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MIXED DUALITY IN NONDIFFERENTIABLE G-INVEX MULTIOBJECTIVE PROGRAMMING

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Abstract. We consider a class of nondifferentiable multiobjective programs with inequality and equality constraints in which each component of the objective function contains a term involving the support function of a compact convex set. We introduce G-Mixed duality theorem for our nondifferentiable multiobjective programs. Also, we derive G-Mond-Weir type and G-Wolfe type duality theorems as special cases of our duality results. Our duality generalize and improve the results in Antczak [3] to the nondifferentiable case.

1. Introduction and Preliminaries

It was introduced that many results in nonlinear programming involving convex functions actually hold for a wider class of functions, called invex. Craven and Glover [4] proved duality theorems for the so-called cone invex programs. Egudo [5] established some duality results for differentiable multiobjective programming problems with invex functions. Jeyakumar [6] defined ρ-invexity for nonsmooth optimization problems, and Kuk et al. [10] defined the concept of V-ρ-invexity for vector valued functions, which is a generalization of the V-invex function [7, 11]. Mond and Schechter [12] studied nondifferentiable symmetric duality, in which the objective function contains a support function. Kim et al. [8] established necessary and sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable multiobjective fractional programming problems.

Very recently, Antczak [1] introduced a new class of nonconvex functions, called G-invex functions. Furthermore, they introduced new F. John-type and Karush-Kuhn-Tucker-type problems, called G-F.John and G-Karush-Kuhn-Tucker problems, respectively. Subsequently, Antczak [3] formulated new various vector dual problems for differentiable nonconvex multiobjective programming problems. In this way, they introduced various vector G-dual problems in the format of Mond-Weir, vector G-dual problem in the sense of Wolfe, and various vector mixed G-dual problems for the considered multiobjective programming problem.

In this paper, we obtain an extension of the results in Antczak [2, 3] from the differentiable to the nondifferentiable case. We introduce a mixed vector dual programming problem and establish the weak , strong and converse duality theorems.

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In order to prove duality theorems for the nondifferentiable multiobjective programming problems involving vector G-invex functions, we employ necessary optimality conditions, the so-called G-Karush-Kuhn-Tucker necessary optimality conditions, in Kim et al. [9]. As special cases of our duality results, G-Mond-Weir type and G-Wolfe type duality theorems are given.

We provide some definitions and some results that we shall use in the sequel. The following convention for equalities and inequalities will be used throughout the paper.

For any $x = (x_1, x_2, \ldots, x_n)^T, y = (y_1, y_2, \ldots, y_n)^T$, we define:

$$x = y \text{ if and only if } x_i = y_i \text{ for all } i = 1, 2, \ldots, n;$$
$$x < y \text{ if and only if } x_i < y_i \text{ for all } i = 1, 2, \ldots, n;$$
$$x \leqq y \text{ if and only if } x_i \leqq y_i \text{ for all } i = 1, 2, \ldots, n;$$
$$x \leq y \text{ if and only if } x_i \leqq y_i \text{ and } x \neq y, n > 1.$$

We also will use the same notation for row and column vectors when the interpretation is obvious. We say that a vector $z \in \mathbb{R}^n$ is negative if $z \leqq 0$ and strictly negative if $z < 0$.

Definition 1.1 ([12])**.** Let C be a compact convex set in $\mathbb{R}^n$. The support function $s(x|C)$ is defined by

$$s(x|C) := max\{x^T y : y \in C\}.$$

The support function $s(x|C)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that

$$s(y|C) \geq s(x|C) + z^T(y - x) \text{ for all } y \in D.$$

Equivalently,

$$z^T x = s(x|C).$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) := \{z \in C : z^T x = s(x|C)\}.$$

Let $f = (f_1, \ldots, f_k) : X \to \mathbb{R}^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subset \mathbb{R}^n$, and $I_{f_i}(X), i = 1, \ldots, k$, be the range of f_i, that is, the image of X under f_i.

Definition 1.2. Let $f : X \to \mathbb{R}^k$ be a vector-valued differentiable function defined on a nonempty set $X \subset \mathbb{R}^n$ and $u \in X$. If there exist a differentiable vector-valued function $G_f = (G_{f_1}, \ldots, G_{f_k}) : \mathbb{R} \to \mathbb{R}^k$ such that any its component $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$ is a strictly increasing function on its domain. If there exists a vector-valued function $\eta : X \times X \to \mathbb{R}^n$ such that, for all $x \in X (x \neq u)$ and for any $i = 1, \ldots, k$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geqq G'_{f_i}(f_i(u)) \nabla f_i(u) \eta(x, u) \ (>), \tag{1.1}$$

then f is said to be a (strictly) vector G_f-invex function at u on X(with respect to η) (or shortly, G-invex function at u on X). If (1.1) is satisfied for each $u \in X$, then f is vector G_f-invex on X with respect to η.

Remark 1.3 ([2]). In order to define an analogous class of (strictly) vector G_f-incave functions with respect to η, the direction of the inequality in the definition of these function should be changed to the opposite one.

Remark 1.4 ([2]). In the case when $G_{f_i}(a) \equiv a,\ i = 1, \ldots, k$, for any $a \in I_{f_i}(X)$, we obtain a definition of a vector-valued invex function.

