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ANOTHER VERSION OF THE VON NEUMANN-JORDAN CONSTANT

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Abstract. We define another version of the von Neumann-Jordan constant, study some of its properties and relate it to the fixed point property for Lipschitzian mappings.

1. A generalization of von Neumann-Jordan constant

The von Neumann-Jordan constant of a Banach space X was defined by Clarkson [1] as

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x, y \in X, (x,y) \neq (0,0)\right\}.$$

It is known that this constant is connected with some geometric structures of Banach spaces, such as normal structure and uniform non-squareness, and thus with the fixed point property, see for example [9] and [11].

Based on the following characterization of Hilbert spaces we will define a family of constants for Banach spaces similar to the von Neumann-Jordan constant and apply this to find conditions for the existence of fixed points for periodic mappings.

Lemma 1.1. *Let X be a Banach space and $\alpha \in (0,1)$. X is a Hilbert space if and only if for every $x, y \in X$,*

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2.$$

Definition 1.2. Let X be a Banach space and $\alpha \in [0,1]$, we set:

$$C_\alpha(X) = \sup\left\{\frac{\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2}{\alpha\|x\|^2 + (1-\alpha)\|y\|^2} : x, y \in X, \alpha\|x\|^2 + (1-\alpha)\|y\|^2 \neq 0\right\}.$$

Closely associated to $C_\alpha(X)$ is the function f given as follows:

Definition 1.3. For $\alpha \in (0, 1/2]$ and $t \in [0,1]$, define $f(t) = \frac{(\alpha+(1-\alpha)t)^2+\alpha(1-\alpha)(1+t)^2}{\alpha+(1-\alpha)t^2}$.

Remark 1.4. f attains its unique maximum $1 + 2\sqrt{\alpha(1-\alpha)}$ at $t_0 = \sqrt{\frac{\alpha}{1-\alpha}}$.

Lemma 1.5. *Let X be a Banach space. Then:*

(a) $C_{\frac{1}{2}}(X) = C_{NJ}(X)$.

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(b) *It is clear that* $C_\alpha(X) = C_{1-\alpha}(X)$ *for every* $\alpha \in [0,1]$.
(c) $C_\alpha(X) = 1$ *for* $\alpha \in (0,1)$ *if and only if* X *is a Hilbert space.*

Proof. (a) and (b) are obvious.

Proof of (c).

Suppose that $C_\alpha(X) = 1$ for $\alpha \in (0,1)$, then for $x, y \in X$, $\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2$. Let $u = \alpha x + (1-\alpha)y$ and $v = \alpha x - \alpha y$. Then we have $\|\alpha u + (1-\alpha)v\|^2 + \alpha(1-\alpha)\|u-v\|^2 \leq \alpha\|u\|^2 + (1-\alpha)\|v\|^2$; that is,

$$\alpha^2\|x\|^2 + \alpha(1-\alpha)\|y\|^2 \leq \alpha\|\alpha x + (1-\alpha)y\|^2 + \alpha^2(1-\alpha)\|x-y\|^2,$$

hence by lemma 1.1 X is a Hilbert space.

If X is a Hilbert space, by lemma 1.1 is clear that $C_\alpha(X) = 1$. □

The following lemma will be useful for calculating $C_\alpha(X^*)$.

Lemma 1.6. *Let* X *be a Banach space and* $\alpha \in (0,1)$. *Let* $Z = X \times X$ *equipped with the norm* $\|(x,y)\|_Z^2 = \alpha\|x\|_X^2 + (1-\alpha)\|y\|_X^2$, *and in* $X^* \times X^*$ *consider*

$$\|(f,g)\|^2 = \frac{\|f\|_{X^*}^2}{\alpha} + \frac{\|g\|_{X^*}^2}{1-\alpha}. \tag{1.1}$$

Then $\|\cdot\|$ *defines the dual norm of* Z.

Proof. Using Lagrange multipliers it is easy to see that the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by $F(u,v) = (au + bv)^2$ subject to the condition $\alpha u^2 + (1-\alpha)v^2 = 1$, attains its maximum at a point (u_0, v_0) and

$$F(u_0, v_0) = \frac{a^2}{\alpha} + \frac{b^2}{1-\alpha}. \tag{1.2}$$

Let X be a Banach space, $\alpha \in (0,1)$, $Z = X \times X$ as in the hypothesis and $f, g \in X^*$. By definition

$$\|(f,g)\|_{Z^*} = \sup\left\{|f(x) + g(y)| : \alpha\|x\|_X^2 + (1-\alpha)\|y\|_X^2 = 1\right\}.$$

Let $x, y \in X$ be such that $\alpha\|x\|_X^2 + (1-\alpha)\|y\|_X^2 = 1$. By (1.2):

$$|f(x) + g(x)|^2 \leq (\|f\|_{X^*}\|x\|_X + \|g\|_{X^*}\|y\|_X)^2 \leq \frac{\|f\|_{X^*}^2}{\alpha} + \frac{\|g\|_{X^*}^2}{1-\alpha},$$

hence $\|(f,g)\|_{Z^*} \leq \|(f,g)\|$.

Let $\varepsilon > 0$ and $x_0, y_0 \in S_X$ be such that $f(x_0) = (1-\varepsilon)\|f\|_{X^*}$ and $g(y_0) = (1-\varepsilon)\|g\|_{X^*}$ and let

$$\begin{aligned} x &= x_0\sqrt{\left(\frac{1-\alpha}{\alpha}\right)\frac{\|f\|_{X^*}^2}{(1-\alpha)\|f\|_{X^*}^2 + \alpha\|g\|_{X^*}^2}} = Ax_0, \\ y &= y_0\sqrt{\left(\frac{\alpha}{1-\alpha}\right)\frac{\|g\|_{X^*}^2}{(1-\alpha)\|f\|_{X^*}^2 + \alpha\|g\|_{X^*}^2}} = By_0. \end{aligned}$$

Then

$$\begin{aligned}
|f(x)+g(y)|^2 =& \left|A(1-\varepsilon)\|f\|+B(1-\varepsilon)\|g\|\right|^2 \\
=& \frac{(1-\varepsilon)^2}{(1-\alpha)\|f\|^2+\alpha\|g\|^2}\left|\sqrt{\frac{1-\alpha}{\alpha}}\|f\|^2+\sqrt{\frac{\alpha}{1-\alpha}}\|g\|^2\right|^2 \\
=& \frac{(1-\varepsilon)^2}{\alpha(1-\alpha)((1-\alpha)\|f\|^2+\alpha\|g\|^2)}\left[(1-\alpha)^2\|f\|^4+2\alpha(1-\alpha)\|f\|^2\|g\|^2+\alpha^2\|g\|^4\right] \\
=& \frac{(1-\varepsilon)^2}{\alpha(1-\alpha)}\left[(1-\alpha)\|f\|^2+\alpha\|g\|^2\right]=(1-\varepsilon)^2\left[\frac{\|f\|^2}{\alpha}+\frac{\|g\|^2}{1-\alpha}\right],
\end{aligned}$$

which proves the equality. □

Proposition 1.7. *For every Banach space X and $\alpha \in [0,1]$, $C_\alpha(X) = C_\alpha(X^*)$.*

The proof is similar to Kato's and Takahashi's, when they showed that $C_{NJ}(X) = C_{NJ}(X^*)$ [10].

