Journal of Nonlinear and Convex Analysis
Volume 13, Number 1, 2012, 141–156

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

GENERALIZED RETRACTION AND FIXED POINT THEOREMS USING BREGMAN FUNCTIONS IN BANACH SPACES

ESKANDAR NARAGHIRAD*, WATARU TAKAHASHI, AND JEN-CHIH YAO

Abstract. In this paper, using Bregman functions, we first study Bregman generalized nonexpansive retracts in Banach spaces and give a characterization of sunny Bregman generalized nonexpansive retracts. Furthermore, we prove fixed point and convergence theorems for Bregman generalized nonexpansive type mappings in Banach spaces.

1. Introduction

Let E be a smooth Banach space and let J be the normalized duality mapping of E. A mapping $T : E \to E$ is said to be of *generalized nonexpansive type* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx), \ \forall x, y \in E,$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. Recently, Ibaraki and Takahashi [5] proved the following fixed point theorem for generalized nonexpansive type mappings in a Banach space.

Theorem 1.1. *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive type mapping of E into itself. Then the following are equivalent:*

(1) *The set $F(T)$ of fixed points of T is nonempty;*
(2) *$\{T^n x\}$ is bounded for some $x \in E$.*

Let E be a Banach space and let E^* be the dual space of E. Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let $g : E \to \mathbb{R}$ be a convex function. Then the *directional derivative* $d^+g(x)(y)$ of g at $x \in E$ with the direction $y \in E$ is defined by

$$d^+g(x)(y) = \lim_{t \downarrow 0} \frac{g(x + ty) - g(x)}{t}. \tag{1.1}$$

The function g is said to be *Gâteaux differentiable* at x if $d^+g(x) \in E^*$ (see, for example, [2, p. 12] or [11, p. 508]). In this case, we denote $d^+g(x)$ by $\nabla g(x)$. A convex function $g : E \to \mathbb{R}$ is said to be *Gâteaux differentiable* if it is Gâteaux differentiable everywhere. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable

---

2010 *Mathematics Subject Classification.* 47H10, 37C25.

Key words and phrases. Bregman generalized nonexpansive retraction, Bregman function, uniformly convex function, uniformly smooth function, fixed point.

*Correspondence author. The work of E. Naraghirad was conducted with a postdoctoral fellowship at the National Sun Yat-Sen University, Kaohsiung, 804, Taiwan. The work of W. Takahashi was partially supported by Grant-in-Aid for Scientific Research No. 23540188 from Japan Society for the Promotion of Science. The work of J.-C. Yao was partially supported by the Grant NSC 99-2115-M-037-002-MY3.

function. Then the *Bregman distance* [1, 3] corresponding to g is the function $D : E \times E \to \mathbb{R}$ defined by

$$D(x,y) = g(x) - g(y) - \langle x - y, \nabla g(y) \rangle, \ \forall x, y \in E. \tag{1.2}$$

It is clear that $D(x,y) \geq 0$ for all $x, y \in E$. In the case when E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all $x \in E$, we have that $\nabla g(x) = 2Jx$ for all $x \in E$ and hence

$$\begin{aligned} D(x,y) &= \|x\|^2 - \|y\|^2 - \langle x - y, \nabla g(y) \rangle \\ &= \|x\|^2 - \|y\|^2 - \langle x - y, 2Jy \rangle \\ &= \|x\|^2 - \|y\|^2 - \langle x, 2Jy \rangle + 2\|y\|^2 \\ &= \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \\ &= \phi(x,y) \end{aligned}$$

for all $x, y \in E$. Let C be a nonempty subset of E. A mapping $T : C \to C$ is said to be of *Bregman generalized nonexpansive type* if

$$D(Tx, Ty) + D(Ty, Tx) \leq D(x, Ty) + D(y, Tx), \ \forall x, y \in C. \tag{1.3}$$

In this paper, we first study Bregman generalized nonexpansive retracts in Banach spaces and give a characterization of sunny Bregman generalized nonexpansive retracts. Furthermore, we generalize the fixed point theorems for generalized nonexpansive type mappings in [5] with Bregman functions in reflexive Banach spaces. We prove fixed point and convergence theorems for Bregman generalized nonexpansive type mappings in Banach spaces.

2. Preliminaries

Let E be a Banach space with the norm $\|.\|$ and the dual space E^*. For any $x \in E$, we denote the value of $x^* \in E^*$ at x by $\langle x, x^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. For any sequence $\{x_n^*\}$ in E^*, we denote the strong convergence of $\{x_n^*\}$ to $x^* \in E^*$ by $x_n^* \to x^*$, the weak convergence by $x_n^* \rightharpoonup x^*$ and the weak* convergence by $x_n^* \rightharpoonup^* x^*$. The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called *smooth*. If the limit (2.1) is attained uniformly in $x, y \in S$, then E is called *uniformly smooth*. The Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S$ and $x \neq y$. It is well-known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [17].

Let $T : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of T by $D(T) = \{x \in E : Tx \neq \emptyset\}$ and $R(T) = \cup_{x \in E} Tx$, respectively. The graph of T

is denoted by $G(T) = \{(x, x^*) \in E \times E^* : x^* \in Tx\}$. The mapping $T \subset E \times E^*$ is said to be *monotone* [13] if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in T$. It is also said to be *maximal monotone* [16] if its graph is not contained in the graph of any other monotone operator on E. If $T \subset E \times E^*$ is maximal monotone, then we can show that the set $T^{-1}0 = \{z \in E : 0 \in Tz\}$ is closed and convex. A function $f : E \to (-\infty, +\infty]$ is said to be *proper* if the domain $D(f) = \{x \in E : f(x) < \infty\}$ is nonempty. It is also called *lower semicontinuous* if $\{x \in E : f(x) \leq r\}$ is closed for all $r \in \mathbb{R}$. We say that f is *upper semicontinuous* if $\{x \in E : f(x) \geq r\}$ is closed for all $r \in \mathbb{R}$. The function f is said to be *convex* if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \tag{2.2}$$

for all $x, y \in E$ and $\alpha \in (0, 1)$. It is also said to be *strictly convex* if the strict inequality holds in (2.2) for all $x, y \in D(f)$ with $x \neq y$ and $\alpha \in (0, 1)$. For a proper lower semicontinuous convex function $f : E \to (-\infty, +\infty]$, the *subdifferential* ∂f of f is defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y), \ \forall y \in E\} \tag{2.3}$$

for all $x \in E$. It is well known that $\partial f \subset E \times E^*$ is maximal monotone [14, 15]. For any proper lower semicontinuous convex function $f : E \to (-\infty, +\infty]$, the *conjugate function* f^* of f is defined by

