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VARIOUS CONVERGENCE RESULTS IN STRONG LAW OF LARGE NUMBERS FOR DOUBLE ARRAY OF RANDOM SETS IN BANACH SPACES

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Abstract. In this paper, we state several convergence results with respect to the slice and Wijsman topologies of strong laws of large numbers for double array of independent (or pairwise independent) random sets in a separable Banach space with or without geometric property. We also provide some typical examples illustrating this study.

1. Introduction

In recent decades, the strong laws of large numbers (SLLN) for sequence of unbounded random sets, gave rise to applications in several fields, such as optimization and control, stochastic and integral geometry, mathematical economics, statistics and related fields. Z.Artstein and S. Hart [2] obtained a SLLN for independent identically distributed (i.i.d.) random sets having values in the closed (possibly unbounded) subsets of $\mathbf{R}^d$ and applied it to a problem of optimal allocations. Later, F. Hiai [19] and C. Hess [15, 16] independently proved similar results for random sets in an infinite dimensional Banach space, with respect to the Kuratowski-Mosco convergence. In [23], H. Inoue and R. L. Taylor derive laws of large numbers for exchangeable random sets in Kuratowski-Mosco sense. Recently, K. A. Fu [13] obtained some SLLN for sequence of identically distributed random sets or fuzzy random sets with $\varphi(\varphi^*)$-mixing dependence in a separable Banach space. The strong laws of large numbers for these two sequences are derived under Kuratowski-Mosco sense. The multivalued SLLN for random sets in Fréchet spaces was proved by P. Raynaud de Fitte [26]. Moreover, C. Hess [18] obtained a convergence result on the Wijsman topology of SLLN for sequence of pairwise independent identically distributed (p.i.i.d.) random sets in a separable Banach space. Let us mention also the work of Attouch-Wets [3] and Castaing-Ezzaki [8] dealing with the law of large numbers for normal integrands. In this context, C. Hess [17] derived from the strong law of large numbers with respect to the Wijsman convergence for sequence of pairwise independent identically distributed integrable random sets and the slice convergence for these objects. In a different context, J. Hoffmann-Jørgensen, G. Pisier proved an important norm a.s. convergence for sequence of independent identically distributed integrable random variables $\{X_n : n \geq 1\}$ in Rademacher

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type p $(1 \le p \le 2)$ Banach space satisfying

$$EX_n = 0 \ \forall n \ge 1 \tag{1.1}$$

$$\sum_{n=1}^{\infty} n^{-p} E\|X_n\|^p < +\infty \ \text{(Chung's condition)}. \tag{1.2}$$

Beck [4] proved that in B-convex space, the strong law of large numbers holds for sequence of independent identically distributed integrable random variables $\{X_n : n \ge 1\}$ satisfying (1.1) and

$$\sup_n E\|X_n\|^2 < +\infty. \tag{1.3}$$

J. Hoffmann-Jørgensen, G. Pisier's result contains Beck's result because a B-convex space is of Rademacher type p with $1 \le p \le 2$.

In this work we present various convergence results in the strong law of large numbers for double array of independent (or pairwise independent) identically distributed closed valued random variables in a separable Banach space with respect to the slice and Wijsman topologies.

Here we also provide two probabilistically constructive methods allowing to prove these type of convergence.

This paper is organized as follows. In Section 2, we introduce some basic notions of random sets and the slice and Wijsman topologies. Section 3 is concerned with Wijsman topology of SLLN for double array of pairwise independent identically distributed closed valued random sets in a separable Banach space. In Section 4, we treat the SLLN in Wijsman topology for double array of independent closed valued random sets in Rademacher type p Banach spaces $(1 \le p \le 2)$, in particular, some illustrating examples are provided. In Section 5, we present the applications of the results obtained in Sections 3-4 to the SLLN in slice topology for double array of independent (or pairwise independent) integrable closed valued random sets.

Our results contain the above mentioned results and some related results in the literature.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space, $(\mathfrak{X}, \|.\|)$ be a separable Banach space and $\mathfrak{X}^*$ be its topological dual. The closed unit ball of $\mathfrak{X}^*$ is denoted by B^* and σ-field of all Borel sets of $\mathfrak{X}$ is denoted by $\mathcal{B}(\mathfrak{X})$. Also denote by $\mathcal{M}(\mathfrak{X})$ (resp. $M(\mathfrak{X})$) the set of all finite measures (resp. probability measures) on $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$. The vector space $\mathcal{M}(\mathfrak{X})$ is endowed with the *narrow topology*, namely the topology $\sigma(\mathcal{M}(\mathfrak{X}), C_b(\mathfrak{X}))$, where $C_b(\mathfrak{X})$ denotes the space of real-valued, bounded continuous functions on $\mathfrak{X}$. Recall that $M(\mathfrak{X})$ is a closed convex subset of $\mathcal{M}(\mathfrak{X})$. For each measurable function $f : \Omega \to \mathfrak{X}$, the distribution of f is denoted by μ_f and defined on $\mathcal{B}(\mathfrak{X})$ by $\mu_f(B) = \mathbf{P}\{f^{-1}(B)\}$ $(B \in \mathcal{B}(\mathfrak{X}))$. In the present paper, $\mathbf{N}^*$ will be denoted the set of positive integers, $\mathbf{R}$ (resp. $\mathbf{R}^+$) the set of real numbers (resp. positive real numbers).

Let $c(\mathfrak{X})$ (resp. $cc(\mathfrak{X})$) (resp. $cb(\mathfrak{X})$) (resp. $cwk(\mathfrak{X})$) (resp. $k(\mathfrak{X})$) (resp. $ck(\mathfrak{X})$) be the family of all nonempty closed (resp. closed convex) (resp. closed bounded convex) (resp. convex weakly compact) (resp. compact) (resp. convex compact) subsets of $\mathfrak{X}$ and $\mathcal{E}$ the Effros σ-field on $c(\mathfrak{X})$. This σ-field is generated by the subsets $U^- = \{F \in c(\mathfrak{X}) : F \cap U \neq \emptyset\}$, where U ranges over the open subsets of $\mathfrak{X}$. On the other hand, for each $A, C \subset \mathfrak{X}$, cl$C$, $w-$cl$C$, co$C$ and $\overline{\text{co}}C$ denote the *norm-closure*, the *weak-closure*, the *convex hull* and the *closed convex hull* of C, respectively; the *distance function* $d(.,C)$ of C, the *gap* between A and C, the *Hausdorff distance* $d_H(A,C)$ of A and C, and the *support function* $\delta^*(.,C)$ of C are defined by

$$\begin{aligned} d(x,C) &= \inf\{\|x-y\| \ : y \in C\}, (x \in \mathfrak{X}), \\ D(A,C) &= \inf\{\|x-y\| : x \in A, y \in C\}, \\ d_H(A,C) &= \max\{\sup_{x\in A} d(x,C), \sup_{y \in C} d(y,A)\}, \\ \delta^*(x^*,C) &= \sup\{\langle x^*, y\rangle \ : y \in C\}, (x^* \in \mathfrak{X}^*). \end{aligned}$$

We note that

$$\delta^*(x^*,C) = \delta^*(x^*,\overline{\text{co}}C). \tag{2.1}$$

A *slice of a ball* is the intersection of a closed ball (of radius r and centered at x_0)

$$B(x_0,r), \ \ (x_0 \in \mathfrak{X}, r > 0)$$

and a closed half space

$$F(z,\alpha) := \{x \in \mathfrak{X} : \langle z, x\rangle \geq \alpha\}, \ \ (z \in \mathfrak{X}^*, z \neq 0, \alpha \in \mathbf{R}).$$

Let t be a topology on $\mathfrak{X}$ and $(C_n)_{n\geq 1}$ be a sequence in $c(\mathfrak{X})$. We put

$$\begin{aligned} t - liC_n &= \{x \in \mathfrak{X} : x = t - \lim x_n, \ x_n \in C_n, \forall n \geq 1\}, \\ t - lsC_n &= \{x \in \mathfrak{X} : x = t - \lim x_k, \ x_k \in C_{n(k)}, \forall k \geq 1\} \end{aligned}$$

where $(C_{n(k)})_{k\geq 1}$ is a subsequence of $(C_n)_{n\geq 1}$. The subsets $t-liC_n$ and $t-lsC_n$ are the *lower limit* and the *upper limit* of $(C_n)_{n\geq 1}$, relative to topology t. We obviously have $t - liC_n \subset t - lsC_n$. Let us denote by s the strong topology of $\mathfrak{X}$.

A map X from Ω into $c(\mathfrak{X})$ is also called a *multifunction* with closed values in $\mathfrak{X}$. The *domain* and the *graph* of X are respectively defined by

$$\text{dom}(X) = \{\omega \in \Omega : X(\omega) \neq \emptyset\} \ \text{ and } \ \text{Gr}(X) = \{(\omega, x) \in \Omega \times \mathfrak{X} : x \in X(\omega)\}.$$

X is said to be *Effros measurable* or *weakly measurable* in the terminology of Himmelberg [21] (or simply "*measurable*") if for every B in $\mathcal{E}$, $X^{-1}(B)$ is a member of $\mathcal{A}$. From the definition of the Effros σ-field it follows that X is measurable if and only if, for any open subset U of $\mathfrak{X}$,

$$X^{-1}(U^-) = \{\omega \in \Omega : X(\omega) \cap U \neq \emptyset\}$$

is a member of $\mathcal{A}$ ($X^{-1}(U^-)$ is also denoted by $X^{-1}(U)$). The sub-σ-field $X^{-1}(\mathcal{E})$ generated by X is denoted by $\mathcal{A}_X$.

A measurable multifunction defined on a probability space is also called a *random set* (r.s.). Like for real or vector valued random variables, the *distribution* μ_X of the measurable multifunction X can be defined on the measurable space $(c(\mathfrak{X}), \mathcal{E})$ by

$$\mu_X(B) = \mathbf{P}\{X^{-1}(B)\}, \ \ \forall B \in \mathcal{E}.$$

C. Hess ([18], Proposition 2.1) showed that the closed valued random sets X and Y have the same distribution on $(c(\mathfrak{X}), \mathcal{E})$ if and only if for any finite subset $F =$

$\{x_1, \ldots, x_k\}$ of $\mathfrak{X}$ (or of some countable dense subset), the $\mathbf{R}^k$-valued random vectors $(d(x,X))_{x\in F}$ and $(d(x,Y))_{x\in F}$ have the same distribution.

A finite set of random sets $\{X_1, \ldots, X_n\}$ is said to be *independent* if

$$\mathbf{P}\{X_1 \in \mathcal{X}_1, X_2 \in \mathcal{X}_2, \ldots, X_n \in \mathcal{X}_n\} = \mathbf{P}\{X_1 \in \mathcal{X}_1\}.\mathbf{P}\{X_2 \in \mathcal{X}_2\} \ldots \mathbf{P}\{X_n \in \mathcal{X}_n\},$$

for all $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n \in \mathcal{E}$. It is known that the finite set of closed valued random variables $\{X_1, \ldots, X_n\}$ is independent if and only if

$$\mathbf{P}\{X_1 \cap K_1 \neq \emptyset, X_2 \cap K_2 \neq \emptyset, \ldots, X_n \cap K_n \neq \emptyset\} = \prod_{i=1}^{n} \mathbf{P}\{X_i \cap K_i \neq \emptyset\}$$

for all $K_1, K_2, \ldots, K_n \in c(\mathfrak{X})$.

A infinite set of random sets $\{X_i, i \in I\}$ is said to be *independent* if every nonempty finite subset $\{X_{i_1}, X_{i_2}, \ldots, X_{i_n}\} \subset \{X_i, i \in I\}$ is independent.

A set of random sets $\{X_i, i \in I\}$ is said to be *pairwise independent* if each pair of random sets $\{X_{i_1}, X_{i_2}\} \subset \{X_i, i \in I\}$ is independent.