Proof. Let X be a Banach space. If $\alpha = 0$ or $\alpha = 1$, the equality holds trivially. Suppose that $\alpha \in (0,1)$. Let $Z = X \times X$ as in the previous lemma.

Define the linear operator $L : Z \to Z$ as $L(x,y) = (\alpha x + (1-\alpha)y, \alpha x - \alpha y)$ for any $(x,y) \in Z$. Let $(x,y) \in Z$, then

$$\|L(x,y)\|_Z^2 \le \|L\|^2\|(x,y)\|_Z^2 = \|L\|^2(\alpha\|x\|_X^2 + (1-\alpha)\|y\|_X^2)$$

and this happens if and only if

$$\frac{\alpha(\|\alpha x+(1-\alpha)y\|_X^2+\alpha(1-\alpha)\|x-y\|_X^2)}{\alpha\|x\|_X^2+(1-\alpha)\|y\|_X^2} \le \|L\|^2.$$

Taking the supremum over $x, y \in X$ we obtain $\alpha C_\alpha(X) \le \|L\|^2$.

Now take $\varepsilon > 0$ and $x, y \in X$ such that $\|L(x,y)\|_Z^2/\|(x,y)\|_Z^2 > \|L\|^2 - \varepsilon$. Then

$$\|L\|^2-\varepsilon < \frac{\alpha(\|\alpha x+(1-\alpha)y\|_X^2+\alpha(1-\alpha)\|x-y\|_X^2)}{\alpha\|x\|_X^2+(1-\alpha)\|y\|_X^2} \le \alpha C_\alpha(X).$$

Hence $\|L\|^2 = \alpha C_\alpha(X) = \|L^t\|^2$ where L^t is the transpose of L, given by $L^t(f,g) = (\alpha f + \alpha g, (1-\alpha)f - \alpha g)$ for every $f, g \in X^*$. Thus

$$\frac{\|L^t(f,g)\|_{Z^*}^2}{\|(f,g)\|_{Z^*}^2} \le \|L\|^2 = \alpha C_\alpha(X).$$

By (1.1) we have:

$$\begin{aligned}
\frac{1}{\alpha}\frac{\|L^t(f,g)\|_{Z^*}^2}{\|(f,g)\|_{Z^*}^2} &= \frac{1}{\alpha}\frac{\frac{\|\alpha f+\alpha g\|_{X^*}^2}{\alpha}+\frac{\|(1-\alpha)f-\alpha g\|_{X^*}^2}{1-\alpha}}{\frac{\|f\|_{X^*}^2}{\alpha}+\frac{\|g\|_{X^*}^2}{1-\alpha}} \\
&= \frac{\alpha(1-\alpha)\|f+g\|_{X^*}^2+\|(1-\alpha)f-\alpha g\|_{X^*}^2}{(1-\alpha)\|f\|_{X^*}^2+\alpha\|g\|_{X^*}^2} \le C_\alpha(X).
\end{aligned}$$

Observe that

$$C_\alpha(X^*) = \sup\left\{\frac{\|\alpha f+(1-\alpha)g\|_{X^*}^2+\alpha(1-\alpha)\|f-g\|_{X^*}^2}{\alpha\|f\|_{X^*}^2+(1-\alpha)\|g\|_{X^*}^2} : f,g\in X^*\right\}$$
$$= \sup\left\{\frac{\|(1-\alpha)f-\alpha g\|_{X^*}^2+\alpha(1-\alpha)\|f+g\|_{X^*}^2}{(1-\alpha)\|f\|_{X^*}^2+\alpha\|g\|_{X^*}^2} : f,g\in X^*\right\},$$

hence

$$C_\alpha(X^*)\le C_\alpha(X).$$

For the other inequality, let us take $(f,g)\in Z^*$ such that

$$\frac{\|L^t(f,g)\|_{Z^*}^2}{\|(f,g)\|_{Z^*}^2} > \|L^t\|^2-\varepsilon\alpha = \alpha C_\alpha(X)-\varepsilon\alpha,$$

that is

$$C_\alpha(X)-\varepsilon < \frac{\alpha(1-\alpha)\|f+g\|_{X^*}^2+\|(1-\alpha)f-\alpha g\|_{X^*}^2}{(1-\alpha)\|f\|_{X^*}^2+\alpha\|g\|_{X^*}^2}\le C_\alpha(X^*),$$

and we conclude $C_\alpha(X)\le C_\alpha(X^*)$. □

In order to be able to calculate the value of $C_\alpha(X)$ we will introduce the following function, which is similar to a function defined by Yang and Wang [12]:

$$\eta(\alpha,t)=\sup\left\{\|\alpha x+(1-\alpha)ty\|^2+\alpha(1-\alpha)\|x-ty\|^2 : x,y\in S_X\right\}.$$

It follows immediately that $\eta(\alpha,t)=\varphi(1-\alpha,t)$. We now list some useful properties of η:

Lemma 1.8. (a) *For every $\alpha\in[0,1]$, $\eta(\alpha,0)=\alpha$.*

(b) *If we take $y=x\in S_X$, we obtain $\eta(\alpha,t)\ge\alpha+(1-\alpha)t^2\ge\alpha$ for any $\alpha\in[0,1]$ and any $t\in[0,1]$. In particular for every α, $\eta(\alpha,1)\ge 1$.*

(c) *By the triangle inequality, we have $\eta(\alpha,t)\le(\alpha+(1-\alpha)t)^2+\alpha(1-\alpha)(1+t)^2=\alpha+(1-\alpha)t^2+4\alpha(1-\alpha)t$ for any $\alpha\in[0,1]$ and any $t\in[0,1]$. In particular $\eta(\alpha,1)\le 1+4\alpha(1-\alpha)\le 2$.*

Moreover the function η is continuous and convex.