$$f^*(x^*) = \sup_{x \in E}\{\langle x, x^* \rangle - f(x)\}$$

for all $x^* \in E^*$. It is well known that $f(x) + f^*(x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in E \times E^*$. It is also known that $(x, x^*) \in \partial f$ is equivalent to

$$f(x) + f^*(x^*) = \langle x, x^* \rangle. \tag{2.4}$$

We also know that if $f : E \to (-\infty, +\infty]$ is a proper lower semicontinuous convex function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous convex function; see [18] for more details on convex analysis. Let $g : E \to \mathbb{R}$ be a convex function. The function g is also said to be *Fréchet differentiable* at $x \in E$ (see, for example, [2, p. 13] or [11, p. 508]) if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|y - x\| \leq \delta$ implies that

$$| \, g(y) - g(x) - \langle y - x, \nabla g(x) \rangle \, | \leq \epsilon \|y - x\|.$$

A convex function $g : E \to \mathbb{R}$ is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere. It is well known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, for example, [2, Proposition 1.1.10]). Also, it is known that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous (see, [11, p. 508]). The mapping ∇g is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ implies that $\nabla g(x_n) \rightharpoonup^* \nabla g(x)$ (for more details, see [2, Theorem 3.2.4] or [11, p. 508]). The function $g : E \to \mathbb{R}$ is said to be *strongly coercive* if

$$\|x_n\| \to \infty \Longrightarrow \frac{g(x_n)}{\|x_n\|} \to \infty.$$

It is also said to be *bounded on bounded sets* if $g(U)$ is bounded for each bounded subset U of E. The following definition is slightly different from that in Butnariu and Iusem [2].

Definition 2.1 ([11]). Let E be a Banach space. The function $g : E \to \mathbb{R}$ is said to be a *Bregman function* if the following conditions are satisfied:

(1) g is continuous, strictly convex and Gâteaux differentiable;
(2) the set $\{y \in E : D(x, y) \le r\}$ is bounded for all $x \in E$ and $r > 0$.

The following lemma follows from Butnariu and Iusem [2] and Zălinscu [19]:

Lemma 2.2. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then*

(1) $\nabla g : E \to E^*$ *is one-to-one, onto and norm-to-weak continuous;*
(2) $\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ *if and only if* $x = y$*;*
(3) $\{x \in E : D(x, y) \le r\}$ *is bounded for all* $y \in E$ *and* $r > 0$*;*
(4) $D(g^*) = E^*$*,* g^* *is Gâteaux differentiable and* $\nabla g^* = (\nabla g)^{-1}$*.*

Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ be a strongly coercive Bregman function and let $D : E \times E \to \mathbb{R}$ be the Bregman distance corresponding to g. Then, $g^* : E^* \to \mathbb{R}$ is convex and Gâteaux differentiable [19]. Let $D_* : E^* \times E^* \to \mathbb{R}$ be the function defined by

$$D_*(x^*, y^*) = g^*(x^*) - g^*(y^*) - \langle \nabla g^*(y^*), x^* - y^*\rangle \tag{2.5}$$

for $x^*, y^* \in E^*$, where ∇g^* is the directional derivative of g^*. It follows from (2.2)-(2.5) and Lemma 2.2 (4) that

$$\begin{aligned} D_*(\nabla g(x), \nabla g(y)) &= g^*(\nabla g(x)) - g^*(\nabla g(y)) - \langle \nabla g^*(\nabla g(y)), \nabla g(x) - \nabla g(y)\rangle \\ &= g^*(\nabla g(x)) - g^*(\nabla g(y)) - \langle y, \nabla g(x) - \nabla g(y)\rangle \\ &= [\langle x, \nabla g(x)\rangle - g(x)] - [\langle y, \nabla g(y)\rangle - g(y)] \\ &\quad -\langle y, \nabla g(x) - \nabla g(y)\rangle \\ &= \langle x, \nabla g(x)\rangle - g(x) - \langle y, \nabla g(y)\rangle + g(y) \\ &\quad -\langle y, \nabla g(x)\rangle + \langle y, \nabla g(y)\rangle \\ &= g(y) - g(x) - \langle y - x, \nabla g(x)\rangle \\ &= D(y, x) \end{aligned} \tag{2.6}$$

for all $x, y \in E$. Let E be a Banach space and let C be a nonempty convex subset of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then, for $x \in E$ and $x_0 \in C$, $D(x_0, x) = \min_{y \in C} D(y, x)$ if and only if

$$\langle y - x_0, \nabla g(x) - \nabla g(x_0)\rangle \le 0, \ \forall y \in C. \tag{2.7}$$

Let us show ($\Longrightarrow$). For any $z \in C$ and $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$, put $y = (1-\lambda)x_0 + \lambda z$. Then, we have that

$$\begin{aligned} D(x_0, x) &\le D((1 - \lambda)x_0 + \lambda z, x) \\ &\Longleftrightarrow g(x_0) - g(x) - \langle x_0 - x, \nabla g(x)\rangle \\ &\le g((1 - \lambda)x_0 + \lambda z) - g(x) - \langle (1 - \lambda)x_0 + \lambda z - x, \nabla g(x)\rangle \\ &\Longleftrightarrow 0 \le g(x_0 + \lambda(z - x_0)) - g(x_0) - \lambda\langle z - x_0, \nabla g(x)\rangle \\ &\Longleftrightarrow 0 \le \tfrac{g(x_0+\lambda(z-x_0))-g(x_0)}{\lambda} - \langle z - x_0, \nabla g(x)\rangle. \end{aligned}$$

Letting $\lambda \downarrow 0$, we have that

$$0 \leq \langle z - x_0, \nabla g(x_0) \rangle - \langle z - x_0, \nabla g(x) \rangle$$

and hence $0 \leq \langle z - x_0, \nabla g(x_0) - \nabla g(x) \rangle$. This implies (2.7). Further, if C is a nonempty closed convex subset of a reflexive Banach space E and $g : E \to \mathbb{R}$ is a strongly coercive Bregman function, then for each $x \in E$, there exists a unique $x_0 \in C$ such that

$$D(x_0, x) = \min_{y \in C} D(y, x).$$

The *Bregman projection* P_C from E onto C is defined by $P_C(x) = x_0$ for all $x \in E$. It is also well known that P_C has the following property:

$$D(y, P_C x) + D(P_C x, x) \leq D(y, x) \tag{2.8}$$

for all $y \in C$ and $x \in E$ (see [2] for more details). Let E be a Banach space and B be the unit ball of E. Let $rB := \{z \in E : \|z\| \leq r\}$ for all $r > 0$. Then a function $g : E \to \mathbb{R}$ is said to be *uniformly convex on bounded sets* ([19, pp. 203, 221]) if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_r(t) = \inf_{x,y \in rB, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha) g(y) - g(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)}$$

for all $t \geq 0$. The function g is also said to be *uniformly smooth on bounded sets* ([19, pp. 207, 221]) if $\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0$ for all $r > 0$, where $\sigma_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_r(t) = \sup_{x \in rB, y \in S, \alpha \in (0,1)} \frac{\alpha g(x + (1-\alpha) t y) + (1-\alpha) g(x - \alpha t y) - g(x)}{\alpha(1-\alpha)}$$

for all $t \geq 0$. We know the following results; see [19, Proposition 3.6.4].