For every sub-σ-field $\mathcal{F}$ of $\mathcal{A}$, consider the space $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X})$ of all measurable functions from $(\Omega, \mathcal{F})$ into $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$. Further, define the two following subsets associated with the multifunction X involving its measurable selections:

$$S(X,\mathcal{F}) = \{f \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X}) : f(\omega) \in X(\omega), \text{ for almost every } \omega \in \mathrm{dom}(X)\}$$
$$M(X,\mathcal{F}) = \{\mu = \mu_f \in M(\mathfrak{X}) : f \in S(X,\mathcal{F})\}.$$

So, $S(X,\mathcal{F})$ is the set of $\mathcal{F}$-measurable selections of X and $M(X,\mathcal{F})$ is the set of all probability measures μ on $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$ such that each $\mu \in M(X,\mathcal{F})$ is the distribution of some $\mathcal{F}$-measurable selection of X. By Kuratowski Ryll-Nardzewski Theorem every Effros measurable multifunction, X admits at least one measurable selection. Moreover, X admits a *Castaing representation*, that is, a sequence (f_n) of measurable selections, such that for every $\omega \in \mathrm{dom}(X)$, $X(\omega)$ is equal to the closure of the countable subset $\{f_n(\omega) : n \geq 1\}$ (see [10]).

It is known that $\mathcal{L}^0(\Omega, \mathcal{A}, \mathbf{P}, \mathfrak{X})$ $(= \mathcal{L}^0(\mathfrak{X}))$ endowed with the topology of convergence in probability is a metrizable topological vector space. Since a sequence converging in probability admits an almost sure converging subsequence it is clear that, for any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$, the set $S(X,\mathcal{F})$ is closed in $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X})$.

For $1 \leq p < \infty$, $\mathcal{L}^p(\Omega, \mathcal{A}, \mathbf{P}, \mathfrak{X})$ $(= \mathcal{L}^p(\mathfrak{X}))$ denotes the subspace of $\mathcal{L}^0(\mathfrak{X})$ whose members $f : \Omega \to \mathfrak{X}$ satisfy

$$\|f\|_p = (E\|f\|^p)^{\frac{1}{p}} = \left(\int_\Omega \|f(\omega)\|^p \mathbf{P}(d\omega)\right)^{\frac{1}{p}} < +\infty.$$

$\mathcal{L}^p(\mathbf{R})$ is denoted by $\mathcal{L}^p$. We denote by $M^1(\mathfrak{X})$ the subset of $M(\mathfrak{X})$ whose members μ satisfy $\int_{\mathfrak{X}} \|x\| d\mu < +\infty$. Given a sub-$\sigma$-field $\mathcal{F}$ of $\mathcal{A}$ and a random set X, define the following subsets of $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X})$ and $M(\mathfrak{X})$ respectively

$$S^p(X,\mathcal{F}) = \{f \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X}) : f(\omega) \in X(\omega), \text{ for almost every } \omega \in \mathrm{dom}(X)\}$$
$$M^p(X,\mathcal{F}) = \{\mu_f : f \in S^p(X,\mathcal{F})\}.$$

Using standard measurable selection arguments, it is not hard to see that, when $\mathcal{A}_X \subseteq \mathcal{F} \subseteq \mathcal{A}$, the set $S^1(X,\mathcal{F})$ is non empty if and only if the positive function

$d(0, X)$ is integrable. In such a situation, we shall say that the multifunction X is *integrable*. Observe that when X is integrable, $\mathbf{P}(\mathrm{dom}(X)) = 1$. On the other hand, X is said to be *integrably bounded* if the function $|X|(\omega) = \sup\{\|x\| : x \in X(\omega)\}$ is integrable. In this case, we have $S^1(X, \mathcal{A}) = S(X, \mathcal{A})$. An integrably bounded multifunction is also integrable, but the converse implication is false. For any measurable multifunction X and any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$, the *multivalued integral* of X over Ω, with respect to $\mathcal{F}$, is defined by

$$E(X, \mathcal{F}) = \{E(f) : f \in S^1(X, \mathcal{F})\},$$

where $E(f) = \int_\Omega f d\mathbf{P}$ is the usual Bochner integral of f. We note that $E(X, \mathcal{A})$ is not always closed. $E(X, \mathcal{A})$ is non empty if and only if X is integrable. Now, consider an integrable multifunction X. Obviously, the inclusion $M^1(X, \mathcal{F}) \subseteq M(X, \mathcal{F})$ holds for any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$. C. Hess (see [18], Lemma 2.4) showed that, the following equality holds true

$$\mathrm{cl}\, M^1(X, \mathcal{F}) = \mathrm{cl}\, M(X, \mathcal{F})$$

the closure being taken in $\mathcal{M}(\mathfrak{X})$ (or $M(\mathfrak{X})$) in the narrow topology.

The Wijsman topology $\mathcal{T}_W$ on $c(\mathfrak{X})$ is the topology of pointwise convergence of distance functions. Recall that a net (C_α) of closed sets is said to *converge to C in the Wijsman topology* if, for every $x \in \mathfrak{X}$, one has

$$d(x, C) = \lim_\alpha d(x, C_\alpha).$$

It is known that for any closed convex subset C of $\mathfrak{X}$ and for any $x \in \mathfrak{X}$, we have

$$d(x, C) = \sup_{z \in B^*} \{\langle z, x\rangle - \delta^*(z, C)\}. \tag{2.2}$$

C. Hess (see [18], Lemma 3.1) showed that, for any closed convex subset C of $\mathfrak{X}$, there exists a countable subset D^* of B^* verifying, for any $x \in \mathfrak{X}$,

$$d(x, C) = \sup_{z \in D^*} \{\langle z, x\rangle - \delta^*(z, C)\}. \tag{2.3}$$

The *slice topology* on $c(\mathfrak{X})$ is the narrow topology $\mathcal{T}_S$ determined by the following family of gap functionals

$$\{D(A, \cdot) : A \text{ is a nonempty slice of a ball}\}.$$

The slice topology is generally stronger than both the Kuratowski-Mosco topology and the Wijsman topology; it coincides with the Kuratowski-Mosco topology if and only if $\mathfrak{X}$ is reflexive [6]. From Theorem 5.2 in [6], we know that the slice topology restricted to $cc(\mathfrak{X})$ coincides with the narrow topology $\mathcal{T}$ determined by the following family of gap functionals

$$\{D(B, \cdot) : B \in cb(\mathfrak{X})\}.$$

A real separable Banach space is of Rademacher type $p\,(1 \le p \le 2)$ if and only if there exists a constant $0 < C < \infty$ such that

$$E\Big\| \sum_{j=1}^n f_j \Big\|^p \le C \sum_{j=1}^n E\|f_j\|^p$$

for every finite collection $\{f_1, f_2, \dots, f_n\}$ of independent mean 0 integrable random elements.

If a real separable Banach space is of Rademacher type p for some $1 < p \le 2$, then it is of Rademacher type q for all $1 \le q < p$. Every real separable Banach space is of Rademacher type (at least) 1 while the $\mathcal{L}_p$-spaces and l_p-spaces are of Rademacher type $2 \wedge p$ for $p \ge 1$. Every real separable Hilbert space and real separable finite-dimensional Banach space is of Rademacher type 2. In particular, the real line $\mathbf{R}$ is of Rademacher type 2. See [22] for details and proofs.

A double array $\{f_{mn} : m \ge 1, n \ge 1\}$ of random elements is said to be *stochastically dominated* by a random element f if for some constant $C < \infty$

$$\mathbf{P}\{\|f_{mn}\| \ge t\} \le C\mathbf{P}\{\|f\| \ge t\}, \quad t \ge 0,\ m \ge 1,\ n \ge 1.$$

A double array $\{X_{mn} : m \ge 1, n \ge 1\}$ of random sets is said to be *stochastically dominated* by a random element X if for some constant $C < \infty$

$$\mathbf{P}\{|X_{mn}| \ge t\} \le C\mathbf{P}\{\|X\| \ge t\}, \quad t \ge 0,\ m \ge 1,\ n \ge 1.$$

This condition is satisfied when the $\{X_{mn} : m \ge 1, n \ge 1\}$ is identically distributed.

For notational convenience, for $a, b \in \mathbf{R}$, $\max\{a, b\}$ is denoted by $a \vee b$, $\min\{a, b\}$ is denoted by $a \wedge b$ and the symbol C denotes a generic positive constant which is not necessarily the same one in each circumstance. The logarithms are to the base 2, for $a \in \mathbf{R}$, $\log(a \vee 1)$ will be denoted by $\log^+ a$.

For basic notions of probability theory and set-valued analysis, we refer to G. Beer [5, 7], C. Castaing [1], C. Hess [14, 17, 18], F. Hiai [20], Y. S. Chow and H. Teicher [11], N. Neveu [25].

Now we proceed to state our main results.

3. The strong laws of large numbers for double array of pairwise independent identically distributed closed valued random sets in separable Banach spaces

We must first recall some facts on set valued integration

Proposition 3.1 (*see* [18])**.** (*i*) *For every integrable r.s. X whose values are in $c(\mathfrak{X})$, the following equality holds true*

$$\overline{\text{co}} M^1(X, \mathcal{A}) = \overline{\text{co}} M^1(X, \mathcal{A}_X)$$

the closure being taken in $\mathcal{M}(\mathfrak{X})$ (or $M(\mathfrak{X})$) in the narrow topology.

(*ii*) *For every integrable r.s. X whose values lie in $c(\mathfrak{X})$, one has*

$$\overline{\text{co}} E(X, \mathcal{A}) = \overline{\text{co}} E(X, \mathcal{A}_X).$$

In particular, if $X(\omega) \in cc(\mathfrak{X})$ or $X(\omega) \in cwk(\mathfrak{X})$ for all $\omega \in \Omega$ then $E(X) = E(X, \mathcal{A}_X)$.

Proposition 3.2 (*see* [18])**.** *Let X and Y be two random sets with closed values in $\mathfrak{X}$. Then, the two following statements are equivalent:*
(*i*) *X and Y have the same distribution on the measurable space $(c(\mathfrak{X}), \mathcal{E})$.*
(*ii*) *In $M(\mathfrak{X})$, the following equality holds true*

$$M(X, \mathcal{A}_X) = M(Y, \mathcal{A}_Y).$$

Moreover, if X and Y are integrable then each of the above statements is equivalent to
(iii) In $M^1(\mathfrak{X})$, the following equality holds true

$$M^1(X, \mathcal{A}_X) = M^1(Y, \mathcal{A}_Y).$$

Consequently, if X and Y have the same distribution, one has

$$E(X, \mathcal{A}_X) = E(Y, \mathcal{A}_Y).$$

The two following propositions constitute a key ingredient for proving the main convergence result in this section.

Proposition 3.3. *Let $\{x_{mn} : m \geq 1, n \geq 1\}$ be a double array of elements in a Banach space $\mathfrak{X}$ such that*

$$(i) \text{ For each } m \geq 1, \ \frac{1}{n}\sum_{j=1}^{n} x_{mj} \to x \quad \text{as } n \to \infty,$$

$$(ii) \text{ For each } n \geq 1, \ \frac{1}{m}\sum_{i=1}^{m} x_{in} \to x \quad \text{as } m \to \infty,$$

$$(iii) \ \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \to x \qquad \text{as } m \wedge n \to \infty,$$

where x is a member of $\mathfrak{X}$. Then,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \to x \ \text{ as } \ m \vee n \to \infty.$$

Proof. By (iii), there exists $n_0 \in \mathbf{N}^*$ such that

$$\Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} - x\Big\| < \epsilon \ \text{ for all } m \geq n_0, n \geq n_0. \tag{3.1}$$

By (i), for each $l = 1, 2, \ldots, n_0$ (the row of l^{th}), there exists $r(l) \in \mathbf{N}^*$ such that for all $r > r(l)$ then

$$\Big\|\frac{1}{r}\sum_{j=1}^{r} x_{lj} - x\Big\| < \epsilon. \tag{3.2}$$

By (ii), for each $s = 1, 2, \ldots, n_0$ (the column of s^{th}), there exists $k(s) \in \mathbf{N}^*$ such that for all $k > k(s)$ then

$$\Big\|\frac{1}{k}\sum_{i=1}^{k} x_{is} - x\Big\| < \epsilon. \tag{3.3}$$

Set $N = \max\{n_0, k(1), k(2), \ldots, k(n_0), i(1), i(2), \ldots, i(n_0)\}$. It suffices to show that for every $m \vee n > N$ then

$$\Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} - x\Big\| < \epsilon. \tag{3.4}$$

To do this, we consider the following cases.