Proposition 1.9. *Let X be a Banach space and $\alpha\in[0,1]$. Let us consider $\eta(\alpha,t)$ as a function of the variable t. Then η is a continuous function on $[0,1)$ and is convex and nondecreasing on $[0,1]$.*

Proof. Let $0\le t_1<t_2\le 1$ and $\beta\in[0,1]$. Since $x=\beta x+(1-\beta)x$ and considering that $h(x)=x^2$ is a convex function, for any $x,y\in S_X$

$$\begin{aligned}
&\|\alpha x+(1-\alpha)(\beta t_1+(1-\beta)t_2)y\|^2+\alpha(1-\alpha)\|x-(\beta t_1+(1-\beta)t_2)y\|^2\\
\le&\left(\beta\|\alpha x+(1-\alpha)t_1y\|+(1-\beta)\|\alpha x+(1-\alpha)t_2y\|\right)^2\\
&+\alpha(1-\alpha)\left(\beta\|x-t_1y\|+(1-\beta)\|x-t_2y\|\right)^2\\
\le&\beta\|\alpha x+(1-\alpha)t_1y\|^2+(1-\beta)\|\alpha x+(1-\alpha)t_2y\|^2\\
&+\alpha(1-\alpha)\left(\beta\|x-t_1y\|^2+(1-\beta)\|x-t_2y\|^2\right)\\
\le&\beta\eta(\alpha,t_1)+(1-\beta)\eta(\alpha,t_2).
\end{aligned}$$

Thus, η is a convex function on $[0,1]$ and is continuous on $(0,1)$. By lemma 1.8 (c), $\eta(\alpha,0)=\alpha$ and

$$0 \leq \eta(\alpha,t)-\eta(\alpha,0) \leq (1-\alpha)t^2+4\alpha(1-\alpha)t.$$

We conclude $\lim\limits_{t\to 0}(\eta(\alpha,t)-\eta(\alpha,0))=0$.

Since η is a convex function on $[0,1]$, in order to prove that η is nondecreasing, it is enough to show that $\eta(\alpha,t)\geq\eta(\alpha,0)$ for every $t\in[0,1]$, and this is true, because $\eta(\alpha,0)=\alpha$ and $\eta(\alpha,t)\geq\alpha$ for every $t\in[0,1]$, by lemma 1.8 (b). □

We have the following lemma:

Lemma 1.10. *Let X be a Banach space and $\alpha\in(0,\frac{1}{2}]$. Then*

$$\sup_{t\in[0,1]}\frac{\eta(1-\alpha,t)}{\alpha t^2+(1-\alpha)} \leq \sup_{t\in[0,1]}\frac{\eta(\alpha,t)}{\alpha+(1-\alpha)t^2}.$$

Proof. Let X be a Banach space and $w,z\in S_X$, $t\in[0,1]$. Note that $\frac{\alpha}{1-\alpha}\leq 1$. Then

$$\begin{aligned}
&\frac{\|\alpha tz+(1-\alpha)w\|^2+\alpha(1-\alpha)\|tz-w\|^2}{(1-\alpha)+\alpha t^2}\\
=&\frac{\left\|\alpha w+(1-\alpha)\frac{\alpha t}{1-\alpha}(-z)\right\|^2+\alpha(1-\alpha)\left\|w-\frac{\alpha t}{1-\alpha}(-z)\right\|^2}{\alpha+(1-\alpha)\left(\frac{\alpha t}{1-\alpha}\right)^2}\\
\leq&\frac{\eta(\alpha,\frac{\alpha t}{1-\alpha})}{\alpha+(1-\alpha)\left(\frac{\alpha t}{1-\alpha}\right)^2}\leq\sup_{s\in[0,1]}\frac{\eta(\alpha,s)}{\alpha+(1-\alpha)s^2}.
\end{aligned}$$

Taking the supremum over $z,w\in S_X$ and then the supremum over $t\in[0,1]$, we have the desired inequality. □

Now we can state the following characterization of $C_\alpha(X)$:

Lemma 1.11. *Let X be a Banach space and $\alpha\in(0,\frac{1}{2}]$. Then*

$$C_\alpha(X)=\sup_{t\in[0,1]}\frac{\eta(\alpha,t)}{\alpha+(1-\alpha)t^2}.$$

Proof. Let X be a Banach space and $\alpha\in(0,\frac{1}{2}]$.

Let $x,y\in S_X$ and $t\in[0,1]$. By definition of $C_\alpha(X)$

$$\sup_{t\in[0,1]}\frac{\eta(\alpha,t)}{\alpha+(1-\alpha)t^2}=\sup_{t\in[0,1]}\left\{\frac{\|\alpha x+(1-\alpha)ty\|^2+\alpha(1-\alpha)\|x-ty\|^2}{\alpha+(1-\alpha)t^2}\right\}\leq C_\alpha(X).$$

Let $x, y \in X$ and suppose first that $\|x\| \geq \|y\| > 0$, then

$$\begin{aligned}
&\frac{\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2}{\alpha\|x\|^2 + (1-\alpha)\|y\|^2} \\
&= \frac{\left\|\alpha \frac{x}{\|x\|} + (1-\alpha)\frac{\|y\|}{\|x\|}\frac{y}{\|y\|}\right\|^2 + \alpha(1-\alpha)\left\|\frac{x}{\|x\|} - \frac{\|y\|}{\|x\|}\frac{y}{\|y\|}\right\|^2}{\alpha + (1-\alpha)\left(\frac{\|y\|}{\|x\|}\right)^2} \\
&\leq \frac{\eta(\alpha, \frac{\|y\|}{\|x\|})}{\alpha + (1-\alpha)\left(\frac{\|y\|}{\|x\|}\right)^2} \leq \sup_{t\in[0,1]} \frac{\eta(\alpha,t)}{\alpha + (1-\alpha)t^2}.
\end{aligned}$$

Now suppose that $\|y\| \geq \|x\| > 0$; by lemma 1.10

$$\begin{aligned}
&\frac{\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2}{\alpha\|x\|^2 + (1-\alpha)\|y\|^2} \\
&= \frac{\left\|\alpha \frac{\|x\|}{\|y\|}\frac{x}{\|x\|} + (1-\alpha)\frac{y}{\|y\|}\right\|^2 + \alpha(1-\alpha)\left\|\frac{\|x\|}{\|y\|}\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2}{\alpha\left(\frac{\|x\|}{\|y\|}\right)^2 + (1-\alpha)} \\
&\leq \frac{\eta(1-\alpha, \frac{\|x\|}{\|y\|})}{\alpha\left(\frac{\|x\|}{\|y\|}\right)^2 + (1-\alpha)} \leq \sup_{t\in[0,1]} \frac{\eta(1-\alpha,t)}{\alpha t^2 + (1-\alpha)} \leq \sup_{t\in[0,1]} \frac{\eta(\alpha,t)}{\alpha + (1-\alpha)t^2}.
\end{aligned}$$

Note that if $x = 0$ or $y = 0$, then the same inequality holds, because for $t = 0$ we have $\frac{\eta(\alpha,0)}{\alpha} = 1$. Taking the supremum over $x, y \in X$,

$$C_\alpha(X) \leq \sup_{t\in[0,1]} \frac{\eta(\alpha,t)}{\alpha + (1-\alpha)t^2}.$$

□

Corollary 1.12. *Let X be a Banach space and $\alpha \in [0,1]$. Then $1 \leq C_\alpha(X) \leq 1 + 2\sqrt{\alpha(1-\alpha)}$.*

Proof. For $\alpha = 0$ or $\alpha = 1$ we have equality. Let us consider $\alpha \in (0, 1/2]$. Using lemma 1.8 and remark 1.4:

(1.3)

$$1 \leq \frac{\eta(\alpha,t)}{\alpha + (1-\alpha)t^2} \leq \frac{(\alpha + (1-\alpha)t)^2 + \alpha(1-\alpha)(1+t)^2}{\alpha + (1-\alpha)t^2} = f(t) \leq 1 + 2\sqrt{\alpha(1-\alpha)}.$$

By the previous lemma and by lemma 1.5 (b), we conclude that for $\alpha \in (0,1)$, $1 \leq C_\alpha(X) \leq 1 + 2\sqrt{\alpha(1-\alpha)}$. □

The lower bound for $C_\alpha(X)$ is attained by Hilbert spaces. The upper bound is also attained by Banach spaces which are not uniformly nonsquare, as we will see in proposition 1.16.

