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CONTINUITY PROPERTIES OF THE SOLUTION MAP TO A PARAMETRIC DISCRETE OPTIMAL CONTROL PROBLEM

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This paper is dedicated to
Professor Pham Huu Sach on the occasion of his 70th birthday

Abstract. This paper studies the lower semicontinuity and the Aubin property of the solution map to a parametric dynamic programming problem with linear constraints and convex cost functions. By establishing abstract results on continuous properties of the solution map to a parametric programming and a parametric variational inequality, we obtain the lower semicontinuity and the Aubin property of the solution map to a parametric discrete optimal control problem.

1. Introduction

A wide variety of problems in discrete optimal control can be posed in the following form.

Determine a pair (x, u) of a path $x = (x_0, x_1, \ldots, x_N) \in R^m \times R^m \times \cdots \times R^m$ and a control vector $u = (u_0, u_1, \ldots, u_{N-1}) \in R^n \times R^n \times \cdots \times R^n$ which minimizes the cost

$$f(x, u, \mu) = \sum_{k=0}^{N-1} h_k(x_k, u_k, \mu_k) + h_N(x_N) \tag{1.1}$$

and which satisfies the state equation

$$x_{k+1} = A_k x_k + B_k u_k + w_k \quad (k = 0, 1, \ldots, N-1), \tag{1.2}$$

the constraints

$$\alpha_k \le u_k \le \beta_k \quad (k = 0, 1, \ldots, N-1), \tag{1.3}$$

and the initial condition

$$x_0 = c \in R^m. \tag{1.4}$$

Here

k indexes the discrete time,

x_k is the state of the system which summarizes past information that is relevant for future optimization,

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u_k is the control variable to be selected at time k with the knowledge of the state x_k,

μ_k, w_k are random parameters (also called disturbance or noise) which belong to R^s and R^m respectively, $\mu = (\mu_0, \mu_1, \ldots, \mu_{N-1})$,

N is the horizon or number times control is applied,

α_k and β_k are given vectors in R^n,

$A_k \in M(m,m)$ and $B_k \in M(m,n)$ are given matrices, $M(m,n)$ denotes the set of $m \times n$ matrices.

A classical example of problem (1.1)-(1.4) is the *inventory control problem*, where x_k plays a stock available at the beginning of the kth period, u_k plays a stock order at the beginning of the kth period and w_k is the demand during the kth period with given probability distribution, and where the cost function has the form $\Sigma_{k=0}^{N-1} cu_k + h(x_k + u_k - w_k)$ together with state equation $x_{k+1} = x_k + u_k - w_k$ (see [3] for details). For more information on discrete optimal control problem we refer the readers to [1], [4], [5], [6], [8], [14] and references are given therein.

Put $X = R^{(N+1)m}, U = R^{Nn}$, $Z = X \times U$, $M = R^{Ns}$ and $W = R^{Nm}$. For each $(\mu, w) \in M \times W$, we denote by $S(\mu, w)$ the solution set of problem (1.1)-(1.4) corresponding parameters $\mu = (\mu_0, \mu_1, ..., \mu_{N-1}) \in M$ and $w = (w_0, w_1, \ldots, w_{N-1}) \in W$. Thus

$$S : M \times W \to 2^Z$$

is a set-valued map which is called the solution map to the problem (1.1)-(1.4).

The study of the continuity of the solution map to the problem (1.1)-(1.4) are of importance in analysis and optimization. Such problems were studied by [9] and [19] some years ago. In particular, when h_k are strongly convex and of class C^2, Malanowski [9] showed that the solution map is single and differentiable in parameters. However, when the cost functions h_k are not strongly convex, the situation becomes more complicated. In this cases the solution map is not single in general. The aim of this paper is to deal with the situation, where the solution map is a set-valued map. Namely, we shall derive some new sufficient conditions under which the solution map to(1.1)-(1.4) has the Aubin property or the lower semicontinuity.

In order to obtain the result, we shall reduce the problem to a programming problem or a parametric variational inequality and use tools of variational analysis to establish some abstract results on the continuous properties of solution maps. We then apply the obtained results to problem(1.1)-(1.4).