1. If $m > N, n \le n_0$ then by (3.3) we have

$$\begin{aligned}\Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}-x\Big\| &= \Big\|\frac{1}{n}\sum_{j=1}^{n}\Big(\frac{1}{m}\sum_{i=1}^{m}x_{ij}-x\Big)\Big\| \le \frac{1}{n}\sum_{j=1}^{n}\Big\|\frac{1}{m}\sum_{i=1}^{m}x_{ij}-x\Big\| \\ &< \frac{1}{n}\sum_{j=1}^{n}\epsilon = \epsilon.\end{aligned}$$

2. If $m > N, n > n_0$ or $m > n_0, n > N$ then the conclusion (3.4) follows immediately from (3.1).

3. If $m \le n_0, n > N$ then using the arguments as in the proof of the first case and applying (3.2) we get

$$\begin{aligned}\Big\|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}-x\Big\| &= \Big\|\frac{1}{m}\sum_{i=1}^{m}\Big(\frac{1}{n}\sum_{j=1}^{n}x_{ij}-x\Big)\Big\| \le \frac{1}{m}\sum_{i=1}^{m}\Big\|\frac{1}{n}\sum_{j=1}^{n}x_{ij}-x\Big\| \\ &< \frac{1}{m}\sum_{i=1}^{m}\epsilon = \epsilon.\end{aligned}$$

This completes the proof of the proposition. □

Proposition 3.4. *Let $\mathfrak{X}$ be a separable Banach space and let $\{X_{mn} : m \ge 1, n \ge 1\}$ be a double array of pairwise independent closed valued random sets having the same distribution as an integrable r.s. X such that*

$$E(|X| \log^+ |X|) < \infty.$$

Let C' be the set of all convex combinations of $E(X, \mathcal{A}_X)$, with rational coefficients. Then, for each $y \in C'$, there exists a negligible subset $N(y)$ of Ω and a double array $\{g_{mn} : m \ge 1, n \ge 1\}$ in $\mathcal{L}^1(\mathfrak{X})$ verifying:

(i) for each $m \ge 1, n \ge 1$, $g_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$

(ii) for any $\omega \in \Omega \setminus N(y)$,

$$y = \lim_{m\vee n\to\infty}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}g_{ij}(\omega).$$

Proof. Consider $y \in C'$. From the definition of C' we have $y = \sum_{j=1}^{k}\lambda_j y_j$ where k is a positive integer, λ_j are positive rational numbers with $\sum_{j=1}^{k}\lambda_j = 1$ and where, for each $j \ge 1$, $y_j \in E(X, \mathcal{A}_X)$. Obviously, for every $j = 1, \dots, k$, it is possible to write $\lambda_j = \frac{d_j}{d}$ where d and the d_j are positive integers satisfying $d = \sum_{j=1}^{k} d_j$. Put $z_1 = y_1, \dots, z_{d_1} = y_1$, $z_{d_1+1} = y_2$, $\dots$, $z_{d_1+d_2} = y_2$, $\dots$, $z_{d_1+\dots+d_{k-1}+1} = y_k$, $\dots$, $z_d = y_k$. Since then, we have $y = \frac{1}{d}\sum_{i=1}^{d} z_i$. For each $1 \le j \le d$, $z_j = E(f_j)$ of $f_j \in S^1(X, \mathcal{A}_X)$.

The proof will be performed in several steps.

Step 1. By Proposition 3.2(*iii*), we can choose $f_{ij} \in S^1(X_{ij}, \mathcal{A}_{X_{ij}})$, $1 \le i \le d, 1 \le j \le d$ such that

$$E(f_{ij}) = \begin{cases} E(f_{i+j-1}) & \text{if } \ i+j \le d+1, \\ E(f_{i+j-1-d}) & \text{if } \ i+j > d+1. \end{cases}$$

Let $z_{ij} = E(f_{ij}), 1 \le i \le d, 1 \le j \le d$. It is easy to check that

$$\frac{1}{d}\sum_{i=1}^{d} z_i = \frac{1}{d^2}\sum_{i=1}^{d}\sum_{j=1}^{d} z_{ij}, \tag{3.5}$$

$$\frac{1}{d}\sum_{i=1}^{d} z_i = \frac{1}{d}\sum_{i=1}^{d} z_{ij} \text{ for each } j = 1, 2, \dots, d, \tag{3.6}$$

$$\frac{1}{d}\sum_{i=1}^{d} z_i = \frac{1}{d}\sum_{j=1}^{d} z_{ij} \text{ for each } i = 1, 2, \dots, d, \tag{3.7}$$

hold.

By Proposition 3.2(iii), for each $m \ge 1, n \ge 1$, there exists a double array $\{g_{mn} : m \ge 1, n \ge 1\}$ of $g_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$ such that $\{g_{(s-1)d+i,(t-1)d+j} : s \ge 1, t \ge 1\}$ is a double array of random elements having the same distribution as f_{ij} for each $i = 1, \dots, d$ and $j = 1, \dots, d$.

For any $m \ge 1, n \ge 1$, there exist the integers s_m, p_m, t_n and q_n satisfying

$$m = s_m d + p_m, \quad s_m \ge 0, \ 1 \le p_m \le d, \tag{3.8}$$

$$n = t_n d + q_n, \quad t_n \ge 0, \ 1 \le q_n \le d. \tag{3.9}$$

From the above relationships, we deduce that the sequences (p_m) and (q_n) are bounded, whereas from (3.8) and (3.9) we deduce

$$\lim_{m\to\infty} s_m = \infty \text{ and } \lim_{n\to\infty} t_n = \infty.$$

Furthermore it is not difficult to shows for all $\omega \in \Omega$ that following equality holds

$$\begin{aligned} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(\omega) = & \frac{s_m t_n}{mn}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{s_m t_n}\sum_{l=1}^{s_m}\sum_{r=1}^{t_n} g_{(l-1)d+i,(r-1)d+j}(\omega) \\ & + \frac{t_n}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{d}\frac{1}{t_n}\sum_{r=1}^{t_n} g_{s_m d+i,(r-1)d+j}(\omega) \\ & + \frac{s_m}{mn}\sum_{i=1}^{d}\sum_{j=1}^{q_n}\frac{1}{s_m}\sum_{l=1}^{s_m} g_{(l-1)d+i,t_n d+j}(\omega) \\ & + \frac{1}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{q_n} g_{s_m d+i,t_n d+j}(\omega). \end{aligned} \tag{$*$}$$

The proof will be performed as follows.

Step 2. **Claim 1:** There is a negligible subset $N_{11}(y)$ such that for every $\omega \in \Omega \setminus N_{11}(y)$,

(3.10)

$$\frac{s_m t_n}{mn}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{s_m t_n}\sum_{l=1}^{s_m}\sum_{r=1}^{t_n} g_{(l-1)d+i,(r-1)d+j}(\omega) \to \frac{1}{d^2}\sum_{i=1}^{d}\sum_{j=1}^{d} z_{ij} \text{ as } m \wedge n \to \infty.$$

Since our assumption that $\{X_{mn} : m \ge 1, n \ge 1\}$ is a double array of p.i.i.d. closed valued random sets, then $\{g_{(s-1)d+i,(t-1)d+j} : s \ge 1, t \ge 1\}$ is a double array of p.i.i.d. random elements in $\mathcal{L}^1(\mathfrak{X})$ with

$$E(\|g_{ij}\| \log^+ \|g_{ij}\|) \leq E(|X_{ij}| \log^+ |X_{ij}|) < \infty.$$

Hence, applying Etemadi's strong law of large numbers (see [12]) to each double array of vector valued random elements $\{g_{(s-1)d+i,(t-1)d+j} : s \geq 1, t \geq 1\}$ in $\mathcal{L}^1(\mathfrak{X})$ yields a negligible subset $N_{11}(y)$ such that for every $\omega \in \Omega \setminus N_{11}(y)$, for any $1 \leq i \leq d$, $1 \leq j \leq d$,

$$\frac{1}{s_m t_n} \sum_{l=1}^{s_m} \sum_{r=1}^{t_n} g_{(l-1)d+i,(r-1)d+j}(\omega) \to z_{ij} \text{ as } m \vee n \to \infty. \tag{3.11}$$

Since we have

$$\lim_{m\to\infty} \frac{s_m}{m} = \lim_{m\to\infty} \Big(\frac{1}{d} + \frac{1}{m} - \frac{p_m}{md}\Big) = \frac{1}{d} \text{ and } \lim_{n\to\infty} \frac{t_n}{n} = \lim_{n\to\infty} \Big(\frac{1}{d} + \frac{1}{n} - \frac{q_n}{nd}\Big) = \frac{1}{d}, \tag{3.12}$$

then (3.10) follows.

Step 3. **Claim 2:**

$$\lim_{m\wedge n\to\infty} \Big(\frac{t_n}{mn} \sum_{i=1}^{p_m} \sum_{j=1}^{d} \frac{1}{t_n} \sum_{r=1}^{t_n} g_{s_m d+i,(r-1)d+j}(\omega) + \frac{s_m}{mn} \sum_{i=1}^{d} \sum_{j=1}^{q_n} \frac{1}{s_m} \sum_{l=1}^{s_m} g_{(l-1)d+i,t_n d+j}(\omega)\Big) = 0 \text{ a.s.} \tag{3.13}$$

Since $\{g_{(s-1)d+i,(r-1)d+j} : r \geq 1\}$ is a sequence of p.i.i.d random elements in $\mathcal{L}^1(\mathfrak{X})$ for each $s \geq 1$, $1 \leq i \leq d$, $1 \leq j \leq d$, applying Etemadi's strong law of large numbers (see [12]) to this sequence yields a negligible subset $N_{12}(y)$ such that for every $\omega \in \Omega \setminus N_{12}(y)$, for any $1 \leq i \leq d$, $1 \leq j \leq d$, $m \geq 1$, we have

$$\frac{1}{t_n} \sum_{r=1}^{t_n} g_{s_m d+i,(r-1)d+j}(\omega) \to z_{ij} \text{ as } n \to \infty. \tag{3.14}$$

Thus, for every $\omega \in \Omega \setminus N_{12}(y)$,

$$\frac{t_n}{mn} \sum_{i=1}^{p_m} \sum_{j=1}^{d} \frac{1}{t_n} \sum_{r=1}^{t_n} g_{s_m d+i,(r-1)d+j}(\omega) \to 0 \text{ as } m \wedge n \to \infty.$$

Similarly, there exists a negligible subset $N_{13}(y)$ such that for every $\omega \in \Omega \setminus N_{13}(y)$,

$$\frac{s_m}{mn} \sum_{i=1}^{d} \sum_{j=1}^{q_n} \frac{1}{s_m} \sum_{l=1}^{s_m} g_{(l-1)d+i,t_n d+j}(\omega) \to 0 \text{ as } m \wedge n \to \infty.$$

Step 4. **Claim 3:** There is a negligible $N_1(y)$ such that for every $\omega \in \Omega \setminus N_1(y)$,

$$\frac{1}{mn} \sum_{i=1}^{p_m} \sum_{j=1}^{q_n} g_{s_m d+i,t_n d+j}(\omega) \to 0 \text{ as } m \wedge n \to \infty. \tag{3.15}$$

We define the negligible subset $N_1(y)$ as the union of the $N_{1j}(y)$ where $j \in \{1,2,3\}$, from (3.11), (3.12) and (3.14) we have that for every $\omega \in \Omega \setminus N_1(y)$, for any $1 \le i \le d,\ 1 \le j \le d$,

$$\begin{aligned}\frac{1}{mn} g_{s_m d+i, t_n d+j}(\omega) &= \frac{s_m t_n}{mn}\Big(\frac{1}{s_m t_n}\sum_{l=1}^{s_m}\sum_{r=1}^{t_n} g_{ld+i,rd+j}(\omega) \\ &- \frac{s_m-1}{s_m}.\frac{1}{(s_m-1)t_n}\sum_{l=1}^{s_m-1}\sum_{r=1}^{t_n} g_{ld+i,rd+j}(\omega) \\ &- \frac{t_n-1}{s_m t_n}.\frac{1}{t_n-1}\sum_{r=1}^{t_n-1} g_{s_m d+i, rd+j}(\omega)\Big) \to \frac{1}{d^2}(z_{ij} - 1.z_{ij} - 0.z_{ij}) = 0 \\ &\qquad \text{as } m \wedge n \to \infty,\end{aligned}$$

whence Claim 3 follows by applying this estimate.