James introduced in [8] the concept of uniform nonsquare spaces:

Definition 1.13. A Banach space X is uniformly nonsquare if there is $\varepsilon \in (0,1)$ such that for any $x, y \in S_X$, $\min\{\|x+y\|, \|x-y\|\} \leq 2(1-\varepsilon)$.

It is known that Hilbert spaces are uniformly nonsquare, but the space ℓ_1 is not uniformly nonsquare.

Lemma 1.14. *Let X be a Banach space and $\alpha \in (0,1)$. X is uniformly nonsquare if and only if there is $\delta \in (0,1)$ such that for any $x, y \in S_X$, $\min\{\|\alpha x + (1-\alpha)y\|, \|x-y\|/2\} \le (1-\delta)$.*

Proof. Suppose that X is uniformly nonsquare and let ε as in the definition 1.13. Let us take $x, y \in S_X$, then $\|x-y\| \le 2(1-\varepsilon)$ or $\|x+y\| \le 2(1-\varepsilon)$. If $\|x-y\| \le 2(1-\varepsilon)$, we have nothing to prove. Suppose that $\|x+y\| \le 2(1-\varepsilon)$; if $\alpha \in (0,1/2]$, then

$$\begin{aligned}\|\alpha x + (1-\alpha)y\| &= \alpha\Big\|x + y + \frac{1-2\alpha}{\alpha}y\Big\| \le \alpha\|x+y\| + \alpha\frac{1-2\alpha}{\alpha} \\ &\le 2\alpha(1-\varepsilon) + 1 - 2\alpha = 1 - 2\alpha\varepsilon.\end{aligned}$$

Similarly, if $\alpha \in [1/2, 1)$, we have that $\|\alpha x + (1-\alpha)y\| \le 1 - 2(1-\alpha)\varepsilon$.

Suppose that there is $\delta \in (0,1)$ such that $\min\{\|\alpha x + (1-\alpha)y\|, \|x-y\|/2\} \le (1-\delta)$ for any $x, y \in S_X$. Let us take $x, y \in S_X$. Suppose that $\|x-y\| > 2(1-\delta)$ then $\|\alpha x + (1-\alpha)y\| \le 1-\delta$ and $\|\alpha y + (1-\alpha)x\| \le 1-\delta$. From this

$$\begin{aligned}\|x+y\| &= \|\alpha x + (1-\alpha)y + \alpha y + (1-\alpha)x\| \\ &\le \|\alpha x + (1-\alpha)y\| + \|\alpha y + (1-\alpha)x\| \le 2(1-\delta).\end{aligned}$$

Hence X is uniformly nonsquare. □

Proposition 1.15. *Let X be a Banach space and $\alpha \in (0,1)$, then the following statements are equivalent:*

(1) *X is not uniformly nonsquare.*
(2) *$\eta(\alpha,t) = (\alpha + (1-\alpha)t)^2 + \alpha(1-\alpha)(1+t)^2$ for every $t \in [0,1]$.*
(3) *$\eta(\alpha,t_0) = (\alpha + (1-\alpha)t_0)^2 + \alpha(1-\alpha)(1+t_0)^2$ for some $t_0 \in (0,1]$.*

Proof. (1) ⇒ (2) If X is not uniformly nonsquare, by lemma 1.14 there are sequences $\{x_n\}$, $\{y_n\} \subset S_X$ such that $\lim_{n\to\infty} \|\alpha x_n + (1-\alpha)y_n\| = 1$ and $\lim_{n\to\infty}\|x_n - y_n\| = 2$. Let $t \in [0,1]$; note that:

$$\begin{aligned}\alpha + (1-\alpha)t \ge &\|\alpha x_n + (1-\alpha)ty_n\| = \|\alpha x_n + (1-\alpha)y_n + (1-\alpha)(t-1)y_n\| \\ \ge &\|\alpha x_n + (1-\alpha)y_n\| - (1-\alpha)(1-t).\end{aligned}$$

Hence $\lim_{n\to\infty}\|\alpha x_n + (1-\alpha)ty_n\| = \alpha + (1-\alpha)t$. On the other hand

$$1 + t \ge \|x_n - ty_n\| = \|x_n - y_n + (1-t)y_n\| \ge \|x_n - y_n\| - (1-t).$$

From this $\lim_{n\to\infty}\|x_n - ty_n\| = 1+t$. Since $\eta(\alpha,t) \le (\alpha + (1-\alpha)t)^2 + \alpha(1-\alpha)(1+t)^2$,

$$\begin{aligned}\eta(\alpha,t) \ge &\lim_{n\to\infty}\left(\|\alpha x_n + (1-\alpha)ty_n\|^2 + \alpha(1-\alpha)\|x_n - ty_n\|^2\right) \\ = &(\alpha + (1-\alpha)t)^2 + \alpha(1-\alpha)(1+t)^2 \ge \eta(\alpha,t).\end{aligned}$$

We conclude that for every $t \in [0,1]$, $\eta(\alpha,t) = (\alpha + (1-\alpha)t)^2 + \alpha(1-\alpha)(1+t)^2$.

(2) ⇒ (3) Is obvious.

