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ON VECTOR OPTIMIZATION PROBLEM AND VECTOR MATRIX GAME EQUIVALENCE

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ABSTRACT. A vector matrix game with more than two skew symmetric matrices, which is an extension of the matrix game, is defined and a nonlinear vector optimization problem is considered. We formulate a dual problem for the nonlinear vector optimization problem and establish equivalence between the dual problem and its corresponding vector matrix game. Moreover, we give a numerical example illustrating such equivalent relations.

1. Introduction

A matrix game is defined by B of real $n \times m$ matrix together with the Cartesian product $S_n \times S_m$ of all n-dimensional probability vectors S_n and all m-dimensional probability vectors S_m, that is, $S_n := \{x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n \mid x_i \geqq 0, \sum_{i=1}^n x_i = 1\}$, where the symbol ^{T} denotes the transpose.

A point $(\bar{x}, \bar{y}) \in S_n \times S_m$ is called an equilibrium point of matrix game B if $x^T B\bar{y} \leqq \bar{x}^T B\bar{y} \leqq \bar{x}^T By$ for all $x \in S_n$ and all $y \in S_m$.

If $n = m$ and B is skew symmetric, then we can check that $(\bar{x}, \bar{y}) \in S_n \times S_n$ is an equilibrium point of game B if and only if $B\bar{x} \leqq 0$ and $B\bar{y} \leqq 0$.

When B is an $n \times n$ skew symmetric matrix, $\bar{x} \in S_n$ is called a solution (an optimal strategy) of matrix game B if $B\bar{x} \leqq 0$ ([3]).

Consider the linear programming problem (LP) together with its dual (LD) as follows:

(LP) Minimize $c^T x$ subject to $Ax \geqq b,\ x \geqq 0$,

(LD) Maximize $b^T y$ subject to $A^T y \leqq c,\ y \geqq 0$,

where $c \in I\!R^n, x \in I\!R^n, b \in I\!R^m, y \in I\!R^m, A = [a_{ij}]$ is an real $m \times n$ matrix.

Now we consider the matrix game associated with the following $(n + m + 1) \times (n + m + 1)$ skew symmetric matrix B:

$$B = \begin{bmatrix} 0 & A^T & -c \\ -A & 0 & b \\ c^T & -b^T & 0 \end{bmatrix}.$$

The following results due to Dantzig ([3]) are well known: Theorems 1.1 and 1.2 give complete equivalent relation linear programming problem and the matrix game B.

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Theorem 1.1. *Let $\bar{x}$ and $\bar{y}$ be optimal solutions to (LP) and (LD) respectively. Let $z^* = 1/(1+\sum_j \bar{x}_j + \sum_i \bar{y}_i)$, $x^* = z^*\bar{x}$, $y^* = z^*\bar{y}$. Then (x^*, y^*, z^*) solves the matrix game B.*

Theorem 1.2. *Let (x^*, y^*, z^*) be a solution (an optimal strategy) of the matrix game B with $z^* > 0$. Let $\bar{x}_j = (x_j^*/z^*)$, $\bar{y}_i = (y_i^*/z^*)$. Then $\bar{x}$ and $\bar{y}$ are optimal solutions to (LP) and (LD), respectively.*

Many authors [1, 2, 4, 7, 8, 9] have extended Theorems 1.1 and 1.2 to several kinds of optimization problems. In particular, the vector versions of Theorems 1.1 and 1.2 for linear vector optimization problems were obtained in [8].

In [8], a vector matrix game with more than two skew symmetric matrices, which is an extension of the matrix game, was defined and equivalence relations, that is vector versions of Theorems 1.1 and 1.2, between a linear vector optimization problem and its corresponding vector matrix game were established. The aim of this paper is to extend the equivalence relations in [8] to nonlinear vector optimization problems.

In this paper, we consider a nonlinear vector optimization problem. We formulate a dual problem for a nonlinear vector optimization problem, and give a weak duality result for the dual problem. Furthermore, we establish equivalence between the dual problem and the corresponding vector matrix game. Lastly, we give a numerical example for showing such equivalent relations.

2. Vector matrix game

Consider the nonlinear vector optimization problem (VOP):

$$\text{(VOP)} \qquad \text{Minimize} \quad f(x) := (f_1(x), \ldots, f_p(x)) \qquad \text{subject to} \quad x \in X,$$

where $X = \{x \in \mathbb{R}^n \mid g(x) \geqq b,\ x \geqq 0\}$, $f : \mathbb{R}^n \to \mathbb{R}^p$, $g : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable functions.

Definition 2.1 ([10]). (1) A point $\bar{x} \in X$ is said to be an efficient solution for (VOP) if there exists no other feasible point $x \in X$ such that $(f_1(x), \ldots, f_p(x)) \leq (f_1(\bar{x}), \ldots, f_p(\bar{x}))$.

(2) A point $\bar{x} \in X$ is said to be a weakly efficient solution for (VOP) if there exists no other feasible point $x \in X$ such that $(f_1(x), \ldots, f_p(x)) < (f_1(\bar{x}), \ldots, f_p(\bar{x}))$.

Now we define solutions for vector matrix game as the following:

Definition 2.2. Let $B_i,\ i = 1, \ldots, p$, be real $n \times n$ skew symmetric matrices.