Combining the above limits and using (3.5) and coming back to $(*)$ we have that for every $\omega \in \Omega \setminus N_1(y)$,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(\omega) \to \frac{1}{d}\sum_{i=1}^{d} z_i \ \text{ as } m \wedge n \to \infty. \tag{3.16}$$

Step 5. Next, for each $n = td + j,\ 1 \le j \le d$. If $m = s_m d + p_m,\ 1 \le p_m \le d$ then for all $\omega \in \Omega$, the following equality holds

$$\frac{1}{m}\sum_{i=1}^{m} g_{in}(\omega) = \frac{s_m}{m}\sum_{i=1}^{d}\frac{1}{s_m}\sum_{h=1}^{s_m} g_{(h-1)d+i,n}(\omega) + \frac{1}{m}\sum_{i=1}^{p_m} g_{s_m d+i,n}(\omega). \tag{$**$}$$

Claim 4:

$$\lim_{m\to\infty}\frac{s_m}{m}\sum_{i=1}^{d}\frac{1}{s_m}\sum_{h=1}^{s_m} g_{(h-1)d+i,n}(\omega) = \frac{1}{d}\sum_{i=1}^{d} z_{ij} \ \text{ a.s.} \tag{3.17}$$

Since $\{g_{(s-1)d+i,n} : s \ge 1\}$ is a sequence of p.i.i.d random elements in $\mathcal{L}^1(\mathfrak{X})$ for each $1 \le i \le d,\ n \ge 1$ then applying again Etemadi's strong law of large numbers (see [12]) to this sequence yields a negligible subset $N_{21}(n,y)$ such that for every $\omega \in \Omega \setminus N_{21}(n,y)$, for any $1 \le i \le d$,

$$\frac{1}{s_m}\sum_{h=1}^{s_m} g_{(h-1)d+i,n}(\omega) \to z_{ij} \ \text{ as } m \to \infty.$$

Thus, for every $\omega \in \Omega \setminus N_{21}(n,y)$,

$$\frac{s_m}{m}\sum_{i=1}^{d}\frac{1}{s_m}\sum_{h=1}^{s_m} g_{(h-1)d+i,n}(\omega) \to \frac{1}{d}\sum_{i=1}^{d} z_{ij} \ \text{ as } m \to \infty.$$

Claim 5:

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{p_m} g_{s_m d+i,n}(\omega) = 0 \ \text{ a.s.} \tag{3.18}$$

Applying Etemadi's strong law of large numbers (see [12]) yields a negligible subset $N_{22}(n, y)$ such that for every $\omega \in \Omega \setminus N_{22}(n, y)$, for any $1 \leq i \leq d$,

$$\frac{1}{m} g_{s_m d+i,n}(\omega) = \frac{s_m}{m}\Big(\frac{1}{s_m}\sum_{l=1}^{s_m} g_{ld+i,n}(\omega) - \frac{s_m - 1}{s_m}\frac{1}{s_m - 1}\sum_{l=1}^{s_m - 1} g_{ld+i,n}(\omega)\Big)$$
$$\to \frac{1}{d}(z_{ij} - 1.z_{ij}) = 0 \text{ as } m \to \infty.$$

Thus, for every $\omega \in \Omega \setminus N_{22}(n, y)$,

$$\frac{1}{m}\sum_{i=1}^{p_m} g_{s_m d+i,n}(\omega) \to 0 \text{ as } m \to \infty.$$

We define the negligible subset $N_2(n, y)$ as the union of the N_{2j} where $j \in \{1, 2\}$, combining the above limits and by (3.6) and coming back to $(**)$ we have that for every $\omega \in \Omega \setminus N_2(n, y)$,

$$\frac{1}{m}\sum_{i=1}^{m} g_{in}(\omega) \to \frac{1}{d}\sum_{i=1}^{d} z_i \text{ as } m \to \infty. \tag{3.19}$$

Similarly, for each $m \geq 1$, there exists a negligible subset $N_3(m, y)$ such that for every $\omega \in \Omega \setminus N_3(m, y)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} g_{mj}(\omega) = \frac{1}{d}\sum_{i=1}^{d} z_i. \tag{3.20}$$

Final Step and Conclusion:

We define the negligible subset $N(y)$ as the union of the $N_1(y)$, $N_2(n, y)$ and $N_3(m, y)$ where $m \geq 1, n \geq 1$. Combining (3.16), (3.19), (3.20) and Proposition 3.3, we have that for every $\omega \in \Omega \setminus N(y)$,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(\omega) \to y \text{ as } m \vee n \to \infty.$$

The proof is therefore completed. □

We also need an useful lemma that is formally derived from (2.2).

Lemma 3.5. *Let $\{C_{mn} : m \geq 1, n \geq 1\}$ be a double array in $c(\mathfrak{X})$. Also consider $C \in c(\mathfrak{X})$ and a countable dense subset D^* of B^* such that*

$$d(x, \overline{\text{co}}C) = \sup_{z\in D^*} \{\langle z, x\rangle - \delta^*(z, \overline{\text{co}}C)\},\ x \in \mathfrak{X}$$

(which is possible by (2.2)). If, for every $z \in D^*$, one has

$$\limsup_{m\vee n\to\infty} \delta^*(z, C_{mn}) \leq \delta^*(z, C) \tag{3.21}$$

then, for every $x \in \mathfrak{X}$,

$$\liminf_{m\vee n\to\infty} d(x, C_{mn}) \geq d(x, \overline{\text{co}}C).$$

Proof. For every $x \in \mathfrak{X}$, we have by (2.1), (2.2), (3.21) and elementary calculations

$$\begin{aligned}
\liminf_{m\vee n\to\infty} d(x, C_{mn}) &\geq \liminf_{m\vee n\to\infty} d(x, \overline{\text{co}}C_{mn}) \\
&\geq \liminf_{m\vee n\to\infty} \Big[\sup_{z\in B^*} \{\langle z, x\rangle - \delta^*(z, C_{mn})\}\Big] \\
&\geq \sup_{z\in B^*} \Big\{ \liminf_{m\vee n\to\infty} [\langle z, x\rangle - \delta^*(z, C_{mn})]\Big\} \\
&\geq \sup_{z\in B^*} \Big\{\langle z, x\rangle - \delta^*(z, C)\Big\} \\
&\geq \sup_{z\in D^*} \Big\{\langle z, x\rangle - \delta^*(z, C)\Big\} = d(x, \overline{\text{co}}C).
\end{aligned}$$

□

Proposition 3.4 allows to prove the "lim sup" part in the Wijsman convergence for the SLLN under consideration.

Proposition 3.6. *Under the same hypotheses as in Proposition* 3.4*, there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$ and $x \in \mathfrak{X}$,*

$$\limsup_{m\vee n\to\infty} d\Big(x, \text{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) \leq d(x, \overline{\text{co}}\,E(X, \mathcal{A})). \tag{3.22}$$

Proof. The proof is similar to the proof of Proposition 3.4 in C. Hess [18]. For the sake of completeness we provide the details.

Let $Z_{mn}(\omega) = \text{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)$ for every $m \geq 1, n \geq 1,\ \omega \in \Omega$. From Proposition 3.1($ii$) we know that $\overline{\text{co}}\,E(X, \mathcal{A}) = \overline{\text{co}}\,E(X, \mathcal{A}_X)$. Further, let D' be a countable dense subset of $E(X, \mathcal{A}_X)$ satisfying $\text{cl}\,D' = \text{cl}\,E(X, \mathcal{A}_X)$, and consider the set C' of all rational convex combinations of members of D'. On the other hand, consider a countable dense subset D of $\mathfrak{X}$ and observe that it suffices to prove (3.22) for all x in D. Indeed, each side of (3.22) defines a Lipschitz function of x (with Lipschitz constant 1). So, consider $x \in D$ and an integer $p \geq 1$. One can find $y' = y'(x, p) \in C'$, depending on x and p, such that

$$\|x - y'\| \leq d(x, \overline{\text{co}}\,E(X, \mathcal{A})) + \frac{1}{p}.$$

Further, Proposition 3.4 applied to y', yields the existence of a negligible subset $N(x, p)$ and of a double array $\{g_{mn} : m \geq 1, n \geq 1\}$ verifying properties (i) and (ii). Then, define the negligible subset N as the union of the $N(x, p)$ where $x \in D$ and $p \geq 1$, and consider $\omega \in \Omega \setminus N$. For every $x \in D$, we have

$$\begin{aligned}
\limsup_{m\vee n\to\infty} d(x, Z_{mn}(\omega)) \quad &\leq \quad \lim_{m\vee n\to\infty} \|x - \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(\omega)\| \\
&= \quad \|x - y'\| \leq d(x, \overline{\text{co}}\,E(X, \mathcal{A})) + \frac{1}{p},
\end{aligned}$$

whence, by the arbitrariness of p, yields the desired conclusion. □

Now, we can state and prove the main result of this section, namely, the multivalued SLLN for double array of pairwise independent identically distributed random sets when $c(\mathfrak{X})$ is endowed with the Wijsman topology.