We consider the following multiobjective programming problem :

$$
\begin{aligned}
(NMP) \quad & \text{Minimize} \quad (\ G_{F_1}(f_1(x) + s(x|C_1)), \ldots, (G_{F_k}(f_k(x) + s(x|C_k)) \ ) \\
& \text{subject to} \quad (\ G_{g_1}(g_1(x)), \ldots, G_{g_m}(g_m(x)) \ ) \leqq 0, \ j \in J \\
& \qquad\qquad\quad (\ G_{h_1}(h_1(x)), \ldots, G_{h_p}(h_p(x)) \ ) = 0, \ t \in T
\end{aligned}
$$

where $F_i : X \to \mathbb{R},\ i \in I = \{1, \ldots, k\},\ g_j : X \to \mathbb{R},\ j \in J = \{1, \ldots, m\},\ h_t : X \to \mathbb{R},\ t \in T = \{1, \ldots, p\}$ are differentiable functions on a nonempty open set $X \subset \mathbb{R}^n$. Moreover $G_{F_i}, i \in I$, are differentiable real-valued strictly increasing functions , $G_{g_j}, j \in J$, are differentiable real-valued strictly increasing functions, and $G_{h_t}, t \in T$, are differentiable real-valued strictly increasing functions. Let $D = \{x \in X : G_{g_j}(g_j(x)) \leqq 0,\ j \in J,\ \ G_{h_t}(h_t(x)) = 0, t \in T\}$ be the set of all feasible solutions for problem (NMP), and $F_i = f_i(\cdot) + (\cdot)^T w_i,\ i = 1, \ldots, k$. Further, we denote by $J(z) := \{j \in J : G_{g_j}(g_j(z)) = 0\}$ the set of inequality constraint functions active at $z \in D$ and by $I(z) := \{i \in I : \lambda_i > 0\}$ the objective functions indices set, for which the corresponding Lagrange multiplier is not equal to 0. For such optimization problems, minimization means in general obtaining weak Pareto solutions in the following sense:

Definition 1.5. A feasible point $\bar{x}$ is said to be a weak Pareto solution (a weakly efficient solution, a weak minimum) of (NMP) if there exists no other $x \in D$ such that

$$G_{f(x)+x^T w}(\ f(x) + s(x|C) \ ) < G_{f(\bar{x})+\bar{x}^T w}(\ f(\bar{x}) + s(\bar{x}|C) \ ).$$

2. Mixed duality theorems

In this section, we introduce a mixed dual programming problem and establish weak and strong duality theorems. Now we propose the following mixed dual (G-VMD) to (NMP).

(G-VMD) Maximize

$$
\Big(G_{F_1}(\ f_1(y) + y^T w_1 \ ) + \sum_{j \in J_0} \xi_j G_{g_j}(g_j(y)) + \sum_{t \in T_0} \mu_t G_{h_t}(h_t(y)), \ldots,
$$

$$
G_{F_k}(\ f_k(y) + y^T w_k \ ) + \sum_{j \in J_0} \xi_j G_{g_j}(g_j(y)) + \sum_{t \in T_0} \mu_t G_{h_t}(h_t(y)) \Big)
$$

subject to

$$
\sum_{i=1}^{k} \lambda_i G'_{F_i}(\ f_i(y) + y^T w_i \ )(\nabla f_i(y) + w_i) + \sum_{j=1}^{m} \xi_j G'_{g_j}(g_j(y)) \nabla g_j(y)
$$

$$+\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(y))\nabla h_t(y)=0,$$

$$\sum_{j\in J_\alpha}\xi_j G_{g_j}(g_j(y))+\sum_{t\in T_\alpha}\mu_t G_{h_t}(h_t(y))\geqq 0,$$

$$y\in X,\quad w_i\in C_i,\ i=1,\dots,k,$$

$$\lambda\in \mathbb{R}^k,\ \lambda\geq 0,\ \lambda^T e=1,$$

$$\xi\in\mathbb{R}^m,\ \xi\geqq 0,\ \mu\in\mathbb{R}^p,$$

where $J_\alpha\subset J=\{1,\dots,m\},\alpha=0,1,\dots,r$ with $\cup_{\alpha=0}^{r}J_\alpha=J$ and $J_\alpha\cap J_\beta=\emptyset$ if $\alpha\neq\beta$, $T_\alpha\subset T=\{1,\dots,p\},\alpha=0,1,\dots,r$ with $\cup_{\alpha=0}^{r}T_\alpha=T$ and $T_\alpha\cap T_\beta=\emptyset$ if $\alpha\neq\beta$.

Let $\Lambda^+=\{\lambda\in\mathbb{R}^p:\ \lambda\geqq 0,\ \lambda^T e=1,\ e=(1,\dots,1)^T\in\mathbb{R}^p\}$ and $G_{F_i}, i\in I$ be differentiable real-valued strictly increasing functions defined in $I_{F_i}(x)$, $G_{g_j}, j\in J$ be differentiable real-valued strictly increasing functions defined in $I_{g_j}(x)$.
Let W_1 denote the set of all feasible solutions for (G-VMD) and $pr_X W_1$ be the projection of the set W_1 on X, that is, $pr_X W_1:=\{y\in X:(y,\lambda,\xi,\mu,w)\in W_1\}$. Moreover, for a given $(y,\lambda,\xi,\mu,w)\in W_1$, we denote by $I(y)$ the set of objective functions indices for which a corresponding Lagrange multiplier is positive, that is, $I(y):=\{i\in I:\lambda_i>0\}$.