(1) A point $\bar{x} \in S_n$ is said to be a vector solution of vector matrix game $B_i,\ i = 1, \ldots, p$ if $(x^T B_1 \bar{x}, \ldots, x^T B_p \bar{x}) \ngeq (\bar{x}^T B_1 \bar{x}, \ldots, \bar{x}^T B_p \bar{x}) \ngeq (\bar{x}^T B_1 x, \ldots, \bar{x}^T B_p x)$ for any $x \in S_n$.

(2) A point $\bar{x} \in S_n$ is said to be a weak vector solution of vector matrix game $B_i,\ i = 1, \ldots, p$ if $(x^T B_1 \bar{x}, \ldots, x^T B_p \bar{x}) \not> (\bar{x}^T B_1 \bar{x}, \ldots, \bar{x}^T B_p \bar{x}) \not> (\bar{x}^T B_1 x, \ldots, \bar{x}^T B_p x)$ for any $x \in S_n$.

Denote riS_n by $\overset{o}{S}_n$, where riS_n is the relative interior of the set S_n.

We proved the characterization of vector solution vector matrix game in [8]. However, for the completeness, we present its proof.

Lemma 2.3. *Let $B_i,\ i=1,\dots,p$, be $n\times n$ skew symmetric matrices. Then $\bar{y}\in S_n$ is a vector solution of vector matrix game $B_i, i=1,\dots,p$ if and only if there exists $\xi\in \overset{o}{S}_p$ such that $(\sum_{i=1}^{p}\xi_i B_i)\bar{y}\leqq 0$.*

Proof. $\bar{y}\in S_n$ is a vector solution of vector matrix game $B_i, i=1,\dots,p$.
$\Longleftrightarrow$ $(y^T B_1\bar{y},\dots,y^T B_p\bar{y})\not\geq 0, \forall y\in S_n$.
$\Longleftrightarrow$ $\bar{y}\in S_n$ is an efficient solution of the following linear vector optimization problem:

$$\begin{aligned}&\text{Maximize} && (y^T B_1\bar{y},\dots,y^T B_p\bar{y})\\ &\text{subject to} && y\in S_n.\end{aligned}$$

$\Longleftrightarrow$ ([6]) $\bar{y}\in S_n$ is a properly efficient solution of the following linear vector optimization problem:

$$\begin{aligned}&\text{Maximize} && (y^T B_1\bar{y},\dots,y^T B_p\bar{y})\\ &\text{subject to} && y\in S_n.\end{aligned}$$

$\Longleftrightarrow$ ([5]) there exists $\xi\in \overset{o}{S}_p$ such that $\bar{y}$ is optimal for the following linear scalar optimization:

$$\begin{aligned}&\text{Maximize} && y^T(\sum_{i=1}^{p}\xi_i B_i)\bar{y}\\ &\text{subject to} && y\in S_n.\end{aligned}$$

$\Longleftrightarrow$ there exists $\xi\in \overset{o}{S}_p$ such that $\forall y\in S_n, y^T(\sum_{i=1}^{p}\xi_i B_i)\bar{y}\leqq 0$.
$\Longleftrightarrow$ there exists $\xi\in \overset{o}{S}_p$ such that $(\sum_{i=1}^{p}\xi_i B_i)\bar{y}\leqq 0$. □

We can easily prove the following lemma.

Lemma 2.4. *Let $B_i,\ i=1,\dots,p$, be $n\times n$ skew symmetric matrices. Then $\bar{y}\in S_n$ is a weak vector solution of vector matrix game $B_i, i=1,\dots,p$ if and only if there exists $\xi\in S_p$ such that $(\sum_{i=1}^{p}\xi_i B_i)\bar{y}\leqq 0$.*

3. Equivalent relations

We consider the vector optimization programming problem (VOP) together with its dual (VOD) as follows:

$$\text{(VOD)}\qquad \text{Maximize}\qquad \Big(f_1(u)-v^T(g(u)-b),\dots,f_p(u)-v^T(g(u)-b)\Big)$$

$$\begin{aligned}&\text{subject to} && \textstyle\sum_{i=1}^{p}\lambda_i\nabla f_i(u)-\nabla(v^Tg)(u)\geqq 0,\\ \text{(3.1)}\qquad & && u^T\Big[\textstyle\sum_{i=1}^{p}\lambda_i\nabla f_i(u)-\nabla(v^Tg)(u)\Big]\leqq 0,\\ & && v\geqq 0,\ \lambda\in \overset{o}{S}_p.\end{aligned}$$

The following Theorems 3.1 and 3.2 are well known, but for the completeness, we give proofs for the theorems.