Theorem 3.7. *Let $\mathfrak{X}$ be a separable Banach space and let $\{X_{mn} : m \geq 1, n \geq 1\}$ be a double array of pairwise independent closed valued random sets having the same distribution as an integrable r.s. X such that*

$$E(|X| \log^+ |X|) < \infty.$$

Then, there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$,

$$\overline{\text{co}}\, E(X, \mathcal{A}) = \mathcal{T}_W - \lim_{m \vee n \to \infty} \text{cl}\, \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)$$

that is, for any $x \in \mathfrak{X}$,

$$d(x, \overline{\text{co}}\, E(X, \mathcal{A})) = \lim_{m \vee n \to \infty} d\Big(x, \text{cl}\, \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)\Big).$$

Proof. Let $Z_{mn}(\omega) = \text{cl}\, \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)$ for every $m \geq 1, n \geq 1, \omega \in \Omega$ and let $C = \overline{\text{co}}\, E(X, \mathcal{A})$. By (2.3) there is a countable set D^* in B^* such that

$$d(x, C) = \sup_{z \in D^*} \{\langle z, x \rangle - \delta^*(z, C)\} \;\; \forall x \in \mathfrak{X}.$$

Let z be fixed in D^*. Since the map $F \to \delta^*(z, F)$ is Effros measurable from $c(\mathfrak{X})$ in $\overline{\mathbf{R}}$, we deduce that $\{\delta^*(z, X_{mn}) : m \geq 1, n \geq 1\}$ is a double array of $\overline{\mathbf{R}}$-valued pairwise independent random variables having the same distribution as $\delta^*(z, X)$. Further, by the inequality

$$E(|\delta^*(z, X)| \log^+ |\delta^*(z, X)|) \leq E(|X| \log^+ |X|) < \infty,$$

we may apply Etemadi's strong law of large numbers (see [12]) to double array of $\overline{\mathbf{R}}$-valued random variables $\{\delta^*(z, X_{mn}) : m \geq 1, n \geq 1\}$. This yields the existence of a negligible subset $N(z)$ of Ω verifying, for every $\omega \in \Omega \setminus N(z)$,

$$\delta^*(z, C) = \lim_{m \vee n \to \infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta^*(z, X_{ij}(\omega)) = \lim_{m \vee n \to \infty} \delta^*(z, Z_{mn}(\omega)).$$

Now, defining the negligible subset N_1 as the union of the $N(z)$, for $z \in D^*$, we deduce that

$$\delta^*(z, C) = \lim_{m \vee n \to \infty} \delta^*(z, Z_{mn}(\omega)) \;\; \forall z \in D^*, \;\; \forall \omega \in \Omega \setminus N_1. \tag{3.23}$$

The above equality and Lemma 3.5 entail

$$\liminf_{m \vee n \to \infty} d(x, Z_{mn}(\omega)) \geq d(x, C) \;\; \forall x \in \mathfrak{X}, \;\; \forall \omega \in \Omega \setminus N_1. \tag{3.24}$$

On the other hand, Proposition 3.6 yields the existence of a negligible subset N_2 such that

$$\limsup_{m \vee n \to \infty} d(x, Z_{mn}(\omega)) \leq d(x, C) \;\; \forall x \in \mathfrak{X}, \;\; \forall \omega \in \Omega \setminus N_2. \tag{3.25}$$

Finish the proof by defining the negligible subset $N = N_1 \cup N_2$ and by combining inequalities (3.24) and (3.25). □

4. The strong laws of large numbers for double array of independent closed valued random sets in Rademacher type p Banach spaces

We will provide several Wijsman convergence results relative to the strong law of large numbers for double array of independent closed valued random sets in Rademacher type p Banach spaces. Before going further, let us recall a useful result due to Stadtmüller and Thanh [29]. That result is an analogue of Toeplitz lemma (see, e.g., Loève [24], p. 250).

Lemma 4.1. *Let $\{x_{mn} : m \geq 1, n \geq 1\}$ be a double array of elements in a Banach space $\mathfrak{X}$ such that*

$$\lim_{m \vee n \to \infty} x_{mn} \;=\; x$$

where x is a member of $\mathfrak{X}$. Then we have

$$\lim_{m \vee n \to \infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = x.$$

The following proposition is a crucial tool for proving the main results in this section.

Proposition 4.2. *Suppose that $\mathfrak{X}$ is a Rademacher type p Banach space, where $1 \leq p \leq 2$. Let $\{X_{mn} : m \geq 1, n \geq 1\}$ be a double array of independent closed valued random sets satisfying the following conditions:*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{E|X_{mn}|^p}{(mn)^p} < \infty$$

and there exists $X \in c(\mathfrak{X})$ such that

$$X \subset s - li_{m \vee n \to \infty}\big(\text{cl}\, E(X_{mn}, \mathcal{A}_{X_{mn}})\big), \tag{4.1}$$

$$\limsup_{m \vee n \to \infty} \delta^*\big(z, \text{cl}\, E(X_{mn}, \mathcal{A})\big) \leq \delta^*(z, X), \quad \forall z \in \mathfrak{X}^*. \tag{4.2}$$

Consider the set C' of all convex combinations of X, with rational coefficients. Then, for each $y \in C'$, there exists a negligible subset $N(y)$ of Ω and a double array $\{g_{mn} : m \geq 1, n \geq 1\}$ in $\mathcal{L}^p(\mathfrak{X})$ satisfying:

(i) for each $m \geq 1, n \geq 1$, $g_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$
(ii) for any $\omega \in \Omega \setminus N(y)$,

$$y = \lim_{m \vee n \to \infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij}(\omega).$$

Proof. Consider $y \in C'$. From the definition of C' we have $y = \sum_{j=1}^{k} \lambda_j y_j$ where k is a positive integer, λ_j are positive rational numbers with $\sum_{j=1}^{k} \lambda_j = 1$ and where, for each $j \geq 1$, $y_j \in X$. Obviously, for every $j = 1, \ldots, k$, it is possible to write $\lambda_j = \frac{d_j}{d}$ where d and the d_j are positive integers satisfying $d = \sum_{j=1}^{k} d_j$. Put $z_1 = y_1, \ldots, z_{d_1} = y_1$, $z_{d_1+1} = y_2$, $\ldots$, $z_{d_1+d_2} = y_2$, $\ldots$, $z_{d_1+\cdots+d_{k-1}+1} = y_k$, $\ldots$, $z_d = y_k$, then we have $y = \frac{1}{d} \sum_{i=1}^{d} z_i$.

The proof will be performed in several steps.
Step 1. The main part of the proof is to apply fairly Proposition 3.3, we choose an array $\{z_{ij} : 1 \le i \le d, 1 \le j \le d\}$ of the members of X such that

$$z_{ij} = \begin{cases} z_{i+j-1} & \text{if } \ i+j \le d+1, \\ z_{i+j-1-d} & \text{if } \ i+j > d+1. \end{cases}$$

It is easy to check that (3.5), (3.6) and (3.7) are true.

By condition (4.1), there exists a double array $\{g_{mn} : m \ge 1, n \ge 1\}$ of $g_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$ such that

(4.3)
$$E\Big(g_{(s-1)d+i,(t-1)d+j}\Big) \to z_{ij} \text{ as } \ s \vee t \to \infty \text{ for each } i = 1, 2, \ldots, d \text{ and } j = 1, 2, \ldots, d.$$

For any $m \ge 1, n \ge 1$, there exist the integers s_m, p_m, t_n and q_n satisfying (3.8) and (3.9). From the above relationships, we deduce that the sequences (p_m) and (q_n) are bounded,

$$\lim_{m\to\infty} s_m = \infty \ \text{ and } \ \lim_{n\to\infty} t_n = \infty.$$

Furthermore for all $\omega \in \Omega$ the following equalities hold

$$(***) \qquad \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(\omega) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(g_{ij}(\omega) - E(g_{ij})) + \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E(g_{ij})$$
$$= \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(g_{ij}(\omega) - E(g_{ij}))$$
$$+ \frac{s_m t_n}{mn}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{s_m t_n}\sum_{l=1}^{s_m}\sum_{r=1}^{t_n} E(g_{(l-1)d+i,(r-1)d+j})$$
$$+ \frac{t_n}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{d}\frac{1}{t_n}\sum_{r=1}^{t_n} E(g_{s_m d+i,(r-1)d+j})$$
$$+ \frac{s_m}{mn}\sum_{i=1}^{d}\sum_{j=1}^{q_n}\frac{1}{s_m}\sum_{l=1}^{s_m} E(g_{(l-1)d+i,t_n d+j})$$
$$+ \frac{1}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{q_n} E(g_{s_m d+i,t_n d+j}).$$

The proof will be performed as follows.
Step 2. **Claim 1:** There exists a negligible subset $N_1(y)$ such that, for every $\omega \in \Omega \setminus N_1(y)$,

$$(4.4) \qquad \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(g_{ij}(\omega) - E(g_{ij})) \to 0 \ \text{ as } \ m \vee n \to \infty.$$

Since $\{X_{mn} : m \ge 1, n \ge 1\}$ is a double array of independent closed valued random sets and $g_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$ then $\{g_{mn} : m \ge 1, n \ge 1\}$ is a double array of independent random elements in $\mathcal{L}^p(\mathfrak{X})$ with

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E\|g_{mn}\|^p}{(mn)^p}\le\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E|X_{mn}|^p}{(mn)^p}<\infty,$$

whence Claim 1 follows by applying Theorem 3.1 in [27].

Step 3. **Claim 2:**

(4.5)

$$\frac{s_mt_n}{mn}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{s_mt_n}\sum_{l=1}^{s_m}\sum_{r=1}^{t_n}E(g_{(l-1)d+i,(r-1)d+j})\to\frac{1}{d^2}\sum_{i=1}^{d}\sum_{j=1}^{d}z_{ij}\ \text{ as }\ m\wedge n\to\infty.$$

By (4.3) and Lemma 4.1, we get

$$\frac{1}{s_mt_n}\sum_{l=1}^{s_m}\sum_{r=1}^{t_n}E(g_{(l-1)d+i,(r-1)d+j})\to z_{ij}\ \text{ as }\ m\vee n\to\infty.$$

By (3.12), we obtain (4.5).

Step 4. **Claim 3:**

$$\lim_{m\wedge n\to\infty}\Big(\frac{t_n}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{d}\frac{1}{t_n}\sum_{r=1}^{t_n}E(g_{s_md+i,(r-1)d+j})+\frac{s_m}{mn}\sum_{i=1}^{d}\sum_{j=1}^{q_n}\frac{1}{s_m}\sum_{l=1}^{s_m}E(g_{(l-1)d+i,t_nd+j})\Big)=0. \tag{4.6}$$

By (4.3), we obtain $E(g_{(s-1)d+i,(t-1)d+j})\to z_{ij}$ as $t\to\infty$ for any $1\le i\le d$, $1\le j\le d$, $s\ge 1$. Hence,

$$\lim_{n\to\infty}\frac{1}{t_n}\sum_{r=1}^{t_n}E(g_{s_md+i,(r-1)d+j})\to z_{ij}\ \text{ for any }\ 1\le i\le d,\ 1\le j\le d,\ m\ge 1.$$

Thus,

$$\frac{t_n}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{d}\frac{1}{t_n}\sum_{r=1}^{t_n}E(g_{s_md+i,(r-1)d+j})\to 0\ \text{ as }\ m\wedge n\to\infty.$$

Similarly,

$$\frac{s_m}{mn}\sum_{i=1}^{d}\sum_{j=1}^{q_n}\frac{1}{s_m}\sum_{l=1}^{s_m}E(g_{(l-1)d+i,t_nd+j})\to 0\ \text{ as }\ m\wedge n\to\infty.$$

Step 5. **Claim 4:**

$$\frac{1}{mn}\sum_{i=1}^{p_m}\sum_{j=1}^{q_n}E(g_{s_md+i,t_nd+j})\to 0\ \text{ as }\ m\wedge n\to\infty. \tag{4.7}$$

By (4.3) and Lemma 4.1, for any $1\le i\le d,\ 1\le j\le d$,

$$\frac{1}{mn}E(g_{s_md+i,t_nd+j})=\frac{s_mt_n}{mn}\Big(\frac{1}{s_mt_n}\sum_{l=1}^{s_m}\sum_{r=1}^{t_n}E(g_{ld+i,rd+j})$$

$$- \frac{s_m - 1}{s_m} . \frac{1}{(s_m - 1)t_n} \sum_{l=1}^{s_m - 1} \sum_{r=1}^{t_n} E(g_{ld+i,rd+j})$$

$$- \frac{t_n - 1}{s_m t_n} . \frac{1}{t_n - 1} \sum_{r=1}^{t_n - 1} E(g_{s_m d+i,rd+j}) \Big) \to \frac{1}{d^2}(z_{ij} - 1.z_{ij} - 0.z_{ij})$$

$$= 0 \text{ as } m \wedge n \to \infty.$$

Thus, we obtain (4.7).