Let us assume that $F : E_1 \rightrightarrows E_2$ is a multifunction between finite dimensional Euclidean spaces, we denote by $\text{dom}F$ and $\text{gph}F$ the effective domain and the graph of F respectively, where

$$\text{dom}F := \{z \in E_1 | F(z) \neq \emptyset\}$$

and the graph

$$\text{gph}F := \{(z, v) \in E_1 \times E_2 | v \in F(z)\}.$$

One says that F have the *Aubin property* around $(z_0, v_0) \in \text{gph}F$ if there exist neighborhoods U_0 of z_0, V_0 of v_0 and a constant $l > 0$ such that

$$F(z') \cap V_0 \subset F(z) + l||z' - z||B_{E_2} \quad z', z \in U_0,$$

where B_{E_2} stands for the closed unit ball in E_2. A multifunction F is said to be *lower semicontinuous* at $z_0 \in E_1$ if for any open set V in E_2 satisfying $F(z_0) \cap V \neq \emptyset$ there exists a neighborhood U_1 of z_0 such that $F(z) \cap V \neq \emptyset$ for all $z \in U_1$.

Let us recall some notions on variational analysis and generalized differentiation that we shall use in this paper. The notions and facts of variational analysis and generalized differentiation can be found in [11], [12] and [18].

Let E be a finite dimensional Euclidean space and Ω is a nonempty set in E. Given a point $\overline{z} \in \Omega$ and $\epsilon \geq 0$, the *set of ϵ-normal* is defined by

$$\widehat{N}_\epsilon(\overline{z}; \Omega) := \left\{ z^* \in E | \limsup_{z \to \overline{z}} \frac{\langle z^*, z - \overline{z} \rangle}{||z - \overline{z}||} \leq \epsilon \right\}. \tag{1.5}$$

When $\epsilon = 0$, the set $\widehat{N}_0(\overline{z}; \Omega)$ is called the *Fréchet normal cone* to Ω at $\overline{z}$ and denoted by $\widehat{N}(\overline{z}; \Omega)$. A vector $z^* \in E$ is called a *limiting normal* to Ω at $\overline{z}$ if there exist sequences $\epsilon_k \to 0^+$, $z_k \to \overline{z}$, and $z_k^* \to z^*$ such that $z_k^* \in \widehat{N}_{\epsilon_k}(\overline{z}_k; \Omega)$ for all k. The collection of such normals is called the *Mordukhovich normal cone* to Ω at $\overline{z}$ and denoted by $N(\overline{z}; \Omega)$. It is obvious that $\widehat{N}(\overline{z}; \Omega) \subset N(\overline{z}; \Omega)$. If Ω is convex then

$$\widehat{N}(\overline{z}; \Omega) = N(\overline{z}; \Omega) = \{z^* \in E | \langle z^*, z - \overline{z} \rangle \leq 0, \forall z \in \Omega\}.$$

Given a set valued map $F : E_1 \rightrightarrows E_2$, the *normal coderivative* of F at $(z, v) \in \mathrm{gph}F$ is the multifunction $D^*F(z, v) : E_2 \to E_1$ defined by

$$D^*F(z, v)(v^*) = \{z^* \in E_1 : (z^*, -v^*) \in N((z, v); \mathrm{gph}F)\}.$$

We now return to problem (1.1)-(1.4). For each

$$\mu = (\mu_0, \mu_1, ..., \mu_{N-1}) \in M \text{ and } w = (w_0, w_1, \ldots, w_{N-1}) \in W$$

we have

$$f(x, u, \mu) = \sum_{k=0}^{N} h_k(x_k, u_k, \mu_k), \tag{1.6}$$

(1.7)
$$K(w) = \{(x, u) \in Z | x_{k+1} = A_k x_k + B_k u_k + w_k, x_0 = c, \alpha_k \leq u_k \leq \beta_k, k = 0, 1, \ldots, N-1\}.$$

Then (1.1)-(1.4) can be formulated in a simpler form:

$$\min \{f(x, u, \mu) | (x, u) \in K(w)\}. \tag{1.8}$$

Throughout of this paper, we assume that $\overline{z} = (\overline{x}, \overline{u})$ is a solution of the problem at $(\overline{\mu}, \overline{w})$, that is $(\overline{x}, \overline{u}) \in S(\overline{\mu}, \overline{w})$ and there exist convex neighborhoods of $\overline{\mu}$, $\overline{x}$ and $\overline{u}$ respectively,

$$M_0 = \prod_{k=0}^{N-1} M_k^0, \quad X_0 = \prod_{k=0}^{N} X_k^0, \quad U_0 = \prod_{k=0}^{N-1} U_k^0$$

such that one of the following conditions hold:

$(H1)$ For each fixed $k \in \{0, 1, \ldots, N-1\}$ and $\lambda \in M_k^0$, the functions $h_k(\cdot, \cdot, \lambda) : X_k^0 \times U_k^0 \to R$ and $h_N : X_N^0 \to R$ are convex.