Theorem 3.1 (Weak Duality). *Let x and (u, v, λ) be feasible for (VOP) and (VOD), respectively. If $\sum_{i=1}^{p} \lambda_i f_i(\cdot) - v^T g(\cdot)$ is pseudoconvex, then the following cannot hold:*

$$(3.2)\qquad \big(f_1(x), \ldots, f_p(x)\big) \leq \big(f_1(u) - v^T(g(u) - b), \ldots, f_p(u) - v^T(g(u) - b)\big).$$

Proof. Suppose that the result (3.2) holds. Since x is feasible for (VOP) and (u, v, λ) is feasible for (VOD),

$$(3.3)\quad \big(f_1(x) - v^T(g(x) - b), \ldots, f_p(x) - v^T(g(x) - b)\big) \leq \big(f_1(u) - v^T(g(u) - b), \ldots, f_p(u) - v^T(g(u) - b)\big).$$

Multiplying (3.3) with λ, we get

$$\sum_{i=1}^{p} \lambda_i f_i(x) - v^T g(x) < \sum_{i=1}^{p} \lambda_i f_i(u) - v^T g(u).$$

By the pseudoconvex of $\sum_{i=1}^{p} \lambda_i f_i(\cdot) - v^T g(\cdot)$,

$$[\sum_{i=1}^{p} \lambda_i \nabla f_i(u) - \nabla(v^T g)(u)]^T (x - u) < 0,$$

which contradicts (3.1). Hence the result holds. □

Theorem 3.2 (Strong Duality). *Let $\bar{x}$ be an efficient solution of (VOP). Suppose that a constraint qualification for (VOP) is satisfied and $\sum_{i=1}^{p} \lambda_i f_i(\cdot) - v^T g(\cdot)$ is pseudoconvex. Then there exist $\bar{\lambda} \in \overset{o}{S}_p$ and $\bar{v} \in \mathbb{R}_m^+$ such that $(\bar{x}, \bar{v}, \bar{\lambda})$ is an efficient solution of (VOD).*

Proof. Let $\bar{x}$ be an efficient solution of (VOP). By Kuhn-Tucker necessary optimality condition, there exist $\bar{\lambda} \in \overset{o}{S}_p$, $\bar{v} \in \mathbb{R}_m^+$ and $\bar{\mu} \in \mathbb{R}_n^+$ such that

$$\begin{aligned}
&\sum_{i=1}^{p} \bar{\lambda}_i \nabla f_i(\bar{x}) - \nabla(\bar{v}^T g)(\bar{x}) - \bar{\mu} = 0,\\
&\bar{v}^T[g(\bar{x}) - b] = 0,\\
&\bar{\mu}^T \bar{x} = 0.
\end{aligned}$$

Thus there exist $\bar{\lambda} \in \overset{o}{S}_p$ and $\bar{v} \in \mathbb{R}_m^+$ such that

$$\begin{aligned}
&\sum_{i=1}^{p} \bar{\lambda}_i \nabla f_i(\bar{x}) - \nabla(\bar{v}^T g)(\bar{x}) \geqq 0,\\
&\bar{x}^T[\sum_{i=1}^{p} \bar{\lambda}_i \nabla f_i(\bar{x}) - \nabla(\bar{v}^T g)(\bar{x})] = 0,\\
&\bar{v}^T[g(\bar{x}) - b] = 0.
\end{aligned}$$

Thus $(\bar{x}, \bar{v}, \bar{\lambda})$ is a feasible solution of (VOD) with $f_i(\bar{x}) = f_i(\bar{x}) - \bar{v}^T(g(\bar{x}) - b)$. By weak duality, $(f_1(\bar{x}) - \bar{v}^T(g(\bar{x}) - b), \dots, f_p(\bar{x}) - \bar{v}^T(g(\bar{x}) - b)) \nleq (f_1(u) - v^T(g(u) - b), \dots, f_p(u) - v^T(g(u) - b))$ for any feasible (u, v, λ) of (VOD). Therefore, $(\bar{x}, \bar{v}, \bar{\lambda})$ is an efficient solution of (VOD). □

Lemma 3.3. *Let $\bar{x}$ and $(\bar{x}, \bar{y}, \bar{\xi})$ be feasible for (VOP) and (VOD), respectively, and assume that $\bar{y}^T(g(\bar{x}) - b) = 0$. If weak duality holds, then $\bar{x}$ is an efficient solution of (VOP) and $(\bar{x}, \bar{y}, \bar{\xi})$ is an efficient solution of (VOD).*

Proof. Let $\bar{x}$ and $(\bar{x}, \bar{y}, \bar{\xi})$ be feasible for (VOP) and (VOD), respectively. By weak duality, $f(x) \nleq f(\bar{x}) - \bar{y}^T(g(\bar{x}) - b)e$ for any feasible x of (VOP). Since $f(\bar{x}) = f(\bar{x}) - \bar{y}^T(g(\bar{x}) - b)e$, $f(x) \nleq f(\bar{x})$ for any feasible x of (VOP). Therefore, $\bar{x}$ is an efficient solution of (VOP). By weak duality, $f(\bar{x}) \nleq f(u) - v^T(g(u) - b)e$ for any feasible (u, v, ξ) of (VOD). Since $f(\bar{x}) = f(\bar{x}) - \bar{y}^T(g(\bar{x}) - b)e$, $f(\bar{x}) - \bar{y}^T(g(\bar{x}) - b)e \nleq f(u) - v^T(g(u) - b)e$ for any feasible (u, v, ξ) of (VOD). Therefore, $(\bar{x}, \bar{y}, \bar{\xi})$ is an efficient solution of (VOD). □

Consider the vector matrix game defined by the following $(n+m+1) \times (n+m+1)$ skew symmetric matrices $B_i(x), i = 1, \dots, p$, related to (VOP) and (VOD):

$$B_i(x) = \begin{bmatrix} 0 & \nabla g(x) & -\nabla f_i(x) \\ -\nabla g(x)^T & 0 & b - g(x) + \nabla g(x)^T x \\ \nabla f_i(x)^T & -(b - g(x))^T - x^T \nabla g(x) & 0 \end{bmatrix}.$$

Now we give equivalent relations between (VOD) and vector matrix game $B_i(x)$, $i = 1, \dots, p$.