Combining the above limits and using (3.5) coming back to $(***)$ we have that for every $\omega \in \Omega \setminus N_1(y)$,

$$\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij}(\omega) \to \frac{1}{d} \sum_{i=1}^{d} z_i \text{ as } m \wedge n \to \infty. \tag{4.8}$$

Step 6. Next, for each $n = td + j,\ 1 \le j \le d$. If $m = s_m d + p_m,\ 1 \le p_m \le d$ then for all $\omega \in \Omega$, the following equalities hold

$$(****) \quad \frac{1}{m} \sum_{i=1}^{m} g_{in}(\omega) = \frac{1}{m} \sum_{i=1}^{m} (g_{in}(\omega) - E(g_{in})) + \frac{1}{m} \sum_{i=1}^{m} E(g_{in})$$

$$= \frac{1}{m} \sum_{i=1}^{m} (g_{in}(\omega) - E(g_{in})) + \frac{s_m}{m} \sum_{i=1}^{d} \frac{1}{s_m} \sum_{h=1}^{s_m} E(g_{(h-1)d+i,n})$$

$$+ \frac{1}{m} \sum_{i=1}^{p_m} E(g_{s_m d+i,n}).$$

Claim 5:

$$\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} (g_{in}(\omega) - E(g_{in})) = 0 \text{ a.s.} \tag{4.9}$$

Since $\{g_{in} : i \ge 1\}$ is a sequence of independent random elements in $\mathcal{L}^p(\mathfrak{X})$ for each $n \ge 1$ and $\sum_{m=1}^{\infty} \frac{E\|g_{mn}\|^p}{m^p} < \infty$, there exists a negligible subset $N_2(n, y)$ such that for every $\omega \in \Omega \setminus N_2(n, y)$,

$$\frac{1}{m} \sum_{i=1}^{m} (g_{in}(\omega) - E(g_{in})) \to 0 \text{ as } m \to \infty.$$

Claim 6:

$$\frac{s_m}{m} \sum_{i=1}^{d} \frac{1}{s_m} \sum_{h=1}^{s_m} E(g_{(h-1)d+i,n}) \to \frac{1}{d} \sum_{i=1}^{d} z_{ij} \text{ as } m \to \infty. \tag{4.10}$$

For any $1 \le i \le d$,

$$\frac{1}{s_m} \sum_{h=1}^{s_m} E(g_{(h-1)d+i,n}) \to z_{ij} \text{ as } m \to \infty.$$

Thus, (4.10) is true.
Claim 7:

$$\frac{1}{m}\sum_{i=1}^{p_m} E(g_{s_m d+i,n}) \to 0 \text{ as } m \to \infty. \tag{4.11}$$

For any $1 \le i \le d$,

$$\frac{1}{m}E(g_{s_m d+i,n}) = \frac{s_m}{m}\Big(\frac{1}{s_m}\sum_{l=1}^{s_m} E(g_{ld+i,n}) - \frac{s_m-1}{s_m}\frac{1}{s_m-1}\sum_{l=1}^{s_m-1} E(g_{ld+i,n})\Big)$$
$$\to \frac{1}{d}(z_{ij} - 1.z_{ij}) = 0 \text{ as } m \to \infty,$$

whence Claim 7 follows by applying above estimate.

Combining the above limits and by (3.6) coming back to $(****)$ we have that for every $\omega \in \Omega \setminus N_2(n,y)$,

$$\frac{1}{m}\sum_{i=1}^{m} g_{in}(\omega) \to \frac{1}{d}\sum_{i=1}^{d} z_i \text{ as } m \to \infty. \tag{4.12}$$

Similarly, for each $m \ge 1$, there exists a negligible subset $N_3(m,y)$ such that for every $\omega \in \Omega \setminus N_3(m,y)$,

$$\frac{1}{n}\sum_{j=1}^{n} g_{mj}(\omega) \to \frac{1}{d}\sum_{i=1}^{d} z_i \text{ as } n \to \infty. \tag{4.13}$$

Final Step and Conclusion:

We define the negligible subset $N(y)$ as the union of the $N_1(y)$, $N_2(n,y)$ and $N_3(m,y)$ where $m \ge 1, n \ge 1$. Combining (4.8), (4.12), (4.13) and Proposition 3.3, we have that for every $\omega \in \Omega \setminus N(y)$,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(\omega) \to y \text{ as } m \vee n \to \infty.$$

The proof is therefore completed. □

Proposition 4.3. *Under the same hypotheses as in Proposition* 4.2*, there exists a negligible subset* N *such that, for any* $\omega \in \Omega \setminus N$ *and* $x \in \mathfrak{X}$*,*

$$\limsup_{m\vee n \to \infty} d(x, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)) \le d(x, \overline{\mathrm{co}}X). \tag{4.14}$$

Proof. Let $Z_{mn}(\omega) = \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)$ for every $m \ge 1, n \ge 1$, $\omega \in \Omega$. Further, let D' be a countable dense subset of X satisfying $\mathrm{cl}\, D' = \mathrm{cl}\, X$, and consider the set C' of all rational convex combinations of members of D'. On the other hand, consider a countable dense subset D of $\mathfrak{X}$ and observe that it suffices to prove (4.14) for all x in D. Indeed, each side of (4.14) defines a Lipschitz function of x (with Lipschitz constant 1). So, consider $x \in D$ and an integer $p \geq 1$. One can find $y' = y'(x,p) \in C'$, depending on x and p, such that

$$\|x - y'\| \leq d(x, \overline{\text{co}}\, X) + \frac{1}{p}.$$

Further, Proposition 4.2 applied to y', yields the existence of a negligible subset $N(x,p)$ and of a double array $\{g_{mn} : m \geq 1, n \geq 1\}$ verifying properties (i) and (ii). Then, define the negligible subset N as the union of the $N(x,p)$ where $x \in D$ and $p \geq 1$, and consider $\omega \in \Omega \setminus N$. For every $x \in D$, we have

$$\limsup_{m \vee n \to \infty} d(x, Z_{mn}(\omega)) \leq \lim_{m \vee n \to \infty} \|x - \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij}(\omega)\| = \|x - y'\| \leq d(x, \overline{\text{co}}\, X) + \frac{1}{p},$$

whence, by the arbitrariness of p, yields the desired conclusion. □

Theorem 4.4. *Under the same hypotheses as in Proposition* 4.2*, there exists a negligible subset* N *such that, for any* $\omega \in \Omega \setminus N$*,*

$$\overline{\text{co}} X = \mathcal{T}_W - \lim_{m \vee n \to \infty} \text{cl}\, \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)$$

that is, for any $x \in \mathfrak{X}$*,*

$$d(x, \overline{\text{co}} X) = \lim_{m \vee n \to \infty} d(x, \text{cl}\, \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)).$$

Proof. We begin by choosing a countable subset D^* of B^*, satisfying (2.3) relatively to the subset $C = \overline{\text{co}}\, X$, that is

$$d(x, C) = \sup_{z \in D^*} \{\langle z, x \rangle - \delta^*(z, C)\} \ \ \forall x \in \mathfrak{X}.$$

Let z be fixed in D^*. Since the map $F \to \delta^*(z, F)$ is Effros measurable from $c(\mathfrak{X})$ in $\overline{\mathbf{R}}$, we deduce that $\{\delta^*(z, X_{mn}) : m \geq 1, n \geq 1\}$ is a double array of $\overline{\mathbf{R}}$-valued independent random variables in $\mathcal{L}^p$ with

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{E|\delta^*(z, X_{mn})|^p}{(mn)^p} \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{E|X_{mn}|^p}{(mn)^p} < \infty.$$

Further, by (4.1), (4.2) and Proposition 3.1,

$$E\Big(\delta^*(z, X_{mn})\Big) = \delta^*(z, \text{cl} E(X_{mn}, \mathcal{A})) \to \delta^*(z, X) \ \text{ as } \ m \vee n \to \infty.$$

Hence, there exists a negligible subset $N(z)$ of Ω verifying, for every $\omega \in \Omega \setminus N(z)$,

$$\delta^*(z, C) = \lim_{m \vee n \to \infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta^*(z, X_{ij}(\omega)) = \lim_{m \vee n \to \infty} \delta^*(z, Z_{mn}(\omega)).$$

Now, defining the negligible subset N_1 as the union of the $N(z)$, for $z \in D^*$, we deduce that

$$\delta^*(z, C) = \lim_{m \vee n \to \infty} \delta^*(z, Z_{mn}(\omega)) \ \ \forall z \in D^*, \ \ \forall \omega \in \Omega \setminus N_1. \tag{4.15}$$

The above equality and Lemma 3.5 entail

$$\liminf_{m\vee n\to\infty} d(x, Z_{mn}(\omega)) \geq d(x, C) \ \ \forall x \in \mathfrak{X}, \ \ \forall \omega \in \Omega \setminus N_1. \tag{4.16}$$

On the other hand, Proposition 4.3 yields the existence of a negligible subset N_2 such that

$$\limsup_{m\vee n\to\infty} d(x, Z_{mn}(\omega)) \leq d(x, C) \ \ \forall x \in \mathfrak{X}, \ \ \forall \omega \in \Omega \setminus N_2. \tag{4.17}$$

Finish the proof by defining the negligible subset $N = N_1 \cup N_2$ and by combining inequalities (4.16) and (4.17). □

If the conditions (4.1) and (4.2) no longer satisfied then we get the following theorem

Theorem 4.5. *Let $1 \leq p \leq 2$ and let $\mathfrak{X}$ be a Rademacher type p Banach space. Consider a double array $\{X_{mn} : m \geq 1, n \geq 1\}$ of independent closed valued random sets such that $0 \in E(X_{mn}, \mathcal{A}_{X_{mn}})$ and for every choice of constants $\alpha > 0, \beta > 0$,*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E|X_{mn}|^p}{(m^\alpha n^\beta)^p} < \infty.$$

Then, there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$ and $x \in \mathfrak{X}$,

$$\limsup_{m\vee n\to\infty} d(x, \mathrm{cl}\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)) \leq d(x, \{0\}). \tag{4.18}$$

Proof. For all $m \geq 1, n \geq 1$, since $0 \in E(X_{mn}, \mathcal{A}_{X_{mn}})$, there exists $f_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$ such that $E(f_{mn}) = 0$.

Moreover, since $\{X_{mn} : m \geq 1, n \geq 1\}$ is a double array of independent closed valued random sets and f_{mn} is a $\mathcal{A}_{X_{mn}}$-measurable selection of X_{mn}, then $\{f_{mn} : m \geq 1, n \geq 1\}$ is independent.

On the other hand, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E\|f_{mn}\|^p}{(m^\alpha n^\beta)^p} \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E|X_{mn}|^p}{(m^\alpha n^\beta)^p} < \infty.$$

By Theorem 3.1 [27], we get

$$\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega) \to 0 \ \text{ a.s. as } \ m \vee n \to \infty.$$

Then, there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$ and $x \in \mathfrak{X}$,

$$\begin{aligned}\limsup_{m\vee n\to\infty} d\Big(x, \mathrm{cl}\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) &\leq \limsup_{m\vee n\to\infty} \Big\|x - \frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega)\Big\| \\ &= d(x, \{0\}).\end{aligned}$$

The proof is completed. □

In the next theorem, we obtain the Marcinkiewicz-Zygmund type law of large numbers for double array of independent closed valued random sets.

Theorem 4.6. *Let $\mathfrak{X}$ be a Rademacher type p $(1 < p \le 2)$ Banach space and let $\{X_{mn} : m \ge 1, n \ge 1\}$ be a double array of independent closed valued random sets with $0 \in E(X_{mn}, \mathcal{A}_{X_{mn}})$ for all $m \ge 1, n \ge 1$. Suppose that $\{X_{mn} : m \ge 1, n \ge 1\}$ is stochastically dominated by a random element X.*

(i) If $E\big(\|X\|^q \log^+ \|X\|\big) < \infty$ for some $q \in (0, p)$, then, there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$ and $x \in \mathfrak{X}$,

$$\limsup_{m\vee n\to\infty} d\Big(x, \mathrm{cl}\frac{1}{(mn)^{\frac{1}{q}}}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) \le d(x, \{0\}). \tag{4.19}$$

(ii) If $E\big(\|X\|(\log^+ \|X\|)^2\big) < \infty$, then, there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$ and $x \in \mathfrak{X}$,

$$\limsup_{m\vee n\to\infty} d\Big(x, \mathrm{cl}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)) \le d(x, \{0\}\Big). \tag{4.20}$$

Proof. Using the same argument as in the proof of Theorem 4.5 provides a selection f_{mn} of X_{mn} such that $E(f_{mn}) = 0$ and $\{f_{mn} : m \ge 1, n \ge 1\}$ is independent.