$(H2)$ For each fixed $k \in \{0, 1, \ldots, N-1\}$, the functions $h_k : X_k^0 \times U_k^0 \times M_k^0 \to R$ and $h_N : X_N^0 \to R$ are continuous.

$(H3)$ For each fixed $k \in \{0, 1, \ldots, N-1\}$, the functions $h_k : X_k^0 \times U_k^0 \times M_k^0 \to R$ and $h_N : X_N^0 \to R$ are of class C^2 and the map

$$\frac{\partial^2 f(\overline{z}, \overline{\mu})}{\partial \mu \partial z} : Z \times M \to Z$$

is surjective.
Here, $\overline{\mu} = (\overline{\mu}_0, \overline{\mu}_1, \ldots, \overline{\mu}_{N-1}), \overline{x} = (\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_N), \overline{u} = (\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_{N-1})$ and M_k^0, X_k^0, U_k^0 are convex neighborhoods of $\overline{\mu}_k$, $\overline{x}_k$, $\overline{u}_k$ respectively,

The rest of the paper consists of two sections. In Section 2, we establish some sufficient conditions under which the solution map is lower semicontinuous. Section 3 is devoted to the Aubin property of the solution map.

2. Lower semicontinuity of the solution map

In this section we shall give a result on the lower semicontinuity of the solution map to problem (1.1)-(1.4). First of all, we notice that condition (3) can be rewritten in the form

$$u_k \leq \beta_k \quad \text{and} \quad -u_k \leq -\alpha_k.$$

Define

$$z = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \\ u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}, \quad b(w) = \begin{bmatrix} w_0 \\ -w_0 \\ w_1 \\ -w_1 \\ \vdots \\ w_{N-1} \\ -w_{N-1} \\ c \\ -c \\ \beta_0 \\ -\alpha_0 \\ \beta_1 \\ -\alpha_1 \\ \vdots \\ -\alpha_{N-1} \end{bmatrix}, \tag{2.1}$$

and

$$C = \begin{bmatrix}
-A_0 & I_{m\times m} & 0 & 0 & \dots & 0 & 0 & -B_0 & 0 & 0 & \dots & 0 \\
A_0 & -I_{m\times m} & 0 & 0 & \dots & 0 & 0 & B_0 & 0 & 0 & \dots & 0 \\
0 & -A_1 & I_{m\times m} & 0 & \dots & 0 & 0 & 0 & -B_1 & 0 & \dots & 0 \\
0 & A_1 & -I_{m\times m} & 0 & \dots & 0 & 0 & 0 & B_1 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & -A_{N-1} & I_{m\times m} & 0 & 0 & 0 & \dots & -B_{N-1} \\
0 & 0 & 0 & 0 & \dots & A_{N-1} & -I_{m\times m} & 0 & 0 & 0 & \dots & B_{N-1} \\
I_m & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
-I_m & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & I_{n\times n} & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & -I_{n\times n} & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & I_{n\times n} & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -I_{n\times n} & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -I_{n\times n}
\end{bmatrix},$$

(2.2)

where $I_{m\times m}$ and $I_{n\times n}$ denote the $m\times m$ and $n\times n$ unit matrices, respectively. Then we have

$$K(w) = \{z \in X \times U | Cz \le b(w)\} \tag{2.3}$$

and so problem (1.8) can be written in the form

$$\min\{f(z,\mu)|z \in K(w)\}. \tag{2.4}$$

Let us put $\Pi = R^{2(N+1)m+2Nn}$, $P = M \times \Pi$ and define a mapping $K_1 : \Pi \to 2^Z$ by

$$K_1(b) = \{z \in Z | Cz \le b\}, \forall b \in \Pi, \tag{2.5}$$

where C is a matrix which is given by (2.2). Thus we have $K(w) = K_1(b(w))$ for all $w \in W$, where $b(w)$ is defined by (2.1). In the sequel, we denote by D_1 the effective domain of K_1 and D the effective domain of K. It is clear that

$$D = \{w \in W | K(w) \neq \emptyset\} = \{w \in W | b(w) \in D_1\}.$$

The following lemma plays an important role in our proofs.

Lemma 2.1. ([10, Theorem 2.2]) *The set valued map $K_1 : \Pi \to 2^Z$ which is defined by (2.5), is Lipschitz on D_1, i.e. there exists a constant $l > 0$, independent of b, such that*

$$K_1(b_1) \subseteq K_1(b_2) + l\|b_1 - b_2\|B_Z$$

for all $b_1, b_2 \in D_1$.