Theorem 2.3. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex function which is bounded on bounded sets. Then the following assertions are equivalent:*

(1) *g is strongly coercive and uniformly convex on bounded sets;*
(2) *$D(g^*) = E^*$, g^* is bounded on bounded sets and uniformly smooth on bounded sets;*
(3) *$D(g^*) = E^*$, g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded sets.*

Theorem 2.4. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*

(1) *g is bounded on bounded sets and uniformly smooth on bounded sets;*
(2) *g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded sets;*
(3) *$D(g^*) = E^*$, g^* is strongly coercive and uniformly convex on bounded sets.*

The following lemma has been proved in [11].

Lemma 2.5. *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function which is uniformly convex on bounded sets. If $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E such that $\lim_{n\to\infty} D(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Generalized retractions and Bregman functions

Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the Bregman distance [1, 3] satisfies that

$$D(x, z) = D(x, y) + D(y, z) + \langle x - y, \nabla g(y) - \nabla g(z)\rangle, \ \forall x, y, z \in E. \tag{3.1}$$

Let C be a nonempty and closed subset of E. A mapping $T : C \to C$ is called *Bregman firmly generalized nonexpansive* [9] if $F(T) \neq \emptyset$ and

$$D(x, Tx) + D(Tx, p) \leq D(x, p) \tag{3.2}$$

for each $x \in C$ and $p \in F(T)$. A mapping $T : C \to C$ is called *Bregman generalized nonexpansive* if $F(T) \neq \emptyset$ and

$$D(Tx, p) \leq D(x, p), \ \forall (x, p) \in C \times F(T). \tag{3.3}$$

A mapping $T : C \to C$ is of *Bregman generalized nonexpansive type* if

$$D(Tx, Ty) + D(Ty, Tx) \leq D(x, Ty) + D(y, Tx), \ \forall x, y \in C. \tag{3.4}$$

A mapping $T : C \to C$ is of *Bregman firmly generalized nonexpansive type* if

$$\begin{aligned} D(x, Tx) + D(y, Ty) \quad & +D(Tx, Ty) + D(Ty, Tx) \\ & \leq D(x, Ty) + D(y, Tx), \ \forall x, y \in C. \end{aligned} \tag{3.5}$$

It is clear that a Bregman firmly generalized noneaxpansive mapping is Bregman generalized nonexpansive in a Banach space (see also [9]). Let C be a nonempty subset of Banach space E. A mapping $R : E \to C$ is said to be *sunny* if

$$R(Rx + t(x - Rx)) = Rx$$

for each $x \in E$ and $t \geq 0$. A mapping $R : E \to C$ is said to be a *retraction* if $Rx = x$ for each $x \in C$. We have the following results for Bregman firmly generalized nonexpansive type mappings.

Lemma 3.1. *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Let C be a nonempty closed subset of E. If $T : C \to C$ is a Bregman firmly generalized nonexpansive type mapping with $F(T) \neq \emptyset$, then T is Bregman firmly generalized nonexpansive.*

Lemma 3.2. *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Let C be a closed subset of E. Then, a mapping $T : C \to C$ is of Bregman firmly generalized nonexpansive type if and only if*

$$\langle (x - Tx) - (y - Ty), \nabla gTx - \nabla gTy\rangle \geq 0, \ \forall x, y \in C.$$

Using ideas in [6], we can also prove the following result.

Lemma 3.3. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Let C be a nonempty closed subset of E and let R be a retraction from E onto C. Then the following assertions are equivalent:*

(1) *R is sunny and Bregman generalized nonexpansive;*
(2) $\langle x - Rx, \nabla g(y) - \nabla g(Rx)\rangle \le 0$, $\forall (x, y) \in E \times C$.

Furthermore, a sunny and Bregman generalized nonexpansive retraction of E onto C is uniquely determined.

Proof. ($\Longrightarrow$) Let R be a sunny and Bregman generalized nonexpansive retraction of E onto C. Let $x \in E$ and let $y \in C = F(R)$. Putting $x_t = Rx + t(x - Rx)$ for all $t \in [0, 1]$, we have that $D(Rx, y) = D(Rx_t, y) \le D(x_t, y)$ and hence

$$D(Rx, y) = \min_{z \in [x, Rx]} D(z, y),$$

where $[x, Rx]$ is the closed line segment joining x and Rx. Using (2.7), we have that

$$0 \le \langle x_t - Rx, \nabla g(Rx) - \nabla g(y)\rangle, \ \forall t \in [0, 1].$$

Putting $t = 1$, we have $0 \le \langle x - Rx, \nabla g(Rx) - \nabla g(y)\rangle$ for all $y \in C$.

($\Longleftarrow$) Let $x \in E$ and let $y \in C = F(R)$. Then, we have from (3.1) that

$$D(x, y) = D(x, Rx) + D(Rx, y) + \langle x - Rx, \nabla g(Rx) - \nabla g(y)\rangle.$$

From the assumption $\langle x - Rx, \nabla g(Rx) - \nabla g(y)\rangle \ge 0$, we have

$$D(x, y) \ge D(x, Rx) + D(Rx, y) \ge D(Rx, y).$$

This implies that R is Bregman generalized nonexpansive. Let us show that R is sunny. Putting $x_t = Rx + t(x - Rx)$ for $x \in E$ and $t > 0$, we have

$$\langle x_t - Rx_t, \nabla g(Rx_t) - \nabla g(Rx)\rangle \ge 0 \text{ and } \langle x - Rx, \nabla g(Rx) - \nabla g(Rx_t)\rangle \ge 0.$$

From $x_t - Rx = t(x - Rx)$, we have

$$\langle x - Rx_t, \nabla g(Rx) - \nabla g(Rx_t)\rangle = t\langle x - Rx, \nabla g(Rx) - \nabla g(Rx_t)\rangle \ge 0$$

and hence $\langle Rx - Rx_t, \nabla g(Rx_t) - \nabla g(Rx)\rangle \ge 0$. This implies that

$$\langle Rx - Rx_t, \nabla g(Rx_t) - \nabla g(Rx)\rangle = 0.$$

Thus, we have from Lemma 2.2 (2) that $Rx = Rx_t = R(Rx + t(x - Rx))$, that is, R is sunny.