Since the event of $\{\|f_{mn}\| \ge t\} \subset \{|X_{mn}| \ge t\}$ for all $t \ge 0, m \ge 1, n \ge 1$, then

$$\mathbf{P}\{\|f_{mn}\| \ge t\} \le \mathbf{P}\{|X_{mn}| \ge t\} \le C\,\mathbf{P}\{\|X\| \ge t\} \text{ for all } t \ge 0, m \ge 1, n \ge 1.$$

Thus $\{f_{mn} : m \ge 1, n \ge 1\}$ is stochastically dominated by a random element X. Using Corollary 3.2 [28], we get

(*i*) If $E\big(\|X\|^q \log^+ \|X\|\big) < \infty$ for some $q \in (0, p)$, then

$$\frac{1}{(mn)^{1/q}}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega) \to 0 \text{ a.s. as } m \vee n \to \infty,$$

so, there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$ and $x \in \mathfrak{X}$,

$$\begin{aligned}\limsup_{m\vee n\to\infty} d\Big(x, \mathrm{cl}\frac{1}{(mn)^{\frac{1}{q}}}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) &\le \limsup_{m\vee n\to\infty} \Big\|x - \frac{1}{(mn)^{1/q}}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega)\Big\| \\ &= d(x, \{0\}).\end{aligned}$$

(*ii*) If $E\big(\|X\|(\log^+ \|X\|)^2\big) < \infty$, then

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega) \to 0 \text{ a.s. as } m \vee n \to \infty,$$

and so, there exists a negligible subset N such that, for any $\omega \in \Omega \setminus N$ and $x \in \mathfrak{X}$,

$$\limsup_{m\vee n\to\infty} d\Big(x, \mathrm{cl}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) \le \limsup_{m\vee n\to\infty} \Big\|x - \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega)\Big\| = d(x, \{0\}).$$

The proof is completed. □

The following examples show that in Theorems 4.5 and 4.6, the conclusion (4.18) (resp. (4.19)) (resp. (4.20)) cannot be replaced by the stronger one

$$\text{(I)} \qquad \mathcal{T}_W - \lim_{m\vee n\to\infty} \frac{1}{m^\alpha n^\beta} \mathrm{cl} \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega) = \{0\} \text{ a.s.}$$

$$\text{(II)} \qquad (\text{resp. } \mathcal{T}_W - \lim_{m\vee n\to\infty} \frac{1}{(mn)^{\frac{1}{q}}} \mathrm{cl} \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega) = \{0\} \text{ a.s.})$$

$$\text{(III)} \qquad (\text{resp. } \mathcal{T}_W - \lim_{m\vee n\to\infty} \frac{1}{mn} \mathrm{cl} \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega) = \{0\} \text{ a.s.}).$$

Example 4.1. Let $\mathfrak{X} = \mathbf{R}$, then $\mathfrak{X}$ is Rademacher type p $(1 < p \leq 2)$ Banach space. Put $X(\omega) = [-1, 1]$ for all $\omega \in \Omega$, then X is a random set and $\mathcal{A}_X = \{\emptyset, \Omega\}$. Put $f(\omega) = 0$ for all $\omega \in \Omega$, we get $f \in S^1(X, \mathcal{A}_X)$ and $E(f) = 0$.

Put $X_{mn} = X$, $f_{mn} = f$ for all $m \geq 1, n \geq 1$. We infer that $\{X_{mn} : m \geq 1, n \geq 1\}$ is a double array of independent random sets, $0 \in E(X_{mn}, \mathcal{A}_{X_{mn}})$. Let $\alpha = 1, \beta = 1$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{E|X_{mn}|^p}{(mn)^p} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(mn)^p} < \infty \text{ for } 1 < p \leq 2.$$

Next, put $g_{mn}(\omega) = g(\omega) = 1$ for all $\omega \in \Omega$ and $m \geq 1, n \geq 1$, we get $g_{mn} \in S^1(X_{mn}, \mathcal{A}_{X_{mn}})$. Then, we have

$$1 = g(\omega) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij}(\omega) \in \mathrm{cl}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega) \text{ for all } \omega \in \Omega.$$

Since then, for $x = 1$,

$$0 \leq \liminf_{m\vee n\to\infty} d(x, \mathrm{cl}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)) \leq \liminf_{m\vee n\to\infty} \|x - g(\omega)\| = 0 \text{ for all } \omega \in \Omega,$$

but $d(x, \{0\}) = 1$. Hence, for all $\omega \in \Omega$,

$$\liminf_{m\vee n\to\infty} d(x, \mathrm{cl}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)) \not\geq d(x, \{0\}) \text{ for } x = 1.$$

Thus $\{X_{mn} : m \geq 1, n \geq 1\}$ satisfies all the conditions of Theorem 4.5 but the conclusion (I) is not true.

Example 4.2. Let $\mathfrak{X} = \mathbf{R}$, then $\mathfrak{X}$ is Rademacher type p $(1 < p \leq 2)$ Banach space. The random variable g, the double array of random sets $\{X_{mn} : m \geq 1, n \geq 1\}$ and the measurable selections g_{mn} of X_{mn} are defined as in the above example. It is easy to check that $\{X_{mn} : m \geq 1, n \geq 1\}$ is stochastically dominated by the random variable g and g satisfies the conditions (i) and (ii) for all $q \in (0, p)$. However, for all $\omega \in \Omega$,

$$\liminf_{m\vee n\to\infty} d\Big(x, \mathrm{cl}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) \not\geq d(x, \{0\}) \text{ for } x = 1.$$

Hence, $\{X_{mn} : m \geq 1, n \geq 1\}$ satisfies all the conditions of Theorem 4.6 but the conclusions (II) (for $q = 1$) and (III) are not true.

Open problem. It is worth to address the question of the validity of Theorems 4.5 and 4.6 when the condition $0 \in E(X_{mn}, \mathcal{A}_{X_{mn}})$ is replaced by the condition $0 \in E(X_{mn}, \mathcal{A})$.

If X_{mn} having the values in $cc(\mathfrak{X})$ or $cwk(\mathfrak{X})$ then by Proposition 3.1(ii), the answer is positive. The following example shows that in the general case, when X_{mn} having the values in $c(\mathfrak{X})$, we can choose the double array of selections $\{f_{mn} : m \geq 1, n \geq 1\}$ such that $E(f_{mn}) = 0$, but $\{f_{mn} : m \geq 1, n \geq 1\}$ doesn't obey the strong law of large numbers. Hence, in this case, the method used in the proof of the Theorems 4.5 and 4.6 is not applicable to the above problem.

Example 4.3. Let $\Omega = [0, 1]$ and $\mathfrak{X} = \mathbf{R}$. Let $(\Omega, \mathcal{L}([0, 1]), dt)$ be a standard probability space, $\mathcal{L}([0, 1])$ is the σ-algebra of Lebesgue measurable sets on $[0, 1]$ and dt is the Lebesgue measure on $[0, 1]$. Put $X = \{\frac{-1}{2}, \frac{1}{2}\}$, then $\mathcal{A}_X = \{\emptyset, \Omega\}$.

Consider a random variable $f : \Omega \to \{\frac{-1}{2}, \frac{1}{2}\}$ such that $\mathbf{P}(f = \frac{1}{2}) = \mathbf{P}(f = \frac{-1}{2}) = \frac{1}{2}$. Obviously, f is a measurable selection of X and $E(f) = \frac{1}{2}.\frac{1}{2} + \frac{-1}{2}.\frac{1}{2} = 0$, $\mathbf{P}(f \neq 0) = 1 > 0$.

Put $X_{mn} = X$, $f_{mn} = f$ for all $m \geq 1, n \geq 1$. We infer that $\{X_{mn} : m \geq 1, n \geq 1\}$ is a double array of independent random sets, $f_{mn} \in S^1(X_{mn}, \mathcal{A})$ and $E(f_{mn}) = 0$ (so $0 \in E(X_{mn}, \mathcal{A})$). However, $\{f_{mn} : m \geq 1, n \geq 1\}$ doesn't obey the strong law of large numbers

$$\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij}(\omega) = f(\omega) \nrightarrow 0 \text{ a.s. as } m \vee n \to \infty.$$

5. Application to slice convergence

Before going further let us mention some simple lemmas.

Lemma 5.1. *Let φ be a real valued lower semicontinuous function defined on a topological space S and let D be a dense subset in S. The following holds:*

$$\sup_{x \in S} \varphi(x) = \sup_{x \in D} \varphi(x).$$

Lemma 5.2. *Assume that $\mathfrak{X}^*$ is separable. Let D_1^* be a dense sequence in the closed unit ball B^* of $\mathfrak{X}^*$. Then for all bounded closed convex subsets B and C in $\mathfrak{X}$, the following holds:*

$$D(B, C) = \sup_{x^* \in B^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, B)\} = \sup_{x^* \in D_1^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, B)\}.$$

Proof. Equality

$$D(B, C) = \sup_{x^* \in B^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, B)\}$$

follows from Hahn-Banach theorem, while the second equality

$$\sup_{x^* \in B^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, B)\} = \sup_{x^* \in D_1^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, B)\}$$

follows from the strong separability of $\mathfrak{X}^*$ and the above lemma, noting that the function $x^* \to -\delta^*(x^*, C) - \delta^*(-x^*, B)$ is strongly continuous on $\mathfrak{X}^*$. □

Lemma 5.3. *Assume that $\mathfrak{X}$ is separable. Let D_1^* be a dense sequence in the closed unit ball B^* of $\mathfrak{X}^*$ with respect to the Mackey topology. Then for all convex weakly compact subsets B and C in $\mathfrak{X}$, the following holds:*

$$D(B, C) = \sup_{x^* \in B^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, B)\} = \sup_{x^* \in D_1^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, B)\}.$$

Proof. Equality

$$D(B, C) = \sup_{x^* \in B^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, B)\}$$

follows from Hahn-Banach theorem, while equality

$$\sup_{x^* \in B^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, B)\} = \sup_{x^* \in D_1^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, B)\}$$

follows from Lemma 5.1, noting that the function $x^* \to -\delta^*(x^*, C) - \delta^*(-x^*, B)$ is continuous on $\mathfrak{X}^*$ with respect to the Mackey topology. □

Corollary 5.4. *Assume that the strong dual $\mathfrak{X}^*$ of $\mathfrak{X}$ is separable and the hypotheses and notations of Theorem 3.7 are satisfied with $E(X, \mathcal{A})$ bounded, then one has*

$$D(B, \overline{\text{co}} E(X, \mathcal{A})) = \lim_{m \vee n \to \infty} D\Big(B, \text{cl} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)\Big) \ \ a.s.$$

for all B in $cb(\mathfrak{X})$, in other words the double array $\{X_{mn} : m \geq 1, n \geq 1\}$ satisfies the strong law of large numbers with respect to the slice convergence.