(3) $\Rightarrow$ (1) Suppose that $\eta(\alpha, t_0) = (\alpha + (1-\alpha)t_0)^2 + \alpha(1-\alpha)(1+t_0)^2$ for some $t_0 \in (0,1]$ and that X is uniformly nonsquare, then there is $0 < \delta < 1$ as in lemma 1.14. Let $x, y \in S_X$, suppose first that $\|x - y\| \le 2(1-\delta)$, then

$$\begin{aligned}\|\alpha x + (1-\alpha)t_0 y\|^2 + \alpha(1-\alpha)\|x - t_0 y\|^2 \le &(\alpha + (1-\alpha)t_0)^2\\ &+ \alpha(1-\alpha)\|t_0 x - t_0 y + (1-t_0)x\|^2\\ \le&(\alpha + (1-\alpha)t_0)^2\\ &+ \alpha(1-\alpha)(t_0\|x-y\| + (1-t_0))^2\\ \le&(\alpha + (1-\alpha)t_0)^2\\ &+ \alpha(1-\alpha)(2t_0(1-\delta) + (1-t_0))^2\\ =&\eta(\alpha, t_0) - 4\alpha(1-\alpha)\delta t_0(1 + (1-\delta)t_0).\end{aligned}$$

Now suppose that $\|\alpha x + (1-\alpha)y\| \le 1 - \delta$, in this case:

$$\begin{aligned}\|\alpha x + (1-\alpha)t_0 y\|^2 + \alpha(1-\alpha)\|x - t_0 y\|^2 \le&\|(1-t_0)\alpha x + t_0(\alpha x + (1-\alpha)y)\|^2\\ &+ \alpha(1-\alpha)(1+t_0)^2\\ \le&((1-t_0)\alpha + t_0\|\alpha x + (1-\alpha)y\|)^2\\ &+ \alpha(1-\alpha)(1+t_0)^2\\ \le&((1-t_0)\alpha + t_0(1-\delta))^2\\ &+ \alpha(1-\alpha)(1+t_0)^2\\ =&\eta(\alpha, t_0) - \delta t_0((2-\delta)t_0 + 2\alpha(1-t_0)).\end{aligned}$$

We conclude that $\eta(\alpha, t_0) < \eta(\alpha, t_0)$, which is a contradiction. □

Proposition 1.16. *Let X be a Banach space and $\alpha \in (0,1)$. Then X is uniformly nonsquare if and only if $C_\alpha(X) < 1 + 2\sqrt{\alpha(1-\alpha)}$.*

Proof. Since $C_\alpha(X) = C_{1-\alpha}(X)$ for every Banach space X, it is enough to prove it for $\alpha \in (0, 1/2]$. If X is not uniformly nonsquare, by proposition 1.15 and by lemma 1.11

$$C_\alpha(X) = \sup_{t\in[0,1]} \frac{(\alpha + (1-\alpha)t)^2 + \alpha(1-\alpha)}{\alpha + (1-\alpha)t^2} = 1 + 2\sqrt{\alpha(1-\alpha)}.$$

Suppose now that $C_\alpha(X) = 1 + 2\sqrt{\alpha(1-\alpha)}$. By lemma 1.11 there is a sequence $\{t_n\} \subset [0,1]$, which we assume converges to $s \in [0,1]$ such that

$$\lim_n \frac{\eta(\alpha, t_n)}{\alpha + (1-\alpha)t_n^2} = C_\alpha(X) = 1 + 2\sqrt{\alpha(1-\alpha)}.$$

Case 1) $s \in [0, 1)$.

By proposition 1.9 since $\eta(\alpha, s)$ is continuous on $[0, 1)$:

$$\lim_n \frac{\eta(\alpha, t_n)}{\alpha + (1-\alpha)t_n^2} = \frac{\eta(\alpha, s)}{\alpha + (1-\alpha)s^2} = 1 + 2\sqrt{\alpha(1-\alpha)},$$

and by (1.3)

$$1 + 2\sqrt{\alpha(1-\alpha)} = \frac{\eta(\alpha, s)}{\alpha + (1-\alpha)s^2} \le f(s) \le 1 + 2\sqrt{\alpha(1-\alpha)}.$$

Thus $f(s) = 1 + 2\sqrt{\alpha(1-\alpha)}$ and by remark 1.4 $s = \sqrt{\frac{\alpha}{1-\alpha}}$.

Since $s \in [0,1)$, $\alpha \in (0, \frac{1}{2})$ and

$$\begin{aligned}\eta\left(\alpha, \sqrt{\frac{\alpha}{1-\alpha}}\right) &= 2\alpha(1 + 2\sqrt{\alpha(1-\alpha)}) \\ &= \left(\alpha + (1-\alpha)\sqrt{\frac{\alpha}{1-\alpha}}\right)^2 + \alpha(1-\alpha)\left(1 + \sqrt{\frac{\alpha}{1-\alpha}}\right)^2,\end{aligned}$$

thus by proposition 1.15, X is not uniformly nonsquare.

Case 2) $s = 1$.

Again by (1.3), for any $n \in \mathbb{N}$:

$$\frac{\eta(\alpha, t_n)}{\alpha + (1-\alpha)t_n^2} \le f(t_n), \tag{1.4}$$

and this implies

$$1 + 2\sqrt{\alpha(1-\alpha)} = \lim_n \frac{\eta(\alpha, t_n)}{\alpha + (1-\alpha)t_n^2} \le \lim_n f(t_n) = f(1) = 1 + 4\alpha(1-\alpha)$$

which cannot hold for $\alpha \in (0, \frac{1}{2})$, because $1 + 4\alpha(1-\alpha) < 1 + 2\sqrt{\alpha(1-\alpha)}$; hence $\alpha = \frac{1}{2}$. By the equality in (1.4) we get $\lim_n \eta(\frac{1}{2}, t_n) = 2$. Therefore, since η is nondecreasing and by lemma 1.8 (c), $\eta(\frac{1}{2}, t) \le 2$, we obtain $\eta(\frac{1}{2}, 1) = 2$ and by proposition 1.15 we conclude that X is not uniformly nonsquare. □

Now we are going to calculate the value of $C_\alpha(X)$ for some spaces.

Example 1.17. Let $X = \ell_2 - \ell_1$ be the space $\mathbb{R}^2$ with the norm $\|\cdot\|_{2,1}$ defined by

$$\|(a,b)\|_{2,1} = \begin{cases} \|(a,b)\|_2 & \text{if} \quad ab \ge 0 \\ \|(a,b)\|_1 & \text{if} \quad ab \le 0. \end{cases}$$

Then $C_\alpha(X) = 1 + \sqrt{\alpha(1-\alpha)}$.

In order to prove this, first note that the set of extreme points of the unit ball is $\mathcal{E}(B_X) = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = 1, ab \ge 0\}$. Using Krein-Milman's theorem, we have that $\eta(\alpha, t) = \sup\{\|\alpha x + (1-\alpha)ty\|^2 + \alpha(1-\alpha)\|x - ty\|^2 : x, y \in \mathcal{E}(B_X)\}$.