Next, we show that a sunny and Bregman generalized nonexpansive retraction is unique. Let R and P be sunny and Bregman generalized nonexpansive retractions of E onto C. Then, we have

$$\langle x - Rx, \nabla g(Px) - \nabla g(Rx)\rangle \le 0 \text{ and } \langle x - Px, \nabla g(Rx) - \nabla g(Px)\rangle \le 0.$$

Thus, we have $\langle Px - Rx, \nabla g(Px) - \nabla g(Rx)\rangle \le 0$ and hence $\langle Px - Rx, \nabla g(Px) - \nabla g(Rx)\rangle = 0$. Then, we have from Lemma 2.2 (2) that $Rx = Px$ for all $x \in E$. □

Using Lemma 3.3, we can prove the following result.

Lemma 3.4. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Let C be a nonempty closed subset of E and let R be a sunny Bregman generalized nonexpansive retraction from E onto C. Let $(x, z) \in E \times C$. Then the following assertions hold:*

(1) *$z = Rx$ if and only if $\langle x - z, \nabla g(y) - \nabla g(z)\rangle \le 0$ for all $y \in C$;*
(2) *$D(Rx, z) + D(x, Rx) \le D(x, z)$.*

Using the techniques developed by Kohsaka and Takahashi [12], we prove the following lemma.

Lemma 3.5. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded sets and uniformly convex on bounded sets. Let C be a nonempty closed Bregman generalized nonexpansive retract of E. Then ∇gC is closed and convex.*

Proof. Let R be a Bregman generalized nonexpansive retraction from E onto C. Since R is a retraction from E onto C, we have $F(R) = C$. We first show that ∇gC is convex. In view of Lemma 2.2 (4), we have $\nabla g^* = (\nabla g)^{-1}$, where g^* is the conjugate function of g. Let x^* and y^* be arbitrary elements of ∇gC, let $\alpha \in (0, 1)$ and put $\beta = 1 - \alpha$. Then we have $x, y \in C$ such that $x^* = \nabla g(x)$ and $y^* = \nabla g(y)$. Since $x, y \in C = F(R)$ and R is Bregman generalized nonexpansive, we have

$$\begin{aligned}
&D(R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), \nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)))\\
&= g(R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y))) - g(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)))\\
&\ -\langle R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)) - \nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), \alpha\nabla g(x) + \beta\nabla g(y)\rangle\\
&= g(R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y))) - g(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)))\\
&\ -[\langle R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), \alpha\nabla g(x)\rangle + \langle R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), \beta\nabla g(y)\rangle\\
&\ -\langle\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), \alpha\nabla g(x)\rangle - \langle\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), \beta\nabla g(y)\rangle]\\
&= g(R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y))) - g(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)))\\
&\ -[\langle R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)) - x, \alpha\nabla g(x)\rangle\\
&\ +\langle x, \alpha\nabla g(x)\rangle + \langle R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)) - y, \beta\nabla g(y)\rangle + \langle y, \beta\nabla g(y)\rangle\\
&\ -\langle\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), \alpha\nabla g(x)\rangle - \langle\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), \beta\nabla g(y)\rangle]\\
&= \alpha g(R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y))) + \beta g(R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)))\\
&\ -\alpha g(x) + \alpha g(x) - \beta g(y) + \beta g(y)\\
&\ -\alpha g(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y))) - \beta g(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)))\\
&\ -[\langle R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)) - x, \alpha\nabla g(x)\rangle\\
&\ +\langle x, \alpha\nabla g(x)\rangle + \langle R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)) - y, \beta\nabla g(y)\rangle + \langle y, \beta\nabla g(y)\rangle\\
&\ -\langle\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), \alpha\nabla g(x)\rangle - \langle\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), \beta\nabla g(y)\rangle]\\
&= \alpha[g(R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y))) - g(x) - \langle R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)) - x, \nabla g(x)\rangle]\\
&\ +\beta[g(R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y))) - g(y) - \langle R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)) - y, \nabla g(y)\rangle]\\
&\ -\alpha[g(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y))) - g(x) - \langle\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)) - x, \nabla g(x)\rangle]\\
&\ -\beta[g(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y))) - g(y) - \langle\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)) - y, \nabla g(y)\rangle]\\
&= \alpha D(R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), x) + \beta D(R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), y)\\
&\ -\alpha D(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), x) - \beta D(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), y)\\
&\le \alpha D(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), x) + \beta D(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), y)\\
&\ -\alpha D(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), x) - \beta D(\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), y)\\
&= 0.
\end{aligned}$$

Thus, we conclude that

$$D(R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)), \nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y))) = 0.$$

It follows from Lemma 2.5 that

$$R\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y) = \nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)).$$

Therefore, we obtain $\nabla g^*(\alpha\nabla g(x) + \beta\nabla g(y)) \in C$ and hence

$$\alpha x^* + \beta y^* = \alpha\nabla g(x) + \beta\nabla g(y) \in \nabla gC.$$

This proves that ∇gC is convex.

We next show that ∇gC is closed. Let $\{x_n^*\}$ be a sequence of ∇gC converging strongly to $x^* \in E^*$. Then there exist $x \in E$ and $x_n \in C$ such that $x^* = \nabla g(x)$ and $x_n^* = \nabla g(x_n)$ for all $n \in \mathbb{N}$. This implies that

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(x)\| = 0. \tag{3.6}$$

From Theorem 2.3, we have $\lim_{n\to\infty} \|x_n - x\| = 0$ and hence $\{x_n\}$ is bounded. Since R is Bregman generalized nonexpansive, it follows from (3.3) and (3.6) that

$$\begin{aligned}
D(Rx, x_n) &\leq D(x, x_n) \\
&= g(x) - g(x_n) - \langle x - x_n, \nabla g(x_n)\rangle \\
&= g(x) - g(x_n) - \langle x - x_n, \nabla g(x)\rangle - \langle x - x_n, \nabla g(x_n) - \nabla g(x)\rangle \\
&\leq g(x) - g(x_n) - \langle x - x_n, \nabla g(x)\rangle + \|x - x_n\|\|\nabla g(x_n) - \nabla g(x)\| \\
&\to g(x) - g(x) - \langle x - x, \nabla g(x)\rangle = 0
\end{aligned}$$

as $n \to \infty$. Thus we have $\lim_{n\to\infty} D(Rx, x_n) = 0$ and $\lim_{n\to\infty} D(x, x_n) = 0$. On the other hand, since $\{x_n\}$ is bounded, we have from Lemma 2.5 that $\lim_{n\to\infty} \|Rx_n - x\| = 0$ and $\lim_{n\to\infty} \|x_n - x\| = 0$. Thus, we conclude that $Rx = x$ and hence $x \in C$. Since R is a retraction of E onto C, we have $x^* = \nabla g(x) = \nabla g(Rx) \in \nabla gC$. Thus ∇gC is closed, which completes the proof. □

The following result can be derived from Lemmas 3.3, 3.4 and 3.5.