From this lemma we get

Corollary 2.2. *The set-valued map $K : W \to 2^Z$ which is defined by $K(w) = K_1(b(w))$ for all $w \in W$, is Lipschitz continuous on its effective domain $D \subset W$.*

Proof. Notice that for all $w, w' \in D$ we have $\|b(w') - b(w)\| = \sqrt{2}\|w' - w\|$. Hence Lemma 2.1 implies that there exists a constant $l > 0$ such that

$$K(w) = K_1(b(w)) \subseteq K_1(b(w')) + l\|b(w) - b(w')\|B_Z \subseteq K(w') + l\sqrt{2}\|w - w'\|B_Z$$

for all $w, w' \in D$. We obtain the desired conclusion. □

Recall that a function $\varphi : Z \to R$ is strictly convex if for any $z_1, z_2 \in Z$ with $z_1 \neq z_2$ and for all $t \in (0,1)$, one has

$$\varphi(tz_1 + (1-t)z_2) < t\varphi(z_1) + (1-t)\varphi(z_2).$$

We are now ready to state the first result

Theorem 2.3. *Suppose that $\overline{z} = (\overline{x}, \overline{u})$ is a solution of problem* (1.1)-(1.4) *corresponding to parameter $(\overline{\mu}, \overline{w}) \in M \times D$ and assumptions (H_1), (H_2) are fulfilled. Assume furthermore that the mappings $h_k(\cdot, \cdot, \overline{w}_k)$ are strictly convex for all $k = 0, 1, \ldots, N-1$.*

Then there exist a neighborhood $M_1 \subset M_0$ of $\overline{\mu}$, a neighborhood $W_1 \subset W$ of $\overline{w}$ and an open bounded neighborhood $Q_1 \subset X_0 \times U_0 := Z_0$ of $(\overline{x}, \overline{u})$ such that the solution map

$$\hat{S} : M_1 \times (W_1 \cap D) \to 2^{Q_1}$$

which is defined by $\hat{S}(\mu, w) = S(\mu, w) \cap Q_1$, is nonempty valued and lower semicontinuous at $(\overline{\mu}, \overline{w})$.

Proof. Notice that for each $\lambda \in M_k^0$, $h_k(\cdot, \lambda)$ is convex, so $f(\cdot, \mu)$ is convex for each fixed $\mu \in M_0$. Moreover, since $h_k(\cdot, \overline{\mu}_k)$ is strictly convex, $f(\cdot, \overline{\mu})$ is strictly convex.

By Corollary 2.2, there exist positive constants k, ϵ_0 and β_0 such that

$$K(w') \cap (\overline{z} + \epsilon_0 B_Z) \subseteq K(w) + k\|w' - w\| B_Z, \forall w', w \in B(\overline{w}, \beta_0) \cap D, \tag{2.6}$$

here $B(\overline{w}, \beta_0)$ is a ball with centered $\overline{w}$, radius β_0.

We now choose positive constants s and δ such that

$$\overline{z} + sB_Z \subset \big(\overline{z} + \epsilon_0 B_Z\big) \cap Z_0, k\|\overline{w} - w\| < s$$

for all $w \in B(\overline{w}, \delta) \subset B(\overline{w}, \beta_0)$. Hence (2.6) implies

$$K(w') \cap (\overline{z} + sB_Z) \subseteq K(w) + k\|w' - w\| B_Z, \forall w', w \in B(\overline{w}, \delta) \cap D. \tag{2.7}$$

For each $\epsilon > 0$ we define K_ϵ by

$$K_\epsilon(w) = K(w) \cap (\overline{z} + \epsilon B_Z). \tag{2.8}$$

Choose a number β such that $0 < \beta < \min\{\delta, \frac{s}{4k}\}$. According to [2, Lemma 2.3], $K_s(\cdot)$ is Lipschitz continuous. Namely, we have

$$K(w') \cap (\overline{z} + sB_Z) \subseteq K(w) \cap \big(\overline{z} + sB_Z\big) + 5k\|w' - w\| B_Z, \forall w', w \in B(\overline{w}, \beta) \cap D. \tag{2.9}$$