Theorem 3.4. *Let $\bar{x}$ and $(\bar{x}, \bar{y}, \bar{\xi})$ be feasible for (VOP) and (VOD), respectively, with $\bar{y}^T(g(\bar{x}) - b) = 0$. Let $z^* = 1/(1 + \sum_i \bar{x}_i + \sum_j \bar{y}_j), x^* = z^*\bar{x}$ and $y^* = z^*\bar{y}$. Then (x^*, y^*, z^*) is a vector solution of vector matrix game $B_i(\bar{x}), i = 1, \dots, p$.*

Proof. Let $\bar{x}$ and $(\bar{x}, \bar{y}, \bar{\xi})$ be feasible for (VOP) and (VOD), respectively. Then the following holds:

$$\sum_{i=1}^{p} \bar{\xi}_i \nabla f_i(\bar{x}) - \nabla(\bar{y}^T g)(\bar{x}) \geqq 0, \tag{3.4}$$

$$\bar{x}^T \Big[\sum_{i=1}^{p} \bar{\xi}_i \nabla f_i(\bar{x}) - \nabla(\bar{y}^T g)(\bar{x})\Big] \leqq 0, \tag{3.5}$$

$$g(\bar{x}) \geqq b, \tag{3.6}$$

$$\bar{y}^T(g(\bar{x}) - b) = 0, \tag{3.7}$$

$$\bar{x} \geqq 0, \ \bar{y} \geqq 0, \ \bar{\xi} \in \overset{o}{S}_p. \tag{3.8}$$

Since $z^* > 0$ by (3.8) and using (3.4) and (3.6), we get:

$$z^* \Big[\nabla(\bar{y}^T g)(\bar{x}) - \sum_{i=1}^{p} \bar{\xi}_i \nabla f_i(\bar{x})\Big] \leqq 0, \tag{3.9}$$

$$z^*(b - g(\bar{x})) \leqq 0. \tag{3.10}$$

From (3.5) and (3.7), we obtain

$$\begin{aligned}&z^*\Big[\textstyle\sum_{i=1}^p \bar{\xi}_i \nabla f_i(\bar{x})^T\bar{x} - (b-g(\bar{x}))^T\bar{y} - \bar{x}^T\nabla(\bar{y}^Tg)(\bar{x})\Big]\\ &= z^*\Big[\textstyle\sum_{i=1}^p \bar{\xi}_i \nabla f_i(\bar{x})^T\bar{x} - \nabla(\bar{y}^Tg)(\bar{x})^T\bar{x}\Big]\\ &= z^*\bar{x}^T\Big[\textstyle\sum_{i=1}^p \bar{\xi}_i \nabla f_i(\bar{x}) - \nabla(\bar{y}^Tg)(\bar{x})\Big]\\ &\leqq 0.\end{aligned} \tag{3.11}$$

From (3.9), (3.10) and (3.11) we have following inequality

$$\Big(\sum_{i=1}^p \bar{\xi}_i B_i(\bar{x})\Big)\begin{pmatrix} z^*\bar{x}\\ z^*\bar{y}\\ z^*\end{pmatrix} \leqq 0.$$

By Lemma 2.3, (x^*, y^*, z^*) is a vector solution of the vector matrix game $B_i(\bar{x}), i = 1, \ldots, p$. □

Theorem 3.5. *Let (x^*, y^*, z^*) with $z^* > 0$ be a vector solution of vector matrix game $B_i(\bar{x}), i = 1, \ldots, p$, where $\bar{x} = x^*/z^*$. Let $\bar{y} = y^*/z^*$. Then $\bar{x}$ is feasible for (VOP) and there exists $\bar{\xi} \in \overset{o}{S}_p$ such that $(\bar{x}, \bar{y}, \bar{\xi})$ is feasible for (VOD) and $\bar{y}^T(g(\bar{x}) - b) = 0$.*

Proof. Let (x^*, y^*, z^*) with $z^* > 0$ be a vector solution of vector matrix game $B_i(\bar{x}),\ i = 1, \ldots, p$. Then by Lemma 2.3, there exists $\bar{\xi} \in \overset{o}{S}_p$ such that

$$\Big(\sum_{i=1}^p \bar{\xi}_i B_i(\bar{x})\Big)\begin{pmatrix} x^*\\ y^*\\ z^*\end{pmatrix} \leqq 0.$$

Thus we get:

$$\nabla g(\bar{x})y^* - \sum_{i=1}^p \bar{\xi}_i \nabla f_i(\bar{x}) z^* \leqq 0, \tag{3.12}$$

$$-\nabla g(\bar{x})^T x^* + [b - g(\bar{x}) + \nabla g(\bar{x})^T\bar{x}]z^* \leqq 0, \tag{3.13}$$

$$\sum_{i=1}^p \bar{\xi}_i \nabla f_i(\bar{x})^T x^* - [(b-g(\bar{x}))^T + \bar{x}^T\nabla g(\bar{x})]y^* \leqq 0, \tag{3.14}$$

$$x^* \geqq 0,\ y^* \geqq 0,\ z^* > 0. \tag{3.15}$$

Dividing (3.12), (3.13) and (3.14) by $z^* > 0$, we have

$$\nabla g(\bar{x})\bar{y} - \sum_{i=1}^p \bar{\xi}_i \nabla f_i(\bar{x}) \leqq 0, \tag{3.16}$$

$$b - g(\bar{x}) \leqq 0, \tag{3.17}$$

$$\sum_{i=1}^p \bar{\xi}_i \nabla f_i(\bar{x})^T\bar{x} - [(b-g(\bar{x}))^T + \bar{x}^T\nabla g(\bar{x})]\bar{y} \leqq 0. \tag{3.18}$$