Lemma 3.6. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous and strongly coercive function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let C_* be a nonempty closed convex subset of E^* and let P_{C_*} be the Bregman projection from E^* onto C_*. Then the mapping R defined by $R = (\nabla g)^{-1} P_{C_*} \nabla g$ is a sunny Bregman generalized nonexpansive retraction from E onto $(\nabla g)^{-1} C_*$.*

Proof. We first prove that $(\nabla g)^{-1} C_*$ is closed. Let $\{x_n\}$ be a sequence in $\nabla g^* C_*$ such that $x_n \to x$. Then, we have $\nabla g(x_n) \in C_*$. Since ∇g is continuous, we have $\nabla g(x_n) \to \nabla g(x)$ and hence $\nabla g(x) \in C_*$. So, we have $x \in (\nabla g)^{-1} C_*$. Thus, $(\nabla g)^{-1} C_*$ is closed. If $x \in E$, then we have

$$R(x) = (\nabla g)^{-1} P_{C_*} \nabla g(x) \in (\nabla g)^{-1} P_{C_*} E^* = (\nabla g)^{-1} C_*$$

and hence R is a mapping of E into $(\nabla g)^{-1} C_*$. Furthermore, for any $x \in (\nabla g)^{-1} C_*$ we have $\nabla g(x) \in C_*$ and hence $P_{C_*} \nabla g(x) = \nabla g(x)$. Thus, we have

$$Rx = (\nabla g)^{-1} P_{C_*} \nabla g(x) = (\nabla g)^{-1} \nabla g(x) = x.$$

Then, R is onto and $Rx = x$ for all $x \in (\nabla g)^{-1} C_*$. It is obvious that

$$R^2 x = R(Rx) = Rx = x$$

for all $x \in E$ and hence R is a retraction. We finally show that R is sunny and Bregman generalized nonexpansive. Since R is a retraction of E onto $(\nabla g)^{-1} C_*$, we have $F(R) = (\nabla g)^{-1} C_*$. Thus $F(R)$ is nonempty. On the other hand, we know from (2.8) that

$$D_*(y^*, P_{C_*} x^*) + D_*(P_{C_*} x^*, x^*) \leq D_*(y^*, x^*)$$

for all $(x^*, y^*) \in E^* \times C_*$, which is equivalent to

$$D_*(\nabla g(y), P_{C_*}\nabla g(x)) + D_*(P_{C_*}\nabla g(x), \nabla g(x)) \leq D_*(\nabla g(y), \nabla g(x))$$

for all $(x, y) \in E \times (\nabla g)^{-1}C_*$. Thus, we have

$$D(Rx, y) + D(x, Rx) \leq D(x, y)$$

for all $(x, y) \in E \times (\nabla g)^{-1}C_*$. Then, we have that for all $(x, y) \in E \times (\nabla g)^{-1}C_*$,

$$\begin{aligned} 0 &\leq D(x, y) - \{D(Rx, y) + D(x, Rx)\} \\ &= g(x) - g(y) - \langle x - y, \nabla g(y)\rangle - \{g(Rx) - g(y) - \langle Rx - y, \nabla g(y)\rangle\} \\ &\quad -\{g(x) - g(Rx) - \langle x - Rx, \nabla g(Rx)\rangle\} \\ &= \langle Rx, \nabla g(y)\rangle - \langle x, \nabla g(y)\rangle + \langle x, \nabla g(Rx)\rangle\} - \langle Rx, \nabla g(Rx)\rangle \\ &= \langle x - Rx, \nabla g(Rx) - \nabla g(y)\rangle. \end{aligned}$$

By Lemma 3.3 we have that R is sunny and Bregman generalized nonexpansive. This completes the proof. □

Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Let C be a nonempty closed subset of E. We know from Lemma 3.3 that a sunny Bregman generalized nonexpansive retraction of E onto C is uniquely determined. Then, such a sunny Bregman generalized nonexpansive retraction of E onto C is denoted by R_C. A nonempty subset C of E is said to be a *sunny Bregman generalized nonexpansive retract* (resp. a *Bregman generalized nonexpansive retract*) of E if there exists a sunny Bregman generalized nonexpansive retraction (resp. a Bregamn generalized nonexpansive retraction) of E onto C. The set of all fixed points of such a sunny Bregman generalized nonexpansive retraction of E onto C is, of course, C. We obtain the following result by using Lemmas 2.2 (4), 3.5 and 3.6.

Theorem 3.7. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous and strongly coercive function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let C be a nonempty closed subset of E. Then the following statements are equivalent:*

(1) *C is a sunny Bregman generalized nonexpansive retract of E;*
(2) *C is a Bregman generalized nonexpansive retract of E;*
(3) *∇gC is closed and convex.*

In this case, the unique sunny Bregman generalized nonexpansive retraction from E onto C is given by $(\nabla g)^{-1}P_{C_*}\nabla g$, where P_{C_*} is the Bregman projection from E^* onto ∇gC.

