \to P_{cl}(X)$ be a multivalued operator. The fixed point problem is well-posed for T with respect to H_d if:

(a_2) $(SF)_T = \{x^*\}$

(b_2) If $(x_n)_{n\in\mathbb{N}} \subset Y$ and $H_d(x_n, T(x_n)) \to 0$ as $n \to +\infty$, then $x_n \to x^*$ as $n \to +\infty$.

Remark 5.3. Notice that (b_1) implies (b_2) and (a_1) implies (a_2). Moreover, if $F_T = (SF)_T = \{x^*\}$ then the well-posedness of the fixed point problem for T with respect to D_d implies the well-posedness of the fixed point problem for T with respect to H_d.

We have the following abstract results.

Theorem 5.4. *Let* (X, d) *be a complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued operator such that* $F_T = \{x^*\}$. *Suppose there exists a function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* ψ *is increasing, continuous in* 0 *with* $\psi(0) = 0$ *and, for each* $x \in X$, *we have* $d(x, x^*) \leq \psi(D_d(x, T(x)))$. *Then, the fixed point problem is well-posed for* T *with respect to* D_d. *In particular, if* T *is a multivalued* (ψ, D_d)*-Picard operator, then the fixed point problem for* T *is well-posed with respect to* D_d.

Proof. Let $x_n \in X$, $n \in \mathbb{N}$ such that $D(x_n, T(x_n)) \to 0$, as $n \to +\infty$. Then $d(x_n, x^*) \leq \psi(D(x_n, T(x_n))) \to 0$, as $n \to +\infty$. □

Theorem 5.5. *Let* (X, d) *be a complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued operator such that* $(SF)_T = \{x^*\}$. *Suppose there exists a function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* ψ *is increasing, continuous in* 0 *and* $\psi(0) = 0$ *and, for each* $x \in X$, *we have* $d(x, x^*) \leq \psi(H(x, T(x)))$. *Then the fixed point problem is well-posed for* T *with respect to* H_d. *In particular, if* T *is a multivalued* (ψ, H_d)*-Picard operator, then the fixed point problem for* T *is well-posed with respect to* H_d.

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence such that $H(x_n, T(x_n)) \to 0$ as $n \to +\infty$. Then $d(x_n, x^*) \leq \psi(H(x_n, T(x_n))) \to 0$ as $n \to +\infty$. □

6. Ulam-Hyers stability of the (strict) fixed point problem

We start this section by presenting the concept of (generalized) Ulam-Hyers stability for the (strict) fixed point problem.

Definition 6.1. Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued operator. The strict fixed point inclusion

$$\{x\} = T(x),\ x \in X \tag{1}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in X$ of the inequation

$$H(y, T(y)) \leq \varepsilon \tag{2}$$

there exists a solution $x^* \in X$ of the strict fixed point inclusion (1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the strict fixed point inclusion (1) is said to be Ulam-Hyers stable.

The following theorem is an abstract result concerning the Ulam-Hyers stability of the strict fixed point inclusion (1) for multivalued operators with closed values.

Theorem 6.2. *Let* (X, d) *be a metric space and* $T : X \to P_{cl}(X)$ *be a multivalued* (ψ, H_d)*-Picard operator. Then, the strict fixed point inclusion (1) is generalized Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and $y^* \in X$ be a solution of (2), i.e., $H(y^*, T(y^*)) \leq \varepsilon$. Since T is a multivalued (ψ, H_d)-Picard operator, we have

$$d(x, x^*) \leq \psi(H(x, T(x))),\quad \text{for all } x \in X.$$

Hence, $d(y^*, x^*) \leq \psi(H(y^*, T(y^*))) \leq \psi(\varepsilon)$. □

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