By using (3.15), we get

$$\bar{x} \geqq 0,\ \bar{y} \geqq 0. \tag{3.19}$$

From (3.16) and (3.19), we obtain

$$\nabla(\bar{y}^T g)(\bar{x})\bar{x} - \sum_{i=1}^{p} \bar{\xi}_i \nabla f_i(\bar{x})^T \bar{x} \leqq 0. \tag{3.20}$$

From (3.18) and (3.20), $(b - g(\bar{x}))^T \bar{y} \geqq 0$. Using (3.17) and (3.19), we obtain $(b - g(\bar{x}))^T \bar{y} \leqq 0$. It implies that

$$(b - g(\bar{x}))^T \bar{y} = 0. \tag{3.21}$$

From (3.18) and (3.21), $\bar{x}^T \left[\sum_{i=1}^{p} \bar{\xi}_i \nabla f_i(\bar{x}) - \nabla(\bar{y}^T g)(\bar{x})\right] \leqq 0$. Using (3.20), we obtain $\bar{x}^T \left[\sum_{i=1}^{p} \bar{\xi}_i \nabla f_i(\bar{x}) - \nabla(\bar{y}^T g)(\bar{x})\right] = 0$. Therefore $\bar{x}$ is feasible for (VOP) and $(\bar{x}, \bar{y}, \bar{\xi})$ is feasible for (VOD). □

By Theorem 3.5 and Lemma 3.3, we give the following corollary.

Corollary 3.6. *Let (x^*, y^*, z^*) with $z^* > 0$ be a vector solution of vector matrix game $B_i(\bar{x}), i = 1, \ldots, p$, where $\bar{x} = x^*/z^*$. Let $\bar{y} = y^*/z^*$. If weak duality holds, $\bar{x}$ is an efficient solution of (VOP) and there exists $\xi \in \overset{o}{S}_p$ such that $(\bar{x}, \bar{y}, \bar{\xi})$ is an efficient solution of (VOD).*

Corollary 3.7. *The converse of Theorem 3.4 holds, and the converse of Theorem 3.5 holds.*

Proof. Let (x^*, y^*, z^*) be a vector solution of the vector matrix game $B_i(\bar{x}), i = 1, \ldots, p$ and S_x be the set of vector solutions of the vector matrix game $B_i(x)$, $i = 1, \ldots, p$. Let $x^* = \frac{\bar{x}}{1+\sum_i \bar{x}_i + \sum_j \bar{y}_j}$, $y^* = \frac{\bar{y}}{1+\sum_i \bar{x}_i + \sum_j \bar{y}_j}$, $z^* = \frac{1}{1+\sum_i \bar{x}_i + \sum_j \bar{y}_j}$. Since $z^* > 0$ and $(x^*, y^*, z^*) \in S_{\frac{x^*}{z^*}}$ for some y^*, it follows from Theorem 3.5 that $\frac{x^*}{z^*}$ is an efficient solution of (VOP) and there exists $\bar{\xi} \in \overset{o}{S}_p$ such that $(\frac{x^*}{z^*}, \frac{y^*}{z^*}, \bar{\xi})$ is an efficient solution of (VOD), $\bar{y}^T(g(\bar{x}) - b) = 0$. Therefore, the converse of Theorem 3.4 holds.

Let $\bar{x}$ be feasible for (VOP) and there exists $\bar{\xi} \in \overset{o}{S}_p$ such that $(\bar{x}, \bar{y}, \bar{\xi})$ be feasible for (VOD) and $\bar{y}^T(g(\bar{x}) - b) = 0$. Let $z^* = \frac{1}{1+\sum_i \bar{x}_i + \sum_j \bar{y}_j}$, $x^* = \bar{x} z^*$, $y^* = \bar{y} z^*$. Then $z^* > 0$ and by Theorem 3.4, $(x^*, y^*, z^*) \in S_{\frac{x^*}{z^*}}$ for some y^*. Therefore, the converse of Theorem 3.5 holds. □

By Corollary 3.7, we give the following corollary.

Corollary 3.8. $\left\{\frac{x^*}{z^*} \mid z^* > 0,\ (x^*, y^*, z^*) \in S_{\frac{x^*}{z^*}} \text{ for some } y^*\right\} = \{(\bar{x}, \bar{y}) \mid \bar{x} :$ *efficient solution of (VOP), there exists $\bar{\xi} \in \overset{o}{S}_p$ such that $(\bar{x}, \bar{y}, \bar{\xi})$: efficient solution of (VOD), $\bar{y}^T(g(\bar{x}) - b) = 0\}$.*

Now we give an example illustrating Theorems 3.4 and 3.5.