Proof. Since E is reflexive, by Lemma 2.2 (4) we have $\nabla g^* = (\nabla g)^{-1}$. The implication (1) $\Longrightarrow$ (2) is obvious. In view of Lemma 3.5, we have (2) $\Longrightarrow$ (3). Assume now that (3) holds. Since ∇gC is closed and convex, in view of Lemma 3.6, we conclude that $R = (\nabla g)^{-1}P_{C_*}\nabla g$ is a sunny Bregman generalized nonexpansive retraction from E onto $C = (\nabla g)^{-1}\nabla gC$, which completes the proof. □

Lemma 3.8. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous and strongly coercive function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let C be a nonempty closed subset of E such that ∇gC is closed and convex. If $T : C \to C$ is a Bregman* *generalized nonexpansive mapping such that $F(T) \neq \emptyset$, then $F(T)$ is closed and $\nabla gF(T)$ is closed and convex.*

Proof. First, let us prove that $\nabla gF(T)$ is closed. Let $\{x_n^*\}$ be a sequence of $\nabla gF(T)$ such that $x_n^* \to x^*$ for some $x^* \in E^*$. Since ∇gC is closed and convex, we have $x^* \in \nabla gC$. This implies that there exist $x \in C$ and $\{x_n\} \subset F(T)$ such that $x^* = \nabla gx$ and $x_n^* = \nabla gx_n$ for all $n \in \mathbb{N}$. Since E is reflexive, by Lemma 2.2 (4) we have $\nabla g^* = (\nabla g)^{-1}$. In view of Theorem 2.3 (3), we obtain that ∇g^* is uniformly norm-to-norm continuous on bounded sets. Thus we obtain $x_n \to x$ as $n \to \infty$. Since g is continuous and $\{\nabla g(x_n)\}$ is bounded, we conclude that

$$\begin{aligned} D(Tx, x_n) &\leq D(x, x_n) \\ &= g(x) - g(x_n) - \langle x - x_n, \nabla g(x_n)\rangle \\ &\to g(x) - g(x) - 0 \\ &= 0. \end{aligned}$$

This means that $\lim_{n\to\infty} D(Tx, x_n) = 0$ and $\lim_{n\to\infty} D(x, x_n) = 0$. On the other hand, we have from Lemma 2.5 that $\|x_n - x\| \to 0$ and $\|x_n - Tx\| \to 0$. Then, we have $Tx = x$. Thus we have $x^* = \nabla gx \in \nabla gF(T)$. So, we get that $\nabla gF(T)$ is closed. Since ∇g is norm-to-norm continuous, we have that $F(T)$ is closed. A similar argument as mentioned in the proof of Lemma 3.5 shows that $\nabla gF(T)$ is convex. This completes the proof. □

The following result is deduced from Theorem 3.7 and Lemma 3.8.

Proposition 3.9. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bonded sets. Let C be a nonempty closed subset of E such that ∇gC is closed and convex. If $T : C \to C$ is a Bregman generalized nonexpansive mapping such that $F(T) \neq \emptyset$, then $F(T)$ is a sunny Bregman generalized nonexpansive retract of E.*

4. Fixed point theorems

In this section, we prove fixed point theorems for Bregman generalized nonexpansive type mappings in a Banach space.

Theorem 4.1. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded sets and uniformly convex on bonded sets. Let C be a nonempty closed Bregman generalized nonexpansive retract of E and let $T : C \to C$ be a Bregman generalized nonexpansive type mapping. Then the following statements are equivalent:*

(1) *$F(T)$ is nonempty;*
(2) *$\{T^n x\}$ is bounded for some $x \in C$.*

Proof. The implication (1) $\Longrightarrow$ (2) is obvious. We prove the implication (2) $\Longrightarrow$ (1). Let there exist $x \in C$ such that $\{T^n x\}$ is bounded. By the definition of T, we get

$$\begin{aligned} D(T^{k+1}x, Ty) &+ D(Ty, T^{k+1}x) \\ &\leq D(T^k x, Ty) + D(y, T^{k+1}x), \ \forall k \in \mathbb{N} \cup \{0\}, \ y \in C. \end{aligned} \tag{4.1}$$

In view of (3.1), we conclude that

$$D(Ty, T^{k+1}x) = D(Ty, y) + D(y, T^{k+1}x) + \langle Ty - y, \nabla gy - \nabla gT^{k+1}x\rangle.$$

This implies that

$$D(y, T^{k+1}x) - D(Ty, T^{k+1}x) = -D(Ty, y) + \langle y - Ty, \nabla gy - \nabla gT^{k+1}x\rangle. \tag{4.2}$$

In view of (4.1) and (4.2), we obtain

$$\begin{aligned} 0 \le D(T^k x, Ty) \quad & -D(T^{k+1}x, Ty) - D(Ty, y) \\ & + \langle y - Ty, \nabla gy - \nabla gT^{k+1}x\rangle, \ \forall k \in \mathbb{N} \cup \{0\}, \ y \in C. \end{aligned}$$

Summing these inequalities with respect to $k = 0, 1, 2, ..., n-1$ and then dividing by n, we get

$$0 \le \frac{1}{n} D(x, Ty) - \frac{1}{n} D(T^n x, Ty) - D(Ty, y) + \langle y - Ty, \nabla gy - S_n^* x\rangle, \tag{4.3}$$

where $S_n^* x := \frac{1}{n}\sum_{k=1}^{n} \nabla gT^k x$. Since C is a Bregman generalized nonexpansive retract, in view of Lemma 3.5, we conclude that ∇gC is closed and convex. This implies that $\{S_n^* x\}$ is a well-defined sequence in ∇gC. The function g is bounded on bounded subsets of E and therefore ∇g is also bounded on bounded subsets of E^* (see, for example, [2, Proposition 1.1.11] for more details). Since $\{T^n x\}$ is bounded, $\{\nabla gT^n x\}$ is also bounded. So, we have $\{S_n^* x\}$ is bounded. Since E^* is reflexive, we have that $\{S_n^* x\}$ has a subsequence $\{S_{n_i}^* x\}$ such that $S_{n_i}^* x \rightharpoonup p^*$ for some $p^* \in \nabla gC$. Letting $n_i \to \infty$ in (4.3), we obtain

$$0 \le -D(Ty, y) + \langle y - Ty, \nabla gy - p^*\rangle. \tag{4.4}$$

Put $p := (\nabla g)^{-1} p^*$. Then $p \in C$ and letting $y = p$ in (4.4), we conclude that

$$0 \le -D(Tp, p) + \langle p - Tp, \nabla gp - \nabla gp\rangle.$$

This implies that $D(Tp, p) \le 0$. From Lemma 2.5, we have $Tp = p$. Thus we have $F(T)$ is nonempty, which completes the proof. □

The following theorem is an easy consequence of Lemma 3.5 and Theorem 4.1.