We define the so called vector G-Lagrange function. The vector G-Lagrange function $L_G: X\times R^k\times R^m\times R^p\times R^k\to R^k$

$$\begin{aligned}L_G(y,\lambda,\xi,\mu,w)=&diag\ \lambda(G_{F_1}(\ f_1(y)+s(x|C_1)\ ),\dots,G_{F_k}(\ f_k(y)+s(x|C_k)\ )^T\\&+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))e+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))e\\=&\big(\lambda_1 G_{F_1}(\ f_1(y)+s(x|C_1)\ )+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y)),\\&\dots,\\&\lambda_k G_{F_k}(\ f_k(y)+s(x|C_k)\ )+\sum_{j=1}^{m}\xi_j G_{g_j}(g_j(y))+\sum_{t=1}^{p}\mu_t G_{h_t}(h_t(y))\big)\end{aligned}$$

where

$$diag\ \lambda=\begin{bmatrix}\lambda_1 & 0 & 0 & \dots & 0\\ 0 & \lambda_2 & 0 & \dots & 0\\ 0 & & \ddots & & \vdots\\ \vdots & & & \ddots & 0\\ 0 & \dots & \dots & 0 & \lambda_k\end{bmatrix}$$

Theorem 2.1 (G-Weak Duality). *Let x and (y,λ,ξ,μ,w) be any feasible solutions for (NMP) and (G-VMD), respectively. Further, assume that $f(\cdot)+(\cdot)^T w$ is G_F-invex with respect to η at $y\in pr_X W_1$ on $D\cup pr_X W_1$, g is G_g-invex with respect to η at $y\in pr_X W_1$ on $D\cup pr_X W_1$, $h_t,\ t\in T^+(y)$ is G_{h_t}-invex with respect to η at*

$y \in pr_X W_1$ on $D \cup pr_X W_1$ and h_t, $t \in T^-(y)$ is G_{h_t}-incave with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$. Then the following holds:

$$\begin{aligned}(2.2)\quad &(G_{F_1}(f_1(x)+s(x|C_1)),\dots,G_{F_k}(f_k(x)+s(x|C_k)))\\ &\not< (G_{F_1}(f_1(y)+y^Tw_1)+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(y))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(y)),\dots,\\ &\qquad G_{F_k}(f_k(y)+y^Tw_k)+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(y))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(y))).\end{aligned}$$

Proof. Let x and (y,λ,ξ,μ,w) be any feasible solutions for (NMP) and (G-VMD), respectively. We proceed by contradiction. Suppose that

$$\begin{aligned}&(G_{F_1}(f_1(x)+s(x|C_1)),\dots,G_{F_k}(f_k(x)+s(x|C_k)))\\ &< (G_{F_1}(f_1(y)+y^Tw_1)+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(y))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(y)),\dots,\\ &\qquad G_{F_k}(f_k(y)+y^Tw_k)+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(y))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(y))).\end{aligned}$$

Since $G_{F_i}, i \in I$, are differentiable real-valued strictly increasing functions defined in $I_{F_i}(x)$. By definition of support function, we get

$$\begin{aligned}&(G_{F_1}(f_1(x)+x^Tw_1),\dots,G_{F_k}(f_k(x)+x^Tw_k))\\ &< (G_{F_1}(f_1(y)+y^Tw_1)+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(y))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(y)),\dots,\\ &\qquad G_{F_k}(f_k(y)+y^Tw_k)+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(y))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(y))).\end{aligned}$$

Therefore, for any $i \in I$

$$\begin{aligned}(2.3)\quad &G_{F_i}(f_i(x)+x^Tw_i) - G_{F_i}(f_i(y)+y^Tw_i)\\ &< \sum_{j\in J_0}\xi_j G_{g_j}(g_j(y))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(y)).\end{aligned}$$

Since $\lambda \geq 0$, then (2.3) gives

$$\begin{aligned}&\sum_{i=1}^k \lambda_i G_{F_i}(f_i(x)+x^Tw_i) - \sum_{i=1}^k \lambda_i G_{F_i}(f_i(y)+y^Tw_i)\\ &< \sum_{i=1}^k \lambda_i\Big[\sum_{j\in J_0}\xi_j G_{g_j}(g_j(y))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(y)) \Big].\end{aligned}$$

From the feasibility of (y,λ,ξ,μ,w) in (G-VMD), we have $\sum_{i=1}^k \lambda_i = 1$. Then the inequality above implies

$$\begin{aligned}(2.4)\quad &\sum_{i=1}^k \lambda_i G_{F_i}(f_i(x)+x^Tw_i) - \sum_{i=1}^k \lambda_i G_{F_i}(f_i(y)+y^Tw_i)\\ &< \sum_{j\in J_0}\xi_j G_{g_j}(g_j(y))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(y)).\end{aligned}$$

From $x \in D$ follows that $G_{g_j}(g_j(x)) \leqq 0,\ j \in J$ and $G_{h_t}(h_t(x)) = 0,\ t \in T$. Thus from the feasibility of $(y, \lambda, \xi, \mu, w)$ in (G-VMD), it follows that

$$\sum_{j \in J_0} \xi_j G_{g_j}(g_j(x)) + \sum_{t \in T_0} \mu_t G_{h_t}(h_t(x)) \leqq 0 \tag{2.5}$$