Let $a, b, c, d \ge 0$, be such that $a^2 + b^2 = c^2 + d^2 = 1$ and let $t \in [0,1]$. First take $x = (a,b)$ and $y = (c,d)$. It is clear that $\|\alpha x + (1-\alpha)ty\|_{2,1} = \|\alpha x + (1-\alpha)ty\|_2$. If $\|x - ty\|_{2,1} = \|x - ty\|_2$, then we have

$$\|\alpha x + (1-\alpha)ty\|_{2,1}^2 + \alpha(1-\alpha)\|x - ty\|_{2,1}^2 = \alpha + (1-\alpha)t^2.$$

If $\|x - ty\|_{2,1} = \|x - ty\|_1$, then

$$\begin{aligned}&\|\alpha x + (1-\alpha)ty\|_{2,1}^2 + \alpha(1-\alpha)\|x - ty\|_{2,1}^2 \\ &\qquad = \alpha + (1-\alpha)t^2 + 2\alpha(1-\alpha)((ad + cb)t - ab - t^2 cd) \\ &\qquad \le \alpha + (1-\alpha)t^2 + 2\alpha(1-\alpha)t\end{aligned}$$

because $ad + cb \le 1$.

If we consider $x = (a,b)$ and $y = (-c,-d)$, we get the same inequality.

Thus $\eta(\alpha,t) \leq \alpha + (1-\alpha)t^2 + 2\alpha(1-\alpha)t$. For $x = (1,0), y = (0,1)$, we get equality.

If $0 < \alpha \leq \frac{1}{2}$, it can be proved that $\sup_{t\in[0,1]} \{\frac{\eta(\alpha,t)}{\alpha+(1-\alpha)t^2}\} = 1 + \sqrt{\alpha(1-\alpha)} = C_\alpha(X)$.

Example 1.18. Let $X = \ell_\infty - \ell_1 = (\mathbb{R}^2, \|\cdot\|_{\infty,1})$, where:

$$\|x\|_{\infty,1} = \begin{cases} \|x\|_\infty & \text{if} \quad x_1x_2 \geq 0, \\ \|x\|_1 & \text{if} \quad x_1x_2 \leq 0, \end{cases}$$

for $x = (x_1, x_2) \in X$. If $1 \leq \alpha \leq \frac{1}{2}$, then $C_\alpha(X) = 1 + \frac{-\alpha+\sqrt{4\alpha-3\alpha^2}}{2}$ and the maximum value is $C_{\frac{1}{3}}(X) = \frac{4}{3}$.

Take $\alpha \in (0, \frac{1}{2}]$. We can see that $\mathcal{E}(B_X)$ consists of 6 points. Using again Krein-Milman's theorem,

$$\eta(\alpha,t) = \begin{cases} \alpha + (1-\alpha)^2t^2 + 2\alpha(1-\alpha) & 0 \leq t \leq \sqrt{\frac{\alpha}{1-\alpha}}, \\ \alpha(1-\alpha) + (1-\alpha)t^2 + 2\alpha(1-\alpha) & \sqrt{\frac{\alpha}{1-\alpha}} \leq t \leq 1 \end{cases}$$

thus

$$\sup_{t\in[0,1]} \eta(\alpha,t) = 1 + \frac{-\alpha + \sqrt{4\alpha - 3\alpha^2}}{2} = C_\alpha(X).$$

Example 1.19. Let us consider the ℓ_p spaces, for $p \in [1,2]$. Take $\alpha \in (0, 1/2]$, $x_0 = (0,1,0\dots)$, $y_0 = (1,0,\dots)$ and $t \in [0,1]$. By definition:

$$\begin{aligned} C_\alpha(\ell_p) \geq & \frac{\|\alpha x_0 + (1-\alpha)ty_0\|_p^2 + \alpha(1-\alpha)\|x_0 - ty_0\|_p^2}{\alpha\|x\|_p^2 + (1-\alpha)\|ty_0\|^2} \\ = & \frac{(\alpha^p + (1-\alpha)^p t^p)^{2/p} + \alpha(1-\alpha)(1+t^p)^{2/p}}{\alpha + (1-\alpha)t^2} = f(t), \end{aligned}$$

and evaluating at $t = \sqrt{\frac{\alpha}{1-\alpha}}$ we have

$$C_\alpha(\ell_p) \geq (\alpha^{p/2} + (1-\alpha)^{p/2})^{2/p}. \tag{1.5}$$

If we take $p = 1$ or $p = 2$, we have equality in (1.5) for every $\alpha \in (0, 1/2]$. Note also that

$$\lim_{p\to 1^+} (\alpha^{p/2} + (1-\alpha)^{p/2})^{2/p} = 1 + 2\sqrt{\alpha(1-\alpha)} = C_\alpha(\ell_1).$$

If we take $\alpha = 1/2$, then $C_{1/2}(\ell_p) = C_{NJ}(\ell_p) \geq 2^{2/p-1}$, in fact, the equality was proved by Clarkson (see [2]).

1.1. **Some applications to fixed point theory.** Applying some of the previous results, we want to study the existence of fixed points for k-Lipschitzian rotative mappings:

Definition 1.20. Let $C \subset X$ be a subset of a Banach space X and $T : C \to C$. We say that T is a Lipschitzian mapping if there is $k > 0$ such that $\|Tx - Ty\| \leq k\|x - y\|$ for any $x, y \in C$ and we will write $T \in \mathscr{L}(k)$. If k_0 is the minimum number such that $T \in \mathscr{L}(k)$ we will write $T \in \mathscr{L}_0(k_0)$. If $T \in \mathscr{L}(1)$ we will say that T is nonexpansive.

Definition 1.21. Let $T : C \to C$ where C is a nonempty, closed and convex subset of a Banach space X. We will say that T is an (a, n) rotative mapping if $0 \le a < n$ and for any $x \in C$, $\|x - T^n x\| \le a\|x - Tx\|$. If T is an (a, n) rotative mapping for some $n \in \mathbb{N}$ and some $a < n$, we will say that T is a rotative mapping. If $a = 0$, we will say that T is an n-periodic mapping.

In 1981 K. Goebel and M. Koter, proved the following, see [4] and [5]:

Theorem 1.22. *If C is a nonempty, closed and convex subset of a Banach space, then any nonexpansive and rotative mapping $T : C \to C$ has a fixed point.*

Let us define the following:

$$\gamma_n^X(a) = \inf\{k : \exists C \subset X, T : C \to C, (a, n) - \text{rotative}, T \in \mathscr{L}_0(k), \text{Fix}(T) = \emptyset\},$$

where C is a nonempty, closed and convex subset of a Banach space X. If $a = 0$, we will write γ_n^X instead of $\gamma_n^X(0)$.

In [3, pp.179-180] it was shown that for any $0 \le a < n$, $\gamma_n^X(a) > 1$. The exact value of $\gamma_n^X(a)$ is unknown, even more, it is not known if is bounded.

The next lemma and its proof are similar to a result by J. Górnicki and K. Pupka in [7].

Lemma 1.23. *Let X be a Banach space and $T : X \to X$ a continuous function. If there are $0 < A < 1$, $B > 0$ and $\{u_n\}_{n=0}^{\infty} \subset X$ such that for $n \ge 1$*

$$\mathrm{d}(Tu_n, u_n) \le A\,\mathrm{d}(Tu_{n-1}, u_{n-1})$$

and

$$\mathrm{d}(u_n, u_{n-1}) \le B\,\mathrm{d}(Tu_{n-1}, u_{n-1}),$$

then $z = \lim_{n\to\infty} u_n$ is a fixed point of T.