Theorem 4.2. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded sets and uniformly convex on bounded sets. Let C be a nonempty closed Bregman generalized nonexpansive retract of E and let $T : C \to C$ be a Bregman firmly generalized nonexpansive type mapping. Then the following statements are equivalent:*

(1) *$F(T)$ is nonempty;*
(2) *$\{T^n x\}$ is bounded for some $x \in C$.*

Let C be a nonempty closed subset of a Banach space E and let $T : C \to C$ be a mapping. A point $p \in C$ is said to be a Bregman generalized asymptotic fixed point [8] of T if C contains a sequence $\{x_n\}$ such that $\nabla gx_n \rightharpoonup^* \nabla gp$ and $\|\nabla gx_n - \nabla gTx_n\| \to 0$. The set of all Bregman generalized asymptotic fixed points of T is denoted by $\check{F}(T)$.

Theorem 4.3. *Let E be a reflexive Banach space. Let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded sets and uniformly convex on bounded sets. Let C be a nonempty closed Bregman generalized nonexpansive retract of E and let $T : C \to C$ be a Bregman generalized nonexpansive type mapping. If $F(T) \neq \emptyset$, then $\check{F}(T) = F(T)$.*

Proof. It is clear that $F(T) \subset \check{F}(T)$. Let us show that $\check{F}(T) \subset F(T)$. For any $p \in \check{F}(T)$, there exists a sequence $\{x_n\} \subset C$ such that $\nabla g x_n - \nabla g T x_n \to 0$ and $\nabla g x_n \rightharpoonup \nabla g p$. By the definition of T, we obtain

$$D(Tx_n, Tp) + D(Tp, Tx_n) \le D(x_n, Tp) + D(p, Tx_n). \tag{4.5}$$

In view of (3.1), we conclude that

$$D(Tp, Tx_n) = D(Tp, p) + D(p, Tx_n) + \langle Tp - p, \nabla gp - \nabla g T x_n \rangle$$

and hence

$$D(p, Tx_n) - D(Tp, Tx_n) = -D(Tp, p) + \langle p - Tp, \nabla gp - \nabla g T x_n \rangle. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\begin{aligned} & D(Tx_n, Tp) - D(x_n, Tp) \\ & \quad \le -D(Tp, p) + \langle p - Tp, \nabla gp - \nabla gTx_n \rangle \\ & \quad \le -D(Tp, p) + \langle p - Tp, \nabla gp - \nabla gx_n + \nabla gx_n - \nabla gTx_n \rangle \\ & \quad \le -D(Tp, p) + \langle p - Tp, \nabla gp - \nabla gx_n \rangle \\ & \qquad + \|p - Tp\| \|\nabla g(x_n) - \nabla g(Tx_n)\|. \end{aligned} \tag{4.7}$$

On the other hand, we have that

$$\begin{aligned} & D(Tx_n, Tp) - D(x_n, Tp) \\ = \ & g(Tx_n) - g(Tp) - \langle Tx_n - Tp, \nabla g(Tp) \rangle \\ & -[g(x_n) - g(Tp) - \langle x_n - Tp, \nabla g(Tp) \rangle] \\ = \ & g(Tx_n) - g(Tp) - \langle Tx_n - Tp, \nabla g(Tp) \rangle \\ & -g(x_n) + g(Tp) + \langle x_n - Tp, \nabla g(Tp) \rangle \\ = \ & g(Tx_n) - g(x_n) - \langle Tx_n - x_n, \nabla g(Tp) \rangle \\ = \ & g(Tx_n) - g(x_n) - \langle Tx_n - x_n, \nabla g(Tp) - \nabla g(x_n) + \nabla g(x_n) \rangle \\ = \ & g(Tx_n) - g(x_n) - \langle Tx_n - x_n, \nabla g(x_n) \rangle - \langle Tx_n - x_n, \nabla g(Tp) - \nabla g(x_n) \rangle \\ = \ & D(Tx_n, x_n) - \langle Tx_n - x_n, \nabla g(Tp) - \nabla g(x_n) \rangle \\ \ge \ & -\|Tx_n - x_n\| \|\nabla g(Tp) - \nabla g(x_n)\|. \end{aligned}$$

From $\nabla x_n \rightharpoonup \nabla gp$, we have $\{\nabla g x_n\}$ is bounded. Since the mapping ∇g^* on E^* is uniformly norm to norm continuous on each bounded set and $\|\nabla g x_n - \nabla g T x_n\| \to 0$, we obtain $\|x_n - T x_n\| \to 0$. Thus, we have that

$$\liminf_{n \to \infty} \{D(Tx_n, Tp) - D(x_n, Tp)\} \ge 0. \tag{4.8}$$

In view of (4.7) and (4.8), we get $-D(Tp, p) \ge 0$. This implies that $D(Tp, p) = 0$. From Lemma 2.5 we have $p \in F(T)$, which completes the proof. □

5. Weak convergence theorem

In this section, we prove a weak convergence theorem for Bregman firmly generalized nonexpansive type mappings in a reflexive Banach space.

Theorem 5.1. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous and strongly coercive function which is bounded on bounded sets, and uniformly convex and uniformly smooth on bounded sets. Let C be a nonempty closed Bregman generalized nonexpansive retract of E and let $T : C \to C$ be a Bregman firmly generalized nonexpansive type mapping. If the mapping ∇g is weakly sequentially continuous, then the following statements are equivalent:*

(1) *$F(T)$ is nonempty;*
(2) *$\{T^n x\}$ is bounded for some $x \in C$.*

In this case, $\{T^n x\}$ converges weakly to an element of $F(T)$.

Proof. We know from Theorem 4.2 that (1) $\iff$ (2). Let $x \in C$ and $z \in F(T)$. Since T is a Bregman firmly generalized nonexpansive type mapping from C into itself, we have T is Bregman firmly generalized nonexpansive. This implies that

$$D(T^{n+1}x, z) \leq D(T^n x, T^{n+1}x) + D(T^{n+1}x, z) \leq D(T^n x, z), \ \forall n \in \mathbb{N}. \tag{5.1}$$

Therefore, $\lim_{n\to\infty} D(T^n x, z)$ exists. From (5.1), it follows that

$$D(T^n x, T^{n+1}x) \leq D(T^n x, z) - D(T^{n+1}x, z), \ \forall n \in \mathbb{N}. \tag{5.2}$$

Since $\{D(T^n x, z)\}$ converges, we obtain that

$$\lim_{n\to\infty} D(T^n x, T^{n+1}x) = 0. \tag{5.3}$$

Since $\{T^n x\}$ is bounded, we have from (5.3) and Lemma 2.5 that

$$\lim_{n\to\infty} \|T^n x - T^{n+1}x\| = 0. \tag{5.4}$$

Since $T : C \to C$ is a Bregman firmly generalized nonexpansive type mapping, we have from Lemma 3.2 that

$$\langle T^n x - T^{n+1}x - (y - Ty), \nabla g T^{n+1}x - \nabla g Ty \rangle \geq 0, \ \forall n \in \mathbb{N}, \ y \in C. \tag{5.5}$$