By assumption, $f(\cdot) + (\cdot)^T w$ is G_F-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, $g_j,\ j \in J_0 \cup J_\alpha$ is G_{g_j}-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$, $h_t,\ t \in T^+(y)$ is G_{h_t}-invex with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$ and $h_t,\ t \in T^-(y)$ is G_{h_t}-incave with respect to η at $y \in pr_X W_1$ on $D \cup pr_X W_1$. Then, by Definition 1.2, we have

$$\begin{aligned}
&(G_{F_i}(\ f_i(x) + x^T w_i\ ) - (G_{F_i}(\ f_i(y) + y^T w_i\ ) \\
&\qquad \geqq [G'_{F_i}(f_i(y) + y^T w_i)(\nabla f_i(y) + w_i)]\ \eta(x, y), \\
&G_{g_j}(g_j(x)) - G_{g_j}(g_j(y)) \geqq G'_{g_j}(g_j(y)) \nabla g_j(y)\ \eta(x, y), \\
&G_{h_t}(h_t(x)) - G_{h_t}(h_t(y)) \geqq G'_{h_t}(h_t(y)) \nabla h_t(y)\ \eta(x, y),\ t \in T^+(y) \\
&G_{h_t}(h_t(x)) - G_{h_t}(h_t(y)) \leqq G'_{h_t}(h_t(y)) \nabla h_t(y)\ \eta(x, y),\ t \in T^-(y).
\end{aligned}$$

respectively.

From the feasibility of $(y, \lambda, \xi, \mu, w)$ in (G-VMD), it follows that

$$\begin{aligned}
&\sum_{i=1}^{k} \lambda_i G_{F_i}(\ f_i(x) + x^T w_i\ ) - \sum_{i=1}^{k} \lambda_i G_{F_i}(\ f_i(y) + y^T w_i\ ) \\
&\qquad \geqq \sum_{i=1}^{k} \lambda_i G'_{F_i}(f_i(y) + y^T w_i)(\nabla f_i(y) + w_i) \eta(x, y),
\end{aligned} \tag{2.6}$$

$$\sum_{j \in J_0} \xi_j G_{g_j}(g_j(x)) - \sum_{j \in J_0} \xi_j G_{g_j}(g_j(y)) \geqq \sum_{j \in J_0} \xi_j G'_{g_j}(g_j(y)) \nabla g_j(y) \eta(x, y), \tag{2.7}$$

$$\begin{aligned}
&\sum_{t \in T_0} \mu_t G_{h_t}(h_t(x)) - \sum_{t \in T_0} \mu_t G_{h_t}(h_t(y)) \geqq \sum_{t \in T_0} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y) \eta(x, y), \\
&\qquad t \in T^+(y) \cap T_0,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&\sum_{t \in T_0} \mu_t G_{h_t}(h_t(x)) - \sum_{t \in T_0} \mu_t G_{h_t}(h_t(y)) \leqq \sum_{t \in T_0} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y) \eta(x, y), \\
&\qquad t \in T^-(y) \cap T_0.
\end{aligned} \tag{2.9}$$

By (2.4) and (2.6)

$$\begin{aligned}
&\sum_{j \in J_0} \xi_j G_{g_j}(g_j(y)) + \sum_{t \in T_0} \mu_t G_{h_t}(h_t(y)) \\
&\qquad > \sum_{i=1}^{k} \lambda_i G'_{F_i}(f_i(y) + y^T w_i)(\nabla f_i(y) + w_i) \eta(x, y).
\end{aligned} \tag{2.10}$$

Adding both sides of inequalities (2.7), (2.8) and (2.10), we get

$$\sum_{j\in J_0}\xi_j G_{g_j}(g_j(x)) + \sum_{t\in T_0}\mu_t G_{h_t}(h_t(x)) > \Big[\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(y)+y^T w_i)(\nabla f_i(y)+w_i)$$

$$+\sum_{j\in J_0}\xi_j G'_{g_j}(g_j(y))\nabla g_j(y) + \sum_{t\in T_0}\mu_t G'_{h_t}(h_t(y))\nabla h_t(y)\Big]\eta(x,y). \tag{2.11}$$

From the feasibility of (y,λ,ξ,μ,w) in (G-VMD), we also get

$$0 \geqq \Big[\sum_{j\in J_\alpha}\xi_j G'_{g_j}(g_j(y))\nabla g_j(y) + \sum_{t\in T_\alpha}\mu_t G'_{h_t}(h_t(y))\nabla h_t(y)\Big]\eta(x,y). \tag{2.12}$$

By (2.5), (2.11) and (2.12)

$$\Big[\sum_{i=1}^{k}\lambda_i G'_{F_i}(f_i(y)+y^T w_i)(\nabla f_i(y)+w_i) + \sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(y))\nabla g_j(y)$$

$$+\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(y))\nabla h_t(y)\Big]\eta(x,y) < 0,$$

which contradicts the feasibility of (y,λ,ξ,μ,w) in (G-VMD). □

Theorem 2.2 (G-Weak Duality)**.** *Let x and (y,λ,ξ,μ,w) be any feasible solutions for (NMP) and (G-VMD), respectively. If the G-Lagrange function L_G is invex with respect to η at y on $D\cup pr_X W_1$. Further, assume that $g_j,\ j\in J_\alpha$ is G_{g_j}-invex with respect to η at $y\in pr_X W_1$ on $D\cup pr_X W_1$, $h_t,\ t\in T^+(y)\cap T_\alpha$ is G_{h_t}-invex with respect to η at $y\in pr_X W_1$ on $D\cup pr_X W_1$ and $h_t,\ t\in T^-(y)\cap T_\alpha$ is G_{h_t}-incave with respect to η at $y\in pr_X W_1$ on $D\cup pr_X W_1$.Then the relation (2.2) is fulfilled.*

The proof is similar to the one used for Theorem 2.1.