Example 3.9. Consider the following vector optimization problem (VOP) together with its dual (VOD) as follows:

(VOP) Minimize $(-x, x^2)$

$$\begin{array}{lll} & \text{subject to} & -x \geqq -2, \\ & & x \geqq 0, \\ \text{(VOD)} & \text{Maximize} & (-u + uv - 2v,\ u^2 + uv - 2v) \\ & \text{subject to} & -\lambda_1 + 2\lambda_2 u + v \geqq 0, \\ & & u[-\lambda_1 + 2\lambda_2 u + v] \leqq 0, \\ & & v \geqq 0, \\ & & \lambda = (\lambda_1, \lambda_2) \in \overset{o}{S}_2. \end{array}$$

Now we determine the set of all vector solutions of vector matrix game $B_i(x), i = 1, 2$. Let

$$B_i(x) = \begin{pmatrix} 0 & \nabla g(x) & -\nabla f_i(x) \\ -\nabla g(x)^T & 0 & b - g(x) + \nabla g(x)^T x \\ \nabla f_i(x)^T & -(b - g(x))^T - x^T \nabla g(x) & 0 \end{pmatrix}.$$

Then

$$B_1(x) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad B_2(x) = \begin{pmatrix} 0 & -1 & -2x \\ 1 & 0 & -2 \\ 2x & 2 & 0 \end{pmatrix}.$$

Let $x \in \mathbb{R}$ and $(x^*, y^*, z^*) \in S_3$ be a vector solution of vector matrix game $B_i(x),\ i = 1, 2$ if and only if there exists $\xi_1 > 0,\ \xi_2 > 0,\ \xi_1 + \xi_2 = 1$ such that

$$\left(\xi_1 \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix} + \xi_2 \begin{pmatrix} 0 & -1 & -2x \\ 1 & 0 & -2 \\ 2x & 2 & 0 \end{pmatrix} \right) \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\iff$ there exists $\xi_1 > 0,\ \xi_2 > 0,\ \xi_1 + \xi_2 = 1$ such that

$$\begin{pmatrix} -y^* + (\xi_1 - 2x\xi_2) z^* \\ x^* - 2z^* \\ -(\xi_1 - 2x\xi_2) x^* + 2y^* \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus

(i) the case that $\xi_1 - 2x\xi_2 > 0$;
There exist $\xi_1 > \frac{2x}{1+2x},\ \xi_2 = 1 - \xi_1,\ \xi_1 > 0,\ \xi_2 > 0$, and $x \in \mathbb{R}$ such that

$$\begin{cases} (\xi_1 - 2x\xi_2) z^* & \leqq y^* \\ x^* & \leqq 2z^* \\ 2y^* & \leqq (\xi_1 - 2x\xi_2) x^*. \end{cases}$$

$\iff$ $(x^*, y^*, z^*) \in \{(x_1, y_1, z_1) \mid x_1 = \frac{2}{\xi_1 - 2x\xi_2 + 3},\ y_1 = \frac{\xi_1 - 2x\xi_2}{\xi_1 - 2x\xi_2 + 3},\ z_1 = \frac{1}{\xi_1 - 2x\xi_2 + 3},\ x_1 \geqq 0,\ y_1 \geqq 0,\ z_1 \geqq 0,\ \xi_1 > 0,\ \xi_2 > 0,\ \xi_1 + \xi_2 = 1,\ \xi_1 - 2x\xi_2 > 0,\ x \in \mathbb{R}\}$.

(ii) the case that $\xi_1 - 2x\xi_2 = 0$;
There exist $\xi_1 = \frac{2x}{1+2x},\ \xi_2 = \frac{1}{1+2x},\ \xi_1 > 0,\ \xi_2 > 0$, and $x \in \mathbb{R}$ such that

$$\begin{cases} -y^* & \leqq 0 \\ x^* & \leqq 2z^* \\ 2y^* & \leqq 0. \end{cases}$$

$\iff \quad (x^*, y^*, z^*) \in \{(x_1, 0, z_1) \mid 0 \leqq x_1 \leqq \frac{2}{3},\ z_1 = 1 - x_1\}.$

(iii) the case that $\xi_1 - 2x\xi_2 < 0$;
There exist $\xi_1 < \frac{2x}{1+2x},\ \xi_2 = 1 - \xi_1,\ \xi_1 > 0,\ \xi_2 > 0$, and $x \in \mathbb{R}$ such that

$$\begin{cases} x^* & \leqq 2z^* \\ 2y^* & \leqq (\xi_1 - 2x\xi_2)x^*. \end{cases}$$

$\iff \quad (x^*, y^*, z^*) = (0, 0, 1).$

Let $x \in \mathbb{R}$ and S_x be the set of vector solutions of vector matrix game $B_i(x),\ i = 1, 2$. From (i), (ii) and (iii), if $0 < x < 2$, then

$$\begin{aligned} S_x \quad = \quad & \left\{\left(\frac{2}{\xi_1 - 2x\xi_2 + 3}, \frac{\xi_1 - 2x\xi_2}{\xi_1 - 2x\xi_2 + 3}, \frac{1}{\xi_1 - 2x\xi_2 + 3}\right)\right. \\ & \left. \mid \xi_1 > 0,\ \xi_2 > 0,\ \xi_1 + \xi_2 = 1, \xi_1 - 2x\xi_2 > 0\right\} \\ & \cup \left\{(u, 0, 1-u) \mid 0 \leqq u \leqq \frac{2}{3}\right\}, \end{aligned}$$

and if $x = 2$, then

$$\begin{aligned} S_2 \quad = \quad & \left\{\left(\frac{2}{\xi_1 - 4\xi_2 + 3}, \frac{\xi_1 - 4\xi_2}{\xi_1 - 4\xi_2 + 3}, \frac{1}{\xi_1 - 4\xi_2 + 3}\right) \mid \frac{4}{5} < \xi_1 < 1,\ \xi_2 = 1 - \xi_1\right\} \\ & \cup \left\{(u, 0, 1-u) \mid 0 \leqq u \leqq \frac{2}{3}\right\}. \end{aligned}$$