Putting $w' = \overline{w}$ in (2.7), we see that for each $w \in B(\overline{w}, \beta) \cap D$, there exists $z \in K(w)$ such that $\|z - \overline{z}\| \leq k\|w - \overline{w}\| < s$. Consequently, $K(w) \cap B(\overline{z}, s) \neq \emptyset$ for all $w \in B(\overline{w}, \beta) \cap D$. Fixing any $\mu \in M_0$ and $w \in B(\overline{w}, \beta) \cap D$ we consider the problem

$$\begin{cases} f(z, \mu) \to \min \\ z \in K(w) \cap \overline{B}(\overline{z}, s), \end{cases} \tag{2.10}$$

where $\overline{B}(\overline{z}, s)$ is a closed ball with centered $\overline{z}$, radius s. Since $K(w) \cap \overline{B}(\overline{z}, s)$ is a compact set and $f(\cdot, \mu)$ is continuous, (2.10) has a solution $z = z(\mu, w) \in K(w) \cap \overline{B}(\overline{z}, s)$. We claim that there exists a neighborhood $M_0' \subset M_0$ of $\overline{\mu}$ and a neighborhood $W_0' \subset W$ of $\overline{w}$ such that for all $(\mu, w) \in M_0' \times W_0'$, there exists a solution $z(\mu, w)$ of (2.10) satisfying

$$z(\mu, w) \notin \partial B(\overline{z}, s), \tag{2.11}$$

where $\partial B(\overline{z}, s)$ is the boundary of $B(\overline{z}, s)$.

Indeed, suppose that the assertion is false. Then we can find sequences $(\mu^j, w^j) \to (\overline{\mu}, \overline{w})$ and $z^j \in \partial B(\overline{z}, s) \cap K(w^j)$ such that

$$f(z^j, \mu^j) \leq f(z, \mu^j), \forall z \in K(w^j) \cap \overline{B}(\overline{z}, s). \tag{2.12}$$

Since $\partial B(\overline{z}, s)$ is a compact set, we can assume that $z^j \to z^0$. Substituting $w' = w^j, w = \overline{w}$ into (2.9), we see that, for each j, there exists $y^j \in K(\overline{w}) \cap \overline{B}(\overline{z}, s)$ such that

$$\|z^j - y^j\| \leqslant 5k\|w^j - \overline{w}\|.$$

Since $K(\overline{w}) \cap \overline{B}(\overline{z}, s)$ is compact, without loss of generality we may assume that $y^j \to y^0 \in K(\overline{w}) \cap \overline{B}(\overline{z}, s)$. From the above, we have $z^j \to y^0$. Consequently, $z^0 = y^0 \in K(\overline{w}) \cap \overline{B}(\overline{z}, s)$.

Putting $w' = \overline{w}, w = w^j$ into (2.9), we see that for each j there exists a point $v^j \in K(w^j) \cap \overline{B}(\overline{z}, s)$ such that $v^j \to \overline{z}$.

Putting $z = v^j$ in (2.12) and letting $j \to \infty$ we obtain

$$f(z^0, \overline{\mu}) \leq f(\overline{z}, \overline{\mu}).$$

Consequently, we have $f(z^0, \overline{\mu}) = f(\overline{z}, \overline{\mu})$. Since $f(\cdot, \overline{\mu})$ is strictly convex and $z^0 \neq \overline{z}$, we obtain

$$f(\overline{z}, \overline{\mu}) \leq f(\frac{\overline{z} + z^0}{2}, \overline{\mu}) < \frac{1}{2} f(\overline{z}, \overline{\mu}) + \frac{1}{2} f(z^0, \overline{\mu}) = f(\overline{z}, \overline{\mu})$$

which is absurd. Our claim is proved.

We now choose a neighborhood $M_1 \times W_1 \subset M_0 \times B(\overline{w}, \beta)$ of $(\overline{\mu}, \overline{w})$ such that (2.11) is valid and put $Q_1 = B(\overline{z}, s)$. We shall show that M_1, W_1 and Q_1 satisfy the conclusion of the theorem. In fact, fix any $(\mu, w) \in M_1 \times (W_1 \cap D)$ we consider problem (2.10). By (2.11), it has a solution $\hat{z} \in \text{int}\overline{B}(\overline{z}, s)$. Fixing any $z \in K(w)$ we see that for $t \in (0, 1)$ small enough, one has

$$f(\hat{z}, \mu) \leq f(\hat{z} + t(z - \hat{z}), \mu) \leq t f(z, \mu) + (1 - t) f(\hat{z}, \mu).$$