Theorem 2.3 (G-Strong Duality)**.** *Let $\bar{x}$ be a weak Pareto solution for problem (NMP). We assume that $G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}), t=1,\ldots,p$ are linearly independent, and there exists $z^*\in \mathbb{R}^n$ such that $\left\langle G'_{g_j}(g_j(\bar{x})\nabla g_j(\bar{x}), z^*\right\rangle < 0, j\in J(\bar{x})$ and $\left\langle G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}), z^*\right\rangle = 0, t=1,\ldots,p$. Then there exist $\lambda\in\mathbb{R}^k_+,\ \xi\in\mathbb{R}^m_+,\ \mu\in\mathbb{R}^p, w_i\in C_i, i=1,\ldots,k,$ such that $(\bar{x},\lambda,\xi,\mu,w)$ is feasible for (G-VMD) and $\langle\bar{x},w_i\rangle = s(\bar{x}|C_i)$. Moreover, the objective functions of (NMP) and (G-VMD) are equal at these points. If also G-weak duality (Theorem 2.1) between (NMP) and (G-VMD) holds, then $(\bar{x},\lambda,\xi,\mu,w)$ is a weak Pareto solution for (G-VMD).*

Proof. By assumption, $\bar{x}$ is a weak Pareto solution for (NMP). We assume that $G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}), t=1,\ldots,p$ are linearly independent, and there exists $z^*\in\mathbb{R}^n$ such that $\left\langle G'_{g_j}(g_j(\bar{x})\nabla g_j(\bar{x}), z^*\right\rangle < 0, j\in J(\bar{x})$ and $\left\langle G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}), z^*\right\rangle = 0, t = 1,\ldots,p$. That is, the Karush-Kuhn-Tucker constraint qualification be satisfied at $\bar{x}$ [9]. Then there exist $\lambda\in\mathbb{R}^k, \xi\in\mathbb{R}^m$, and $\mu\in\mathbb{R}^p, \lambda\geq 0, \xi\geqq 0, w_i\in C_i, i = 1,\ldots,k$, such that

$$\sum_{i=1}^{k}\lambda_i G'_{F_i}(\ f_i(\bar{x})+\bar{x}^T w_i\ )(\ \nabla f_i(\bar{x})+w_i\ ) + \sum_{j=1}^{m}\xi_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})$$

$$+\sum_{t=1}^{p}\mu_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})=0, \tag{2.13}$$

$$\xi_j G_{g_j}(g_j(\bar{x}))=0,\ j\in J. \tag{2.14}$$

From the assumption, we get

$$\sum_{t\in T_0}\mu_t G_{h_t}(h_t(\bar{x}))=0, \tag{2.15}$$

Using (2.13)-(2.15), we obtain the feasibility of $(\bar{x},\lambda,\xi,\mu,w)$ in (G-VMD) and the objective functions of (NMP) and (G-VMD) are equal at these points. Notice that, $G_{F_i}(f(\bar{x})+s(\bar{x}|C_i)) = G_{F_i}(f_i(\bar{x})+\bar{x}^T w_i) = G_{F_i}(f_i(\bar{x})+\bar{x}^T w_i)+\sum_{j\in J_0}\xi_j(G_{g_j}(g_j(\bar{x})))+\sum_{t\in T_0}\mu_t(G_{h_t}(h_t(\bar{x})))$.

By G-weak duality,

$$\begin{aligned}
&(\ G_{F_1}(f_1(\bar{x})+s(\bar{x}|C_1)\ ),\dots,G_{F_k}(f_k(\bar{x})+s(\bar{x}|C_k)\ )\\
&\quad\not\leq (\ G_{F_1}(f_1(\bar{y})+\bar{y}^T w_1)+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(\bar{y}))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(\bar{y}))\ ,\dots,\\
&\qquad\qquad G_{F_k}(f_k(\bar{y})+\bar{y}^T w_k)+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(\bar{y}))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(\bar{y}))\ ),
\end{aligned}$$

where $(\bar{y},\lambda,\xi,\mu,w)$ is any feasible solution of (G-VMD).
Since $\langle\bar{x},w_i\rangle = s(\bar{x}|C_i)$, we have

$$\begin{aligned}
&(\ G_{F_1}(\ f_1(\bar{x})+\bar{x}^T w_1\ )+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(\bar{x}))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(\bar{x}))\ ,\dots\\
&\quad G_{F_k}(\ f_k(\bar{x})+\bar{x}^T w_k\ )+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(\bar{x}))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(\bar{x}))\ ),\\
&\quad\not\leq (\ G_{F_1}(\ f_1(\bar{y})+\bar{y}^T w_1\ )+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(\bar{y}))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(\bar{y}))\ ,\dots,\\
&\qquad\qquad G_{F_k}(\ f_k(\bar{y})+\bar{y}^T w_k\ )+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(\bar{y}))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(\bar{y}))\ ).
\end{aligned}$$