Proposition 1.24. *Let X be a Banach space. Then*

$$\gamma_2^X \ge \sqrt{\frac{5}{C_{1/2}(X)}}.$$

Proof. Let X be a Banach space, C a nonempty, closed and convex subset of X and $\alpha = 1/2$. Let $T : C \to C$ be a k-Lipschitzian and 2-periodic mapping. For $x \in C$ let us define $u(x) = \alpha x + (1-\alpha)Tx$, using the definition of $C_\alpha(X)$ we have that for every $u, v \in X$, $\|\alpha u + (1-\alpha)v\|^2 \le C_\alpha(X)(\alpha\|u\|^2 + (1-\alpha)\|v\|^2) - \alpha(1-\alpha)\|u-v\|^2$. Then we have:

$$\|x - Tu(x)\| = \|T^2x - Tu(x)\| \le k\|u(x) - Tx\| = \alpha k\|x - Tx\|$$

and

$$\|Tx - Tu(x)\| \le k\|u(x) - x\| = (1-\alpha)k\|x - Tx\|.$$

Hence

$$\begin{aligned}\|u(x)-Tu(x)\|^2 =&\|\alpha(x-Tu(x)+(1-\alpha)(Tx-Tu(x)))\|^2\\ \le& C_\alpha(X)\left(\alpha\|x-Tu(x)\|^2+(1-\alpha)\|Tx-Tu(x)\|^2\right)\\ &-\alpha(1-\alpha)\|x-Tx\|^2\\ \le&\left\{C_\alpha(X)\left[\alpha^3+(1-\alpha)^3\right]k^2-\alpha(1-\alpha)\right\}\|x-Tx\|^2\\ =&\left\{C_{\frac{1}{2}}(X)\left[\frac{k^2}{4}\right]-\frac{1}{4}\right\}\|x-Tx\|^2.\end{aligned}$$

If we set $u_n = u^n(x)$ for $n \in \mathbb{N}$ and $u_0 = x$, applying lemma 1.23, if $C_{1/2}(X)\left[k^2/4\right] -1/4 < 1$ or equivalently if $k < \sqrt{\frac{5}{C_{1/2}(X)}}$, then $\mathrm{Fix}(T) \neq \emptyset$. □

From the above, if we consider a Banach space X such that $1 \le C_{1/2}(X) < \frac{5}{4}$, then $\gamma_2^X > 2$, improving in this case the bound obtained by K. Goebel and E. Złotkiewicz [6] in 1971 who proved that for every Banach space X, $\gamma_2^X \ge 2$.

Proposition 1.25. *Let X be a Banach space and $\alpha = 0.346$. If $1 \le C_\alpha(X) < 1.1136$ then $\gamma_3^X > 1.3821$.*

Proof. Let X be a Banach space, C a nonempty, closed and convex subset of X and $T : C \to C$, $T \in \mathscr{L}(k)$, $T^3 = Id$. For $x \in C$, define

$$\begin{aligned}x_0 &= x \in C,\\ x_1 &= \alpha x_0 + (1-\alpha)Tx_0,\\ x_2 &= \alpha x_0 + (1-\alpha)Tx_1.\end{aligned}$$

By definition of $C_\alpha(X)$ we get

$$\begin{aligned}\|x_2-Tx_2\|^2 =&\|\alpha(x_0-Tx_2)+(1-\alpha)(Tx_1-Tx_2)\|^2\\ \le& C_\alpha(X)\left(\alpha\|x_0-Tx_2\|^2+(1-\alpha)\|Tx_1-Tx_2\|^2\right)\\ &-\alpha(1-\alpha)\|x_0-Tx_1\|^2\\ \le& C_\alpha(X)\left(\alpha k^2\|T^2x_0-x_2\|^2+(1-\alpha)k^2\|x_1-x_2\|^2\right)\\ &-\alpha(1-\alpha)\|x_0-Tx_1\|^2\\ \le& C_\alpha(X)(\alpha k^2\|\alpha(x_0-T^2x_0)+(1-\alpha)(Tx_1-T^2x_0)\|^2\\ &+(1-\alpha)^5k^4\|x_0-Tx_0\|^2)-\alpha(1-\alpha)\|x_0-Tx_1\|^2.\end{aligned}$$

Also

$$\begin{aligned}&\|\alpha(x_0-T^2x_0)+(1-\alpha)(Tx_1-T^2x_0)\|^2\\ &\quad\le C_\alpha(X)(\alpha\|x_0-T^2x_0\|^2+(1-\alpha)\|Tx_1-T^2x_0\|^2)-\alpha(1-\alpha)\|x_0-Tx_1\|^2\\ &\quad\le C_\alpha(X)(\alpha k^4\|Tx_0-x_0\|^2+(1-\alpha)k^2\alpha^2\|x_0-Tx_0\|^2)-\alpha(1-\alpha)\|x_0-Tx_1\|^2,\end{aligned}$$

hence

$$\begin{aligned}\|x_2-Tx_2\|^2 \le&\alpha^2k^6C_\alpha(X)^2\|x_0-Tx_0\|^2+\alpha^3(1-\alpha)k^4C_\alpha(X)^2\|x_0-Tx_0\|^2\\ &+(1-\alpha)^5k^4C_\alpha(X)\|x_0-Tx_0\|^2 \qquad (1.6)\\ &-\alpha(1-\alpha)[\alpha k^2C_\alpha(X)+1]\|x_0-Tx_1\|^2.\end{aligned}$$

Now, let us consider

$$\begin{aligned}\|x_1 - Tx_1\|^2 =&\|\alpha(x_0 - Tx_1) + (1-\alpha)(Tx_0 - Tx_1)\|^2\\ \le&\alpha C_\alpha(X)\|x_0 - Tx_1\|^2 + (1-\alpha)^3k^2C_\alpha(X)\|x_0 - Tx_0\|^2\\ &- \alpha(1-\alpha)\|x_0 - Tx_0\|^2.\end{aligned}$$

Suppose that for some ε, $\|x_1 - Tx_1\|^2 \ge (1-\varepsilon)\|x_0 - Tx_0\|^2$, then

$$-\alpha\|x_0 - Tx_1\|^2 \le \frac{-(1-\varepsilon) + (1-\alpha)^3k^2C_\alpha(X) - \alpha(1-\alpha)}{C_\alpha(X)}\|x_0 - Tx_0\|^2.$$

This together with inequality (1.6) gives:

$$\begin{aligned}\|x_2 - Tx_2\|^2 \le&[\alpha^2k^6C_\alpha(X)^2 + \alpha^3(1-\alpha)k^4C_\alpha(X)^2 + (1-\alpha)^5k^4C_\alpha(X)\\ &+ \alpha(1-\alpha)k^2(-(1-\varepsilon) + (1-\alpha)^3k^2C_\alpha(X) - \alpha(1-\alpha))\\ &+ \frac{1-\alpha}{C_\alpha(X)}(-(1-\varepsilon) + (1-\alpha)^3k^2C_\alpha(X) - \alpha(1-\alpha))]\|x_0 - Tx_0\|^2.\end{aligned}$$

Let $y_0 = x$. Suppose we have $y_0, \dots, y_m$. Let $y_m^0 = y_m$ and $y_m^i = \alpha y_m + (1-\alpha)Ty_m^{i-1}$, for $i = 1, 2$.