This implies that $f(\hat{z}, \mu) \leq f(z, \mu)$. Consequently, $\hat{z}$ is also a solution of the problem

$$\begin{cases} f(z, \mu) \to \min \\ z \in K(w). \end{cases} \tag{2.13}$$

It follows that $\hat{z} \in S(\mu, w) \cap B(\overline{z}, s)$ and so

$$\hat{S}(\mu, w) \neq \emptyset, \ \forall (\mu, w) \in M_1 \times (W_1 \cap D).$$

We obtain the first conclusion. It remains to show that $\hat{S} : M_1 \times (W_1 \cap D) \to 2^{Q_1}$ is lower semicontinuous at $(\overline{\mu}, \overline{w})$. Suppose G is a open set in Q_1 such that $\hat{S}(\overline{\mu}, \overline{w}) \cap G \neq \emptyset$. Note that $G = Q_1 \cap G_1$, where G_1 is a open set in Z. Thus we have $S(\overline{\mu}, \overline{w}) \cap Q_1 \cap G_1 \neq \emptyset$. By uniqueness, we have $\overline{z} \in G_2 := Q_1 \cap G_1$. Choose $\overline{s} \in (0, s)$ and $\overline{\delta} > 0$ such that $B(\overline{z}, \overline{s}) \subset G_2$, $k\|w - \overline{w}\| < \overline{s}$ and $B(\overline{w}, \overline{\delta}) \cap D \subset B(\overline{w}, \beta_0) \cap D$. We then have

$$K(w') \cap (\overline{z} + \overline{s} B_Z) \subseteq K(w) + k\|w' - w\| B_Z, \forall w', w \in B(\overline{w}, \overline{\delta}) \cap D. \tag{2.14}$$

Choosing $0 < \overline{\beta} < \min\{\overline{\delta}, \frac{\overline{s}}{4k}\}$ and using [2, Lemma 2.3] again, we see that

$$K(w') \cap (\overline{z} + \overline{s} B_Z) \subseteq K(w) \cap (\overline{z} + \overline{s} B_Z) + 5k\|w' - w\| B_Z, \forall w', w \in B(\overline{w}, \overline{\beta}) \cap D. \tag{2.15}$$

By using similar arguments as the above, we can show that there exist a neighborhood $M_2 \times W_2 \subset M_1 \times W_1$ such that for all $(\mu, w) \in M_2 \times (W_2 \cap D)$, (2.10) has a solution $z(\mu, w)$ satisfying

$$z(\mu, w) \notin \partial B(\overline{z}, \overline{s}).$$

From this we can show that for each $(\mu, w) \in M_2 \times (W_2 \cap D)$, problem (2.13) has a solution $\hat{z} = \hat{z}(\mu, w) \in B(\overline{z}, \overline{s}) \subset G_2$. Hence $S(\mu, w) \cap G_2 = \hat{S}(\mu, w) \cap G \neq \emptyset$ for all $(\mu, w) \in M_2 \times (W_2 \cap D)$. Consequently, $\hat{S}$ is lower semicontinuous at $(\overline{\mu}, \overline{w})$. The proof of the theorem is complete. □

To illustrate the obtained result we give the following example.

Example 2.4. Let $X = R^3, U = R^2, M = R^3, W = R^2$. We consider the minimum problem with the cost function

$$f(x,u,\mu) = \sum_{k=0}^{1}[x_k^2 + u_k^2 + \mu_k^6(\exp(-x_k^2) + \exp(-u_k^2))] + x_2^2 + \mu_2^6 \tag{2.16}$$

with system equation

$$x_{k+1} = x_k + u_k + w_k, \quad k = 0, 1 \tag{2.17}$$

$$x_0 = 1 \tag{2.18}$$

and constraints

$$-1 \leq u_0, u_1 \leq 1. \tag{2.19}$$

Suppose that $M_0 = B(0,1) \subset R^3$ and $W_0 = B(0,1) \subset R^2$ are neighborhoods of $\overline{\mu} = (0,0,0)$ and $\overline{w} = (0,0)$ respectively. Let $\overline{x} = (1, \frac{2}{5}, \frac{1}{5})$ and $\overline{u} = (-\frac{3}{5}, -\frac{1}{5})$. Then the following assertion are fulfilled:
(a) assumptions of Theorem 2.3 are valid;
(b) $\overline{x} = (1, \frac{2}{5}, \frac{1}{5}), \overline{u} = (-\frac{3}{5}, -\frac{1}{5})$ is a solution of the problem corresponding to $(\overline{\mu}, \overline{w})$;
(c) there exist neighborhoods M_1 and W_1 of $\overline{\mu}$ and $\overline{w}$ respectively, and a neighborhood Q_1 of $(\overline{x}, \overline{u})$ such that the solution map

$$S : M_1 \times (W_1 \cap D) \to 2^{Q_1}$$

is nonempty valued and lower semicontinuous at $(\overline{\mu}, \overline{w})$.