Since $(\bar{x},\lambda,\xi,\mu,w)$ is a feasible solution for (G-VMD), $(\bar{x},\lambda,\xi,\mu,w)$ is a weak Pareto solution for (G-VMD). Hence the result holds. □

Theorem 2.4 (G-Converse Duality)**.** *Let $(\bar{y},\lambda,\xi,\mu,w)$ be a weak Pareto solution for (G-VMD) such that $\bar{y}\in D$. Moreover, we assume that G-Lagrange function L_G is invex with respect to η at $\bar{y}$ on $D\cup pr_X W_1$. Then $\bar{y}$ is a weak Pareto solution for (NMP).*

Proof. Let $(\bar{y},\lambda,\xi,\mu,w)$ be a weak Pareto solution for (G-VMD) such that $\bar{y}\in D$. Suppose contrary to the result, that $\bar{y}$ is not a weak Pareto solution for (NMP), that is, there exists $\tilde{x}\in D$ such that

$$G_{F_i}(\ f_i(\tilde{x})+s(\tilde{x}|C_i)\ )+\sum_{j\in J_0}\xi_j G_{g_j}(g_j(\tilde{x}))+\sum_{t\in T_0}\mu_t G_{h_t}(h_t(\tilde{x}))$$

$$< G_{F_i}(\ f_i(\bar{y}) + \bar{y}^T w_i\ ) + \sum_{j\in J_0} \xi_j G_{g_j}(g_j(\bar{y})) + \sum_{t\in T_0} \mu_t G_{h_t}(h_t(\bar{y})).$$

Since $\langle \tilde{x}, w_i\rangle = s(\tilde{x}|C_i), i = 1, \ldots, k$ and $(\bar{y}, \lambda, \xi, \mu, w)$ is a weak Pareto solution for (G-VMD). Thus we get

$$G_{F_i}(\ f_i(\tilde{x}) + \tilde{x}^T w_i\ ) + \sum_{j\in J_0} \xi_j G_{g_j}(g_j(\tilde{x})) + \sum_{t\in T_0} \mu_t G_{h_t}(h_t(\tilde{x}))$$

$$< G_{F_i}(\ f_i(\bar{y}) + \bar{y}^T w_i\ ) + \sum_{j\in J_0} \xi_j G_{g_j}(g_j(\bar{y})) + \sum_{t\in T_0} \mu_t G_{h_t}(h_t(\bar{y})). \tag{2.16}$$

Hence, by the G-Karush-kuhn-Tucker necessary optimality condition [9].

$$\sum_{j\in J_\alpha} \xi_j G_{g_j}(g_j(\tilde{x})) \leqq \sum_{j\in J_\alpha} \xi_j G_{g_j}(g_j(\bar{y})). \tag{2.17}$$

Since $\tilde{x} \in D$ and $\bar{y} \in D$, then

$$\sum_{t\in T_\alpha} \mu_t G_{h_t}(h_t(\tilde{x})) - \sum_{t\in T_\alpha} \mu_t G_{h_t}(h_t(\bar{y})) = 0. \tag{2.18}$$

By (2.16)-(2.18), we get for any $i = 1, \ldots, k$,

$$G_{F_i}(\ f_i(\tilde{x}) + \tilde{x}^T w_i) + \sum_{j\in J_0} \xi_j G_{g_j}(g_j(\tilde{x})) + \sum_{j\in J_\alpha} \xi_j G_{g_j}(g_j(\tilde{x}))$$
$$+ \sum_{t\in T_0} \mu_t G_{h_t}(h_t(\tilde{x})) + \sum_{t\in T_\alpha} \mu_t G_{h_t}(h_t(\tilde{x}))$$
$$< G_{F_i}(\ f_i(\bar{y}) + \bar{y}^T w_i) + \sum_{j\in J_0} \xi_j G_{g_j}(g_j(\bar{y})) + \sum_{j\in J_\alpha} \xi_j G_{g_j}(g_j(\bar{y}))$$
$$+ \sum_{t\in T_0} \mu_t G_{h_t}(h_t(\bar{y})) + \sum_{t\in T_\alpha} \mu_t G_{h_t}(h_t(\bar{y})). \tag{2.19}$$

Since $\lambda_i \geq 0, i \in I$, and $\cup_{\alpha=0}^r J_\alpha = J, \cup_{\alpha=0}^r T_\alpha = T$, then (2.19) yields

$$\sum_{i=1}^k \lambda_i G_{F_i}(\ f_i(\tilde{x}) + \tilde{x}^T w_i\ ) + \sum_{i=1}^k \lambda_i \Big[\sum_{j=1}^m \xi_j G_{g_j}(g_j(\tilde{x})) + \sum_{t=1}^p \mu_t G_{h_t}(h_t(\tilde{x}))\Big]$$
$$< \sum_{i=1}^k \lambda_i G_{F_i}(\ f_i(\bar{y}) + \bar{y}^T w_i\ ) + \sum_{i=1}^k \lambda_i \Big[\sum_{j=1}^m \xi_j G_{g_j}(g_j(\bar{y})) + \sum_{t=1}^p \mu_t G_{h_t}(h_t(\bar{y}))\Big].$$