Since $\{T^n x\}$ is bounded, there exists a subsequence $\{T^{n_i} x\}$ of $\{T^n x\}$ such that $T^{n_i} x \rightharpoonup p$ as $i \to \infty$. Since ∇g is weakly sequentially continuous, we obtain $\nabla g T^{n_i} x \rightharpoonup \nabla g p$. Since $\nabla g C$ is closed and convex, it is weakly closed and hence $\nabla g p \in \nabla g C$. Thus, we have $p \in C$. On the other hand, since g is uniformly smooth on bounded sets, ∇g is norm-to-norm uniformly continuous on each bounded subset of E. So in view of (5.4) we obtain $\|\nabla g T^n x - \nabla g T^{n+1} x\| \to 0$. This implies that $\nabla g T^{n_i+1} x \rightharpoonup \nabla g p$ as $i \to \infty$. Letting $n_i \to \infty$ in (5.5), we conclude that

$$\langle Ty - y, \nabla g p - \nabla g Ty \rangle \geq 0, \ \forall y \in C. \tag{5.6}$$

Putting $y = p$ in (5.6), we get

$$\langle Tp - p, \nabla g p - \nabla g Tp \rangle \geq 0. \tag{5.7}$$

Since ∇g is strictly monotone, we obtain $Tp = p$. Thus we have $p \in F(T)$. Assume now that $\{T^{n_i} x\}$ and $\{T^{n_j} x\}$ are two subsequences of $\{T^n x\}$ such that $T^{n_i} x \rightharpoonup p_1$

and $T^{n_j}x \rightharpoonup p_2$. The above argument shows that $p_1, p_2 \in F(T)$. Let

$$\lim_{n\to\infty}[D(T^n x, p_1) - D(T^n x, p_2)] = \lambda.$$

By the definition of the Bregman distance, we have that for all $n \in \mathbb{N}$

$$\begin{aligned} D(T^n x, p_1) - D(T^n x, p_2) =& g(T^n x) - g(p_1) - \langle T^n x - p_1, \nabla g p_1\rangle \\ &- [g(T^n x) - g(p_2) - \langle T^n x - p_2, \nabla g(p_2)\rangle] \\ =& g(p_2) - g(p_1) - \langle T^n x - p, \nabla g p_1\rangle + \langle T^n x - p_2, \nabla g(p_2)\rangle. \end{aligned}$$

This together with $T^{n_i}x \rightharpoonup p_1$ and $T^{n_j}x \rightharpoonup p_2$ implies that

$$g(p_2) - g(p_1) + \langle p_1 - p_2, \nabla g p_2\rangle = \lambda \tag{5.8}$$

and

$$g(p_2) - g(p_1) - \langle p_2 - p_1, \nabla g p_1\rangle = \lambda. \tag{5.9}$$

In view of (5.8) and (5.9), we obtain

$$\langle p_1 - p_2, \nabla g p_1 - \nabla g p_2\rangle = 0.$$

Employing Lemma 2.2 (2), we conclude that $p_1 = p_2$. Thus we have $\{T^n x\}$ converges weakly to an element of $F(T)$. □

References

[1] L. M. Bregman, *The relation method of finding the common point of convex sets and its application to the solution of problems in convex programming*, USSR Comput. Math. Math. Phys. **7** (1967), 200–217.

[2] D. Butnariu and A. N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Kluwer Academic Publishers, Dordrecht 2000.

[3] Y. Censor and A. Lent, *An iterative row-acton method for interval convex progamming*, J. Optim. Theory Appl. **34** (1981), 321–358.

[4] G. Chen and M. Teboulle, *Convergence analysis of a proximal-like minimization algorithm using Bregman functions*, SIAM J. Optim. **3** (1993), 538–543.

[5] T. Ibaraki and W. Takahashi, *Fixed point theorems for nonlinear mappings of nonexpansive type in Banach spaces*, J. Nonlinear Convex Anal. **10** (2009), 21–32.

[6] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory **149** (2007), 1–14.

[7] T. Ibaraki and W. Takahashi, *Convergence theorems for new projections in Banach spaces*, RIMS Kokyuroku **1484** (2006), 150–160 (in Japanese).

[8] T. Ibaraki and W. Takahashi, *Generalized nonexpansive mappings and a proxima-type algorithm in Banach spaces*, Contemp. Math., 513, Amer. Math. Soc., Providence, RI, 2010.

[9] T. Ibaraki and W. Takahashi, *Strong convergence theorems for a finite family of nonlinear operators of firmly nonexpansive type in Banach spaces*, in Nonlinear Analysis and Convex Analysis, S.-B. Hsu., H.-C. Lai, L.-J. Lin, W. Takahashi, T. Tanaka, J.-C. Yao (eds), Yokahama Publishers, Yokahama, 2009, pp. 49–62.

[10] W. Inthakon, S. Dhompongsa and W. Takahashi, *Strong convergence theorems for maximal monotone operators and generalized nonexpansive mappings in Banach spaces*, J. Nonlinear Convex Anal. **11** (2010), 45–63.

[11] F. Kohsaka and W. Takahashi, *Proximal point algorithms with Bregman functions in Banach spaces*, J. Nonlinear Convex Anal. **6** (2005), 505–523.

[12] F. Kohsaka and W. Takahashi, *Generalized nonexpansive retractions and proximal-type algorithms in Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 197–209.

[13] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976) 877–898.

[14] R.T. Rockafellar, *Characterization of subdifferentials of convex functions*, Pacific. J. Math. **17** (1966), 497–510.
[15] R.T. Rockafellar, *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. **33** (1970), 209–216.
[16] R.T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.
[17] W. Takahashi, *Nonlinear Functional Analysis, Fixed Point Theory and its Applications*, Yokahama Publishers, Yokahama, 2000.
[18] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokahama Publishers, Yokahama, 2000 (in Japanese).
[19] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing Co. Inc., River Edge NJ, 2002.

Manuscript received March 30, 2011
revised November 8, 2011

Eskandar Naraghirad
Department of Mathematics, Yasouj University, Yasouj 75918, Iran
E-mail address: `eskandarrad@gmail.com`

Wataru Takahashi
Department of Mathematical and Computer Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan
E-mail address: `wataru@is.titech.ac.jp`

Jen-Chih Yao
Center for General Education, Kaohsiung Medical University, Kaohsiung 807, Taiwan
E-mail address: `yaojc@kmu.edu.tw`