Solution. It is easy to see that $f(x, u, \overline{\mu})$ is strongly convex. Besides, for all $\mu \in B(0,1)$, $f(x, u, \mu)$ is convex. Hence assumptions of the theorem are fulfilled. It remains to show that $\overline{x} = (1, \frac{2}{5}, \frac{1}{5}), \overline{u} = (-\frac{3}{5}, -\frac{1}{5})$ is a solution of the problem corresponding to $(\overline{\mu}, \overline{w})$. Let us define

$$I(x,k) = \min_{u_k, u_{k+1}, \ldots, u_{N-1}} \sum_{j=k}^{N-1} h_j(x_j, u_j, \overline{\mu}_j) + h_N(x_N),$$

where x is the state at stage k, $x_k = x$ and $u_j \in [-1, 1]$. Then we have the Bellman equation

$$I(x,k) = \min_u [h(x,u,k) + I(x+u, k+1)],$$

$$I(x,N) = \min_{u_N} h_N(x_N) = x^2, (N = 2).$$

Hence we have

$$I(x,1) = \min_{u\in[-1,1]}[x^2 + u^2 + I(x+u,2)] = \min_{u\in[-1,1]}[x^2 + u^2 + (x+u)^2].$$

It follows that

$$\overline{u}_1 = \begin{cases} -\frac{1}{2}x & \text{if } -2 < x < 2 \\ -1 & \text{if } x \geq 2 \\ 1 & \text{if } x \leq -2 \end{cases} \tag{2.20}$$

and

$$I(x,1) = \begin{cases} \frac{3}{2}x^2 & \text{if } -2 < x < 2 \\ 2x^2 - 2x + 2 & \text{if } x \geq 2 \\ 2x^2 + 2x + 2 & \text{if } x \leq -2. \end{cases} \tag{2.21}$$

Similarly, we get

$$I(x,0) = \min_{u\in[-1,1]}[x^2 + u^2 + I(x+u,1)].$$

We consider the following cases.
Case 1. $-2 < x+u < 2$. Then one has

$$I(x,1) = \min_{u\in[-1,1]}[x^2 + u^2 + \frac{3}{2}(x+u)^2].$$

In this case we have

$$\overline{u}_0 = \begin{cases} -\frac{3}{5}x & \text{if } -\frac{5}{3} < x < \frac{5}{3} \\ -1 & \text{if } \frac{5}{3} \leq x < 3 \\ 1 & \text{if } -3 < x \leq -\frac{5}{3}. \end{cases} \tag{2.22}$$

Case 2. $x+u \geq 2$. Then one has

$$I(x,1) = \min_{u\in[-1,1]}[x^2 + u^2 + 2(x+u)^2 - 2(x+u) + 2].$$

In this case we have $\overline{u}_0 = -1$ if $x \geq 2$.
Case 3. $x+u \leq -2$. Then one has

$$I(x,1) = \min_{u\in[-1,1]}[x^2 + u^2 + 2(x+u)^2 + 2(x+u) + 2].$$

In this case we also have $\overline{u}_0 = 1$ if $x \leq -3$. In summary we get

$$\overline{u}_0 = \begin{cases} -\frac{3}{5}x & \text{if } -\frac{5}{3} < x < \frac{5}{3} \\ -1 & \text{if } x \geq \frac{5}{3} \\ 1 & \text{if } x \leq -\frac{5}{3}. \end{cases} \tag{2.23}$$

Since $x_0 = 1$ we obtain from (2.23) that $\overline{u}_0 = -\frac{3}{5}\overline{x}_0 = -\frac{3}{5}$ and so $\overline{x}_1 = \overline{x}_0 + \overline{u}_0 = 1 - \frac{3}{5} = \frac{2}{5}$. Since $\overline{x}_1 = \frac{2}{5}$, we obtain from (2.20) that $\overline{u}_1 = -\frac{1}{2}\overline{x}_1 = -\frac{1}{2}\frac{2}{5} = -\frac{1}{5}$ and so $\overline{x}_2 = \overline{x}_1 + \overline{u}_1 = \frac{2}{5} - \frac{1}{5} = \frac{1}{5}$. Thus we have shown that $\overline{x} = (1, \frac{2}{5}, \frac{1}{5})$ and $\overline{u} = (-\frac{3}{5}, -\frac{1}{5})$. From this we have the $f(\overline{x}, \overline{u}, \overline{\mu}) = \frac{40}{25}$. Finally, assertion (c) follows from the conclusion of Theorem 2.3.