From the feasibility of $(\bar{y}, \lambda, \xi, \mu, w)$ in(G-VMD), we have $\sum_{i=1}^k \lambda_i = 1$. Then, the inequality above implies

$$\sum_{i=1}^k \lambda_i G_{F_i}(\ f_i(\tilde{x}) + \tilde{x}^T w_i\ ) + \Big[\ \sum_{j=1}^m \xi_j G_{g_j}(g_j(\tilde{x})) + \sum_{t=1}^p \mu_t G_{h_t}(h_t(\tilde{x}))\ \Big]$$
$$< \sum_{i=1}^k \lambda_i G_{F_i}(f_i(\bar{y}) + \bar{y}^T w_i\ ) + \Big[\ \sum_{j=1}^m \xi_j G_{g_j}(g_j(\bar{y})) + \sum_{t=1}^p \mu_t G_{h_t}(h_t(\bar{y}))\ \Big]. \tag{2.20}$$

By assumption, the G-Lagrange function L_G is invex with respect to η at $\bar{y}$ on $D \cup pr_X W_1$. Then, by Remark 1.4, it follows that

$$L_G(\tilde{x}, \lambda, \xi, \mu, w) - L_G(\bar{y}, \lambda, \xi, \mu, w) \geqq \nabla L_G(\bar{y}, \lambda, \xi, \mu, w)\ \eta(\tilde{x}, \bar{y}).$$

Hence, from the definition of the G-Lagrange function L_G, it follows for any $i = 1, \ldots, k$,

$$\begin{aligned} &\Big(\lambda_i G_{F_i}(\ f_i(\tilde{x}) + \tilde{x}^T w_i\ ) + \Big[\sum_{j=1}^m \xi_j G_{g_j}(g_j(\tilde{x})) + \sum_{t=1}^p \mu_t G_{h_t}(h_t(\tilde{x})) \Big] \Big) \\ &\quad - \Big(\lambda_i G_{F_i}(\ f_i(\bar{y}) + \bar{y}^T w_i\ ) + \Big[\sum_{j=1}^m \xi_j G_{g_j}(g_j(\bar{y})) + \sum_{t=1}^p \mu_t G_{h_t}(h_t(\bar{y})) \Big] \Big) \\ &\geq \Big[\lambda_i G'_{F_i}(\ f_i(\bar{y}) + y^T w_i\ )(\nabla f_i(\bar{y}) + w_i) \\ &\quad + \sum_{j=1}^m \xi_j G'_{g_j}(g_j(\bar{y}))\nabla g_j(\bar{y}) + \sum_{t=1}^p \mu_t G'_{h_t}(h_t(\bar{y}))\nabla h_t(\bar{y}) \Big]\ \eta(\tilde{x}, \bar{y}). \end{aligned}$$

Adding both sides of the inequalities above and using $\sum_{i=1}^k \lambda_i = 1$, we get

$$\begin{aligned} &\sum_{i=1}^k \lambda_i G_{F_i}(\ f_i(\tilde{x}) + \tilde{x}^T w_i\ ) + \Big[\sum_{j=1}^m \xi_j G_{g_j}(g_j(\tilde{x})) + \sum_{t=1}^p \mu_t G_{h_t}(h_t(\tilde{x})) \Big] \\ &\quad - \sum_{i=1}^k \lambda_i G_{F_i}(\ f_i(\bar{y}) + \bar{y}^T w_i\ ) - \Big[\sum_{j=1}^m \xi_j G_{g_j}(g_j(\bar{y})) + \sum_{t=1}^p \mu_t G_{h_t}(h_t(\bar{y})) \Big] \\ &\quad \geq \Big[\sum_{i=1}^k \lambda_i G'_{F_i}(\ f_i(\bar{y}) + \bar{y}^T w_i\ )(\nabla f_i(\bar{y}) + w_i) \\ &\quad + \sum_{j=1}^m \xi_j G'_{g_j}(g_j(\bar{y}))\nabla g_j(\bar{y}) + \sum_{t=1}^p \mu_t G'_{h_t}(h_t(\bar{y}))\nabla h_t(\bar{y}) \Big]\ \eta(\tilde{x}, \bar{y}). \end{aligned} \tag{2.21}$$

By (2.26) and (2.27), we obtain the following inequality

$$\begin{aligned} &\Big[\sum_{i=1}^k \lambda_i G'_{F_i}(f_i(\bar{y}) + \bar{y}^T w_i)(\nabla f_i(\bar{y}) + w_i\ ) + \sum_{j=1}^m \xi_j G'_{g_j}(g_j(\bar{y}))\nabla g_j(\bar{y}) \\ &\qquad + \sum_{t=1}^p \mu_t G'_{h_t}(h_t(\bar{y}))\nabla h_t(\bar{y}) \Big]\ \eta(\tilde{x}, \bar{y}) < 0, \end{aligned}$$

which contradicts the dual constraint of problem(G-VMD). Thus, the conclusion of theorem is proved. □