Proof. Indeed, by Proposition 3.6 there exists a negligible subset N_1 such that, for any $\omega \in \Omega \setminus N_1$ and $x \in \mathfrak{X}$,

$$\limsup_{m \vee n \to \infty} d\Big(x, \text{cl} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)\Big) \leq d(x, \overline{\text{co}}\, E(X, \mathcal{A})).$$

Whence using the definition of the gap functional, we get formally, the "lim sup" part of the slice convergence under consideration,

$$\begin{aligned} \limsup_{m \vee n \to \infty} D\Big(B, \text{cl} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)\Big) &= \limsup_{m \vee n \to \infty} \inf_{x \in B} d\Big(x, \text{cl} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)\Big) \\ &\leq \inf_{x \in B} \limsup_{m \vee n \to \infty} d\Big(x, \text{cl} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)\Big) \\ &\leq \inf_{x \in B} d(x, \overline{\text{co}}\, E(X, \mathcal{A})) = D(B, \overline{\text{co}}\, E(X, \mathcal{A})) \end{aligned} \tag{5.1}$$

while the "lim inf" part follows easily from the techniques of the proof of Theorem 3.7 and the separability of $\mathfrak{X}^*$. Indeed, by Lemma 5.2 there exists a negligible subset N_2 such that, for any $\omega \in \Omega \setminus N_2$ and every $B \in cb(\mathfrak{X})$,

$$\liminf_{m \vee n \to \infty} D\Big(B, \text{cl} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)\Big) \geq \liminf_{m \vee n \to \infty} D\Big(B, \overline{\text{co}} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}(\omega)\Big)$$

$$= \liminf_{m\vee n\to\infty} \sup_{x^*\in B^*} \Big\{ -\delta^*\Big(x^*, \overline{\mathrm{co}}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) - \delta^*(-x^*, B)\Big\}$$

$$\geq \sup_{x^*\in D_1^*} \liminf_{m\vee n\to\infty} \Big\{ -\delta^*\Big(x^*, \overline{\mathrm{co}}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) - \delta^*(-x^*, B)\Big\}$$

$$\geq \sup_{x^*\in D_1^*} \Big\{ -\limsup_{m\vee n\to\infty} \delta^*\Big(x^*, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) - \delta^*(-x^*, B)\Big\}$$

$$\geq \sup_{x^*\in D_1^*} \{-\delta^*(x^*, \overline{\mathrm{co}}\,E(X,\mathcal{A})) - \delta^*(-x^*, B)\} = D(B, \overline{\mathrm{co}}\,E(X,\mathcal{A})). \tag{5.2}$$

We finish the proof by defining the negligible subset $N = N_1 \cup N_2$ and by combining inequalities (5.1) and (5.2). □

Corollary 5.5. *Assume that the strong dual $\mathfrak{X}^*$ of $\mathfrak{X}$ is separable and the hypotheses and notations of Theorem 4.4 are satisfied with X bounded, then one has*

$$D(B, \overline{\mathrm{co}}X) = \lim_{m\vee n\to\infty} D\Big(B, \mathrm{cl}\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) \quad a.s.$$

for all B in $cb(\mathfrak{X})$, in other words the double array $\{X_{mn} : m \geq 1, n \geq 1\}$ satisfies the strong law of large numbers with respect to the slice convergence.

Proof. Indeed, by Proposition 4.3 there exists a negligible subset N_1 such that, for any $\omega \in \Omega \setminus N_1$ and $x \in \mathfrak{X}$,

$$\limsup_{m\vee n\to\infty} d\Big(x, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) \leq d(x, \overline{\mathrm{co}}X).$$

Whence using the definition of the gap functional, we get formally, the "lim sup" part of the slice convergence under consideration,

$$\limsup_{m\vee n\to\infty} D\Big(B, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) = \limsup_{m\vee n\to\infty} \inf_{x\in B} d\Big(x, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big)$$

$$\leq \inf_{x\in B} \limsup_{m\vee n\to\infty} d\Big(x, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big)$$

$$\leq \inf_{x\in B} d(x, \overline{\mathrm{co}}X) = D(B, \overline{\mathrm{co}}X) \tag{5.3}$$

while the "lim inf" part can be achieved as in the preceding corollary making use of Lemma 5.2. Applying this lemma and (4.15), there exists a negligible subset N_2 such that, for any $\omega \in \Omega \setminus N_2$ and every $B \in cb(\mathfrak{X})$,

$$\liminf_{m\vee n\to\infty} D\Big(B, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) \geq \liminf_{m\vee n\to\infty} D\Big(B, \overline{\mathrm{co}}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big)$$

$$= \liminf_{m\vee n\to\infty} \sup_{x^*\in D_1^*} \Big\{ -\delta^*\Big(x^*, \overline{\mathrm{co}}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) - \delta^*(-x^*, B)\Big\}$$

$$\geq \sup_{x^*\in D_1^*} \liminf_{m\vee n\to\infty} \Big\{ -\delta^*\Big(x^*, \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n X_{ij}(\omega)\Big) - \delta^*(-x^*, B)\Big\}$$

$$\geq \sup_{x^*\in D_1^*} \Big\{ -\limsup_{m\vee n\to\infty} \delta^*\Big(x^*, \frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n X_{ij}(\omega)\Big) - \delta^*(-x^*, B)\Big\}$$

$$\geq \sup_{x^*\in D_1^*} \{-\delta^*(x^*, \overline{\text{co}}X) - \delta^*(-x^*, B)\} = D(B, \overline{\text{co}}X). \tag{5.4}$$

We finish the proof by defining the negligible subset $N = N_1 \cup N_2$ and by combining inequalities (5.3) and (5.4). ☐

Here are some variants of Corollaries 5.4 and 5.5. Let us denote $\mathcal{L}^1_{cwk(\mathfrak{X})}$ the space of all nonempty convex weakly compact valued random sets. From Theorem 3.7 we derive the following

Corollary 5.6. *Assume that the hypotheses and notations of Theorem 3.7 are satisfied with X in $\mathcal{L}^1_{cwk(\mathfrak{X})}$, then one has*

$$D(K, E(X, \mathcal{A})) = \lim_{m\vee n\to\infty} D\Big(K, \text{cl}\,\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n X_{ij}(\omega)\Big) \quad a.s.$$

for all K in $cwk(\mathfrak{X})$, in other words the double array $\{X_{mn} : m \geq 1, n \geq 1\}$ satisfies the strong law of large numbers with respect to the slice convergence.

Proof. Since X is a convex weakly compact valued integrable bounded random sets, then $E(X, \mathcal{A})$ is convex weakly compact, using the James theorem (see, e.g., [9], Proposition 6.2.3 and Remarks 6.2.4). Applying Proposition 3.6 to $\{X_{mn} : m \geq 1, n \geq 1\}$ yields

$$\limsup_{m\vee n\to\infty} d\Big(x, \text{cl}\,\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n X_{ij}(\omega)\Big) \leq d(x, E(X, \mathcal{A})) \;\; \forall x \in \mathfrak{X} \;\text{ a.s.}$$

Whence using the definition of the gap functional, we get formally, the "lim sup" part of the slice convergence under consideration,

$$\limsup_{m\vee n\to\infty} D\Big(K, \text{cl}\,\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n X_{ij}(\omega)\Big) = \limsup_{m\vee n\to\infty} \inf_{x\in K} d\Big(x, \text{cl}\,\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n X_{ij}(\omega)\Big)$$

$$\leq \inf_{x\in K} \limsup_{m\vee n\to\infty} d\Big(x, \text{cl}\,\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n X_{ij}(\omega)\Big)$$

$$\leq \inf_{x\in K} d(x, E(X, \mathcal{A})) = D(K, E(X, \mathcal{A})) \;\text{ a.s.}$$

while the "lim inf" part follows easily from the techniques of the proof of Theorem 3.7 and the Lemmas 5.1, 5.2 and 5.3. Indeed, invoking again the weak compactness assumption, for any convex weakly compact subset C and K in $cwk(\mathfrak{X})$ we have

$$D(K, C) = \sup_{x^*\in B^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, K)\} = \sup_{x^*\in D_1^*} \{-\delta^*(x^*, C) - \delta^*(-x^*, K)\}$$

where D_1^* is a countable dense subset in B^* with respect to the Mackey topology, remembering that the support function $x^* \mapsto -\delta^*(x^*, C) - \delta^*(-x^*, K)$ is continuous on $cwk(\mathfrak{X})^*$ with respect to this topology. Applying Lemma 5.3 and equality (3.7.1) in Theorem 3.7 it is not difficult to get the "lim inf" part

$$\begin{aligned}
\liminf_{m\vee n\to\infty} D\Big(K, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) &\geq \liminf_{m\vee n\to\infty} D\Big(K, \overline{\mathrm{co}}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big)\\
&= \liminf_{m\vee n\to\infty} \sup_{x^*\in B^*}\Big\{-\delta^*\Big(x^*, \overline{\mathrm{co}}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) - \delta^*(-x^*, K)\Big\}\\
&\geq \sup_{x^*\in B^*} \liminf_{m\vee n\to\infty}\Big\{-\delta^*\Big(x^*, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) - \delta^*(-x^*, K)\Big\}\\
&\geq \sup_{x^*\in D_1^*}\Big\{-\limsup_{m\vee n\to\infty}\delta^*\Big(x^*, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) - \delta^*(-x^*, K)\Big\}\\
&\geq \sup_{x^*\in D_1^*}\{-\delta^*(x^*, E(X,\mathcal{A})) - \delta^*(-x^*, K)\} = D(K, E(X,\mathcal{A})) \ \text{ a.s.}
\end{aligned}$$

□

From Theorem 4.4 we derive the following

Corollary 5.7. *Assume that the hypotheses and notations of Theorem 4.4 are satisfied with X in $cwk(\mathfrak{X})$, then one has*

$$D(K,X) = \lim_{m\vee n\to\infty} D\Big(K, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) \ \textit{a.s.}$$

for all K in $cwk(\mathfrak{X})$, in other words the double array $\{X_{mn} : m \geq 1, n \geq 1\}$ satisfies the strong law of large numbers with respect to the slice convergence.

Proof. Indeed, by Proposition 4.3 we have

$$\limsup_{m\vee n\to\infty} d\Big(x, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) \leq d(x,X) \ \ \forall x\in\mathfrak{X} \ \text{ a.s.}$$

Whence using the definition of the gap functional, we get formally, the "lim sup" part of the slice convergence under consideration,

$$\begin{aligned}
\limsup_{m\vee n\to\infty} D\Big(K, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) &= \limsup_{m\vee n\to\infty} \inf_{x\in K} d\Big(x, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big)\\
&\leq \inf_{x\in K}\limsup_{m\vee n\to\infty} d\Big(x, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big)\\
&\leq \inf_{x\in K} d(x,X) = D(K,X) \ \text{ a.s.}
\end{aligned}$$

while the "lim inf" part can be achieved as in the preceding corollary making use the formula

$$D(K,C) = \sup_{x^*\in D_1^*}\{-\delta^*(x^*,C) - \delta^*(-x^*,K)\}$$

which holds for any convex weakly compact K and C in $cwk(\mathfrak{X})$ here D_1^* is a countable dense subset in B^* with respect to the Mackey topology. Applying this equality and (4.15) we get easily

$$\begin{aligned}
\liminf_{m\vee n\to\infty} D\Big(K, \mathrm{cl}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) &\geq \liminf_{m\vee n\to\infty} D\Big(K, \overline{\mathrm{co}}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big)\\
&= \liminf_{m\vee n\to\infty}\sup_{x^*\in B^*}\Big\{-\delta^*\Big(x^*, \overline{\mathrm{co}}\,\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) - \delta^*(-x^*, K)\Big\}\\
&\geq \sup_{x^*\in B^*}\liminf_{m\vee n\to\infty}\Big\{-\delta^*\Big(x^*, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) - \delta^*(-x^*, K)\Big\}\\
&\geq \sup_{x^*\in D_1^*}\Big\{-\limsup_{m\vee n\to\infty}\delta^*\Big(x^*, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\omega)\Big) - \delta^*(-x^*, K)\Big\}\\
&\geq \sup_{x^*\in D_1^*}\{-\delta^*(x^*, X) - \delta^*(-x^*, K)\} = D(K, X) \ \text{ a.s.}
\end{aligned}$$

□

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