Let $\bar{x}$ and $(\bar{x}, \bar{y}, \bar{\xi})$ be feasible for (VOP) and (VOD), respectively, with $\bar{y}^T(g(\bar{x}) - b) = 0$. By definition of efficient solution of (VOP),

$$\{\bar{x} \mid \bar{x} : \text{efficient solution of (VOP)}\} = [0, 2].$$

Moreover, we can easily check that

$$\begin{aligned} & \{(\bar{x}, \bar{y}) \mid \bar{x} : \text{ efficient solution of (VOP), there exists } \bar{\xi} \in \overset{o}{S}_p \text{ such that} \\ & \qquad (\bar{x}, \bar{y}, \bar{\xi}) : \text{efficient solution of (VOD)},\ \bar{y}^T(g(\bar{x}) - b) = 0\} \\ & = \{(\bar{x}, 0) \mid 0 < \bar{x} < 2\} \cup \{(2, \bar{y}) \mid 0 \leqq \bar{y} < 1\}. \end{aligned}$$

Thus for $0 < \bar{x} < 2$,

$$\left(\frac{\bar{x}}{1 + \bar{x} + \bar{y}}, \frac{\bar{y}}{1 + \bar{x} + \bar{y}}, \frac{1}{1 + \bar{x} + \bar{y}}\right) = \left(\frac{\bar{x}}{1 + \bar{x}}, 0, \frac{1}{1 + \bar{x}}\right) \in S_{\bar{x}}$$

and for $\bar{x} = 2$,

$$\left(\frac{\bar{x}}{1 + \bar{x} + \bar{y}}, \frac{\bar{y}}{1 + \bar{x} + \bar{y}}, \frac{1}{1 + \bar{x} + \bar{y}}\right) = \left\{\left(\frac{2}{3 + \bar{y}}, \frac{\bar{y}}{3 + \bar{y}}, \frac{1}{3 + \bar{y}}\right) \mid 0 \leqq \bar{y} < 1\right\} \in S_{\bar{x}}.$$

Therefore, Theorem 3.4 holds.

Let $x \in \mathbb{R}$ and

$$A \quad = \quad \left\{\left(\frac{2}{\xi_1 - 2x\xi_2 + 3}, \frac{\xi_1 - 2x\xi_2}{\xi_1 - 2x\xi_2 + 3}, \frac{1}{\xi_1 - 2x\xi_2 + 3}\right)\right.$$

$$\left. \mid \xi_1 > 0,\ \xi_2 > 0,\ \xi_1 + \xi_2 = 1, \xi_1 - 2x\xi_2 > 0 \right\}$$
$$\cup \left\{ (u, 0, 1-u) \mid 0 \leqq u \leqq \frac{2}{3} \right\}.$$

Then the above calculations implies that

$$\bigcup_{x \in \mathbb{R}} S_x = A.$$

So,

$$\begin{aligned} \left\{ \frac{x^*}{z^*} \mid z^* > 0 \text{ and } (x^*, y^*, z^*) \in S_{\frac{x^*}{z^*}} \text{ for some } y^* \right\} &\subset \left\{ \frac{\bar{x}}{\bar{z}} \mid (\bar{x}, \bar{y}, \bar{z}) \in A \right\} \\ &= [0, 2]. \end{aligned}$$

From (ii) and (iii) of the above calculations, we get

$$(0, 2] \subset \left\{ \frac{x^*}{z^*} \mid z^* > 0 \text{ and } (x^*, y^*, z^*) \in S_{\frac{x^*}{z^*}} \text{ for some } y^* \right\}.$$

Since $S_0 = \left\{ \left(\frac{2}{\xi_1+3}, \frac{\xi_1}{\xi_1+3}, \frac{1}{\xi_1+3} \right) \mid 0 < \xi_1 < 1 \right\}$, hence

$$\left\{ \frac{x^*}{z^*} \mid z^* > 0 \text{ and } (x^*, y^*, z^*) \in S_{\frac{x^*}{z^*}} \text{ for some } y^* \right\} = (0, 2].$$

Moreover, from (i) and (ii) of the above calculations,

$$\begin{aligned} &\left\{ \left(\frac{x^*}{z^*}, \frac{y^*}{z^*} \right) \mid z^* > 0 \text{ and } (x^*, y^*, z^*) \in S_{\frac{x^*}{z^*}} \right\} \\ &\quad = \{(2, \bar{y}) \mid 0 \leqq \bar{y} < 1\} \cup \{(\bar{x}, 0) \mid 0 < \bar{x} < 2\}. \end{aligned}$$