Theorem 2.5 (No-maximal G-Converse Duality)**.** *Let $(\bar{y}, \lambda, \xi, \mu, w)$ be a feasible solution for (G-VMD) such that $\bar{y} \in D$. Moreover, we assume that the G-Lagrange function L_G is (strictly) invex with respect η at $\bar{y}$ on $D \cup pr_X W_1$. Then $\bar{y}$ is a weak Pareto solution for (NMP).*

Theorem 2.6 (G-restricted Converse Duality). *Let $\bar{x}$ and $(\bar{y}, \lambda, \xi, \mu, w)$ be feasible solutions for (NMP) and (G-VMD), respectively, such that*

$$\begin{aligned}(\ G_{F_1}(\ f_1(\bar{x}) + \bar{x}^T w_1 \ ), \ldots, G_{F_k}(\ f_k(\bar{x}) + \bar{x}^T w_k \ ) \ ) \\ = (\ G_{F_1}(\ f_1(\bar{y}) + \bar{y}^T w_1 \ ), \ldots, G_{F_k}(\ f_k(\bar{y}) + \bar{y}^T w_k \ )) \\ +[\ \sum_{j \in J_0} \xi_j G_{g_j}(g_j(\bar{y})) + \sum_{t \in T_0} \mu_t G_{h_t}(h_t(\bar{y})) \ ]e.\end{aligned}$$

Moreover, assume that, for any fixed $\lambda \in \mathbb{R}^k, \lambda \geq 0, \xi \in \mathbb{R}^m, \xi \geqq 0, \mu \in \mathbb{R}^p$, the G-Lagrange function L_G is strictly invex at $\bar{y}$ on $D \cup pr_X W_1$ with respect to η. Then $\bar{x}$ is a weak Pareto solution for (NMP) and $(\bar{y}, \lambda, \xi, \mu, w)$ is a weak Pareto solution for (G-VMD).

The proof of Theorem 2.5 and 2.6 follow directly from the weak duality theorem.(Theorem 2.2)

3. Special cases

As special cases of our duality results between (NMP) and (G-VMD), we give Mond-Weir type and Wolfe type duality theorems.

G-Mond-Weir Type

If $J_0 = T_0 = \emptyset$, $\bigcup_{\alpha=1}^{r} J_\alpha = J$, $\bigcup_{\alpha=1}^{r} T_\alpha = T$, then (G-VMD) reduced to the Mond-Weir type dual (G-VMWD).

(G-VMWD) Maximize $(\ G_{F_1}(\ f_1(y) + y^T w_1 \ ), \ldots, G_{F_k}(\ f_k(y) + y^T w_k \ )$

subject to
$$\begin{aligned}&\sum_{i=1}^{k} \lambda_i G'_{F_i}(\ f_i(y) + y^T w_i \ )(\nabla f_i(y) + w_i \ ) \\ &+ \sum_{j=1}^{m} \xi_j G'_{g_j}(g_j(y)) \nabla g_j(y) + \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y) = 0, \\ &\sum_{j=1}^{m} \xi_j G_{g_j}(g_j(y)) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)) \geqq 0, \\ &y \in X, \quad w_i \in C_i, \ i = 1, \ldots, k, \\ &\lambda \in \mathbb{R}^k, \ \lambda \geq 0, \ \lambda^T e = 1, \\ &\xi \in \mathbb{R}^m, \ \xi \geqq 0, \ \mu \in \mathbb{R}^p.\end{aligned}$$

Since the set of all feasible solutions for problem (G-VMWD) is the same as the set of all feasible solutions for problem (G-VMD), we denote it by W_1.

G-Wolfe Type

If $J_0 = J, T_0 = T$, $\bigcup_{\alpha=1}^{r} J_\alpha = \bigcup_{\alpha=1}^{r} T_\alpha = \emptyset$, then (G-VMD) reduced to the Wolfe type dual (G-VWD).

(G-VWD) Maximize

$$\Big(G_{F_1}\big(f_1(y) + y^T w_1 \big) + \sum_{j=1}^{m} \xi_j G_{g_j}(g_j(y)) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)), \ldots,$$
$$G_{F_k}\big(f_k(y) + y^T w_k \big) + \sum_{j=1}^{m} \xi_j G_{g_j}(g_j(y)) + \sum_{t=1}^{p} \mu_t G_{h_t}(h_t(y)) \Big)$$

subject to
$$\sum_{i=1}^{k} \lambda_i G'_{F_i}\big(f_i(y) + y^T w_i \big)(\nabla f_i(y) + w_i)$$
$$+ \sum_{j=1}^{m} \xi_j G'_{g_j}(g_j(y)) \nabla g_j(y)$$
$$+ \sum_{t=1}^{p} \mu_t G'_{h_t}(h_t(y)) \nabla h_t(y) = 0,$$
$$y \in X, \quad w_i \in C_i, \; i = 1, \ldots, k,$$
$$\lambda \in \mathbb{R}^k, \; \lambda \geq 0, \; \lambda^T e = 1,$$
$$\xi \in \mathbb{R}^m, \; \xi \geqq 0, \; \mu \in \mathbb{R}^p.$$

Let W_2 denote the set of all feasible solutions for (G-VWD) and $pr_X W_2$ be the projection of the set W_2 on X, that is, $pr_X W_2 := \{y \in X : (y, \lambda, \xi, \mu, w) \in W_2\}$.

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