If $\|y_m^1 - Ty_m^1\|^2 < (1-\varepsilon)\|y_m - Ty_m\|^2$, we take $y_{m+1} = y_m^1$; if $\|y_m^1 - Ty_m^1\|^2 \ge (1-\varepsilon)\|y_m - Ty_m\|^2$, let $y_{m+1} = y_m^2$.

Thus, by lemma 1.23, $\{y_m\}$ converges to a fixed point of T, provided that

(1.7)
$$\begin{aligned}&\alpha^2k^6C_\alpha(X)^2 + \alpha^3(1-\alpha)k^4C_\alpha(X)^2 + (1-\alpha)^5k^4C_\alpha(X) - \alpha(1-\alpha)k^2\\ &+\alpha(1-\alpha)^4k^4C_\alpha(X) - \alpha^2(1-\alpha)^2k^2 - \frac{1-\alpha}{C_\alpha(X)} + (1-\alpha)^4k^2 - \frac{\alpha(1-\alpha)^2}{C_\alpha(X)} < 1.\end{aligned}$$

For $C_\alpha(X) = 1$ the best solution with this method is for $\alpha = 0.346$. Taking this α and if X is such that $1 \le C_\alpha(X) < 1.1136$, the last inequality holds for $k < k_0(X)$, where $k_0(X) > 1.3821$, that is, for these spaces X, $\gamma_3^X \ge k_0(X) > 1.3821$ which improves in this case the bound given by J. Górnicki and K. Pupka ([7]) for general Banach spaces. □

Instead of using $C_\alpha(X)$ one could also work with the following constant:

Let X be a Banach space and $\alpha \in (0,1)$. We define

$$D_\alpha(X) = \sup\left\{\frac{\|\alpha x + (1-\alpha)y\|^2 + \|\alpha x - (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 + \alpha(1-\alpha)\|x+y\|^2}{2(\alpha\|x\|^2 + (1-\alpha)\|y\|^2)}\right\}$$

where the supremum is taken over $x, y \in X$ no both zero.

$D_\alpha(X)$ has the following properties which are obtained similarly to those of $C_\alpha(X)$.

(1) For $\alpha \in (0,1)$, $D_{1-\alpha}(X) = D_\alpha(X) \le C_\alpha(X)$.
(2) $D_{1/2}(X) = C_{NJ}(X)$.
(3) $1 \le D_\alpha(X) \le 1 + 2\sqrt{\alpha(1-\alpha)}$.
(4) If H is a Hilbert space, then $D_\alpha(H) = 1$.
(5) X is uniformly nonsquare if and only if $D_\alpha(X) < 1 + 2\sqrt{\alpha(1-\alpha)}$.

Let $\alpha \in (0,1)$ and $t \in [0,1]$. We define

$$\varphi(\alpha,t) = \frac{1}{2} \sup_{x,y\in S_X} \left\{ \|\alpha x + (1-\alpha)ty\|^2 + \|\alpha x - (1-\alpha)ty\|^2 \right.$$
$$\left. + \alpha(1-\alpha)(\|x-ty\|^2 + \|x+ty\|^2) \right\}.$$

Proposition 1.26. *Let $\alpha \in (0,1)$. The function $\varphi(\alpha,t)$ of the variable t is continuous on $[0,1)$ and is convex and nondecreasing on $[0,1]$.*

Corollary 1.27. *Let X be a Banach space and $\alpha \in (0,1/2]$. Then*

$$D_\alpha(X) = \sup_{t\in[0,1]} \frac{\varphi(\alpha,t)}{\alpha+(1-\alpha)t^2}.$$

The problem of this constant is that we don't know if $D_\alpha(X) = D_\alpha(X^*)$ or if there is a Banach space X which is not a Hilbert space with $D_\alpha(X) = 1$. However in this case we were able to calculate $D_\alpha(\ell_p)$ for $p \geq 2$.

Example 1.28. Let $p \geq 2$ and $\alpha \in (0,1/2]$. Then

$$D_\alpha(\ell_p) = \left(\frac{\left(\sqrt{1-\alpha}+\sqrt{\alpha}\right)^p + \left(\sqrt{1-\alpha}-\sqrt{\alpha}\right)^p}{2} \right)^{2/p}.$$

Proof. We will use Clarkson's inequality (see [1]): for $p \geq 2$ and $x, y \in \ell_p$

$$\|x+y\|^p + \|x-y\|^p \leq (\|x\|+\|y\|)^p + |(\|x\|-\|y\|)|^p.$$

Using this inequality and the definition of $\varphi(\alpha,t)$ we have that for $\alpha \in (0,1/2], t \in [0,1]$ and $x, y \in \ell_p$ with $\|x\| = \|y\| = 1$:

$$\frac{1}{2}\left(\|\alpha x + (1-\alpha)yt\|^2 + \|\alpha x - (1-\alpha)yt\|^2 + \alpha(1-\alpha)(\|x-yt\|^2 + \|x+yt\|^2)\right)$$
$$\leq \frac{2^{1-2/p}}{2}\left((\alpha+(1-\alpha)t)^p + |\alpha-(1-\alpha)t|^p + \alpha(1-\alpha)((1-t)^p + (1+t)^p)\right) = h(t).$$

Hence, $\varphi(\alpha,t) \leq h(t)$. If one take $x = (1/2^{1/p}, 1/2^{1/p}, 0, \dots)$ and $y = (1/2^{1/p}, -1/2^{1/p}, 0, \dots)$ we have the equality, thus $\varphi(\alpha,t) = h(t)$ and

$$D_\alpha(\ell_p) = \sup_{t\in[0,1]} \frac{\varphi(\alpha,t)}{\alpha+(1-\alpha)t^2} = \sup_{t\in[0,1]} \frac{h(t)}{\alpha+(1-\alpha)t^2}.$$

The maximum is attained at $t_0 = \sqrt{\frac{\alpha}{1-\alpha}}$, and $h(t_0) = \left(\frac{\left(\sqrt{1-\alpha}+\sqrt{\alpha}\right)^p + \left(\sqrt{1-\alpha}-\sqrt{\alpha}\right)^p}{2} \right)^{2/p}$. □

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