For any $x \in (0, 2]$, x is feasible for (VOP). Let F be the set of all feasible solutions of (VOD). Then we can check that $\{(2, \bar{y}, \bar{\xi}_1, \bar{\xi}_2) \mid 0 \leqq \bar{y} < 1,\ \frac{4}{5} \leqq \bar{\xi}_1 < 1,\ \bar{\xi}_2 = 1 - \bar{\xi}_1\} \subset F$ and $\left\{ (\bar{x}, 0, \bar{\xi}_1, \bar{\xi}_2) \mid 0 < \bar{x} < 2,\ \bar{\xi}_1 = \frac{2\bar{x}}{1+2\bar{x}},\ \bar{\xi}_2 = \frac{1}{2\bar{x}} \right\} \subset F$. For $\bar{y} \in [0, 1)$, $\bar{y}^T(g(2) + 2) = 0$. For $\bar{x} \in (0, 2)$, $0(g(\bar{x}) + 2) = 0$. Therefore, Theorem 3.5 holds. □

In Example 3.9, $\bar{x} = 0$ is an efficient solution, but there does not exist y such that $(0, 0) \notin \{(\bar{x}, \bar{y}) \mid \bar{x}$: efficient solution of (VOP), there exists $\bar{\xi} \in \overset{o}{S}_p$ such that $(\bar{x}, \bar{y}, \bar{\xi})$: efficient solution of (VOD), $\bar{y}^T(g(\bar{x}) - b) = 0\}$. Since $S_0 = \left\{ \left(\frac{2}{\xi_1+3}, \frac{\xi_1}{\xi_1+3}, \frac{1}{\xi_1+3} \right) \mid 0 < \xi_1 < 1 \right\}$, there is no y and z such that $(0, y, z) \in S_0$. Therefore to cover the point $\bar{x} = 0$, we consider weakly efficient solution and weak vector solution of the vector matrix game $B_i(x)$, $i = 1, 2$.

Let us consider formulate the dual $(\mathrm{VOD})_1$, which is based upon the weak efficiency, as follows:

$$
\begin{aligned}
(\mathrm{VOD})_1 \qquad & \text{Maximize} \quad \Big(f_1(u) - v^T(g(u)-b), \ldots, f_p(u) - v^T(g(u)-b)\Big) \\
& \text{subject to} \quad \textstyle\sum_{i=1}^p \lambda_i \nabla f_i(u) - \nabla(v^T g)(u) \geqq 0, \\
& \qquad\qquad u^T\Big[\textstyle\sum_{i=1}^p \lambda_i \nabla f_i(u) - \nabla(v^T g)(u)\Big] \leqq 0, \\
& \qquad\qquad v \geqq 0,\ \lambda \in S_p\,.
\end{aligned}
$$

Following the proofs of Theorems 3.4 and 3.5, we can easily check the following theorems and corollary:

Theorem 3.10. *Let $\bar{x}$ and $(\bar{x}, \bar{y}, \bar{\xi})$ be feasible for (VOP) and $(VOD)_1$, respectively, with $\bar{y}^T(g(\bar{x})-b) = 0$. Let $z^* = 1/(1+\sum_i \bar{x}_i + \sum_j \bar{y}_j)$, $x^* = z^*\bar{x}$ and $y^* = z^*\bar{y}$. Then (x^*, y^*, z^*) is a weak vector solution of the vector matrix game $B_i(\bar{x}), i = 1, \ldots, p$.*

Theorem 3.11. *Let (x^*, y^*, z^*) with $z^* > 0$ be a weak vector solution of the vector matrix game $B_i(\bar{x}), i = 1, \ldots, p$, where $\bar{x} = x^*/z^*$. Let $\bar{y} = y^*/z^*$. Then $\bar{x}$ is feasible for (VOP) and there exists $\bar{\xi} \in S_p$ such that $(\bar{x}, \bar{y}, \bar{\xi})$ is feasible for $(VOD)_1$ and $\bar{y}^T(g(\bar{x})-b) = 0$.*

Corollary 3.12. *Let (x^*, y^*, z^*) with $z^* > 0$ be a weak vector solution of the vector matrix game $B_i(\bar{x}), i = 1, \ldots, p$, where $\bar{x} = x^*/z^*$. Let $\bar{y} = y^*/z^*$. If weak duality holds, $\bar{x}$ is a weakly efficient solution of (VOP) and there exists $\bar{\xi} \in S_p$ such that $(\bar{x}, \bar{y}, \bar{\xi})$ is a weakly efficient solution of $(VOD)_1$.*

In Example 3.9, $\bar{x} = 0$ is a weakly efficient solution and $(0,0) \in \{(\bar{x}, \bar{y}) \mid \bar{x}$: weakly efficient solution of (VOP), there exists $\bar{\xi} \in S_p$ such that $(\bar{x}, \bar{y}, \bar{\xi})$: weakly efficient solution of $(\mathrm{VOD})_1$, $\bar{y}^T(g(\bar{x})-b) = 0\}$. Let $\bar{S}_0$ be the set of all weak vector solutions of the vector matrix game $B_i(0)$, $i = 1, 2$. Then $\bar{S}_0 = \Big\{\Big(\frac{2}{\xi_1+3}, \frac{\xi_1}{\xi_1+3}, \frac{1}{\xi_1+3}\Big) \mid 0 \leqq \xi_1 \leqq 1\}$ and hence there exist y and z such that $(0, y, z) \in \bar{S}_0$.

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