3. The Aubin property of the solution map

In the previous section we have obtained a result on the lower semicontinuity of the solution set to problem (1.1)-(1.4). In this section we continue to study continuous properties of the solution map. Namely, we want to investigate the Aubin property of the solution map of (1.1)-(1.4). Since the Lipschitz continuity is stronger than the lower semicontinuity, assumptions $(H1)$ and $(H2)$ are not enough to establish the property. For this we need assumption $(H3)$ and tools of generalized differentiation to deal with the problem.

Note that, for each fixed couple (μ, w), (2.4) is a convex programming problem under linear constraints. Since f is convex and differentiable in z, we see that z is a solution of the problem if and only if

$$0 \in f'_z(z,\mu) + N(z; K(w)),$$

where $N(z; K(w))$ is the normal cone to $K(w)$ at z in the sense of convex analysis. Putting $\phi(z,\mu) = f'_z(z,\mu)$ we get

$$0 \in \phi(z,\mu) + N(z; K(w)) \tag{3.1}$$

which is called a *parametric variational inequality.*

Recall that given a set $Q \subset Z$, the set

$$Q^* := \{z^* \in Z | \langle z^*, z\rangle \le 0, \forall z \in Q\}$$

is called the polar cone of Q. Let $\Omega \subset Z$ and $\overline{z} \in \Omega$. The *tangent cone* to Ω at $\overline{z}$ which denoted by $T(\overline{z};\Omega)$ and defined by

$$T(\overline{z};\Omega) = N(\overline{z};\Omega)^* = \big\{v \in Z : \langle z^*, v\rangle \le 0, \forall\ z^* \in N(\overline{z};\Omega)\big\}.$$

From now on, we shall write $\bar{b}$ instead of $b(\overline{w})$. For each $(\mu, b) \in M \times \Pi$ we now consider the problem of finding $z = z(\mu, b)$ which satisfies the equation

$$0 \in \phi(\mu, z) + N(z; K_1(b)). \tag{3.2}$$

Let us denote by $S_1(\mu, b)$ the solution set of (3.2) corresponding to $(\mu, b) \in M \times \Pi$. It is clear that $S(\mu, w) = S_1(\mu, b(w))$ for all $(\mu, w) \in M \times D$, where $S(\mu, w)$ is the solution set of (3.1), which is also the solution map of problem (1.1)-(1.4).

Notice that $C = (c_{ij})_{p\times q}$, where $p = 2(N+1)m + 2Nn$ and $q = (N+1)m + Nn$. Put

$$T = \big\{0, 1, ..., p\big\} = T_0 \cup T_1,$$

where

$$T_0 = \{1, 2, ..., 2(N+1)m\}, T_1 = \{2(N+1)m+1, 2(N+1)m+2, ..., 2(N+1)m+2Nn\}.$$

Let us denote by C_i the $i-th$ row of matrix C. For a fixed element $z \in K_1(b)$, the set of active indices at z is given by

$$I(z, b) = \big\{i \in T : C_i z = (b)_i\big\}, \tag{3.3}$$

where $(b)_i$ is the i-th component of b. Here vector b consists of $2(N+1)m + 2Nn$ components, and vector z consists of $(N+1)m + Nn$ components. For convenience we assume that

$$\beta_i = \big(\hat{b}_{2(N+1)m+2in+1}, \hat{b}_{2(N+1)m+2in+2}, ..., \hat{b}_{2(N+1)m+2in+n}\big),$$

$$-\alpha_i = \big(\hat{b}_{2(N+1)m+(2i+1)n+1}, \hat{b}_{2(N+1)m+(2i+1)n+2}, ..., \hat{b}_{2(N+1)m+(2i+1)n+n}\big),$$
$$i = 0, 1, ..., N-1.$$

Here $\hat{b}_k$ are fixed for all $k = 2(N+1)m + 1, ..., 2(N+1)m + 2Nn$. Thus we have

$$b(w) = [w_0, -w_0, w_1, -w_1, ..., w_{N-1}, -w_{N-1}, c, -c, \hat{b}]^T,$$

where $\hat{b} = (\hat{b}_{2(N+1)m+1}, \hat{b}_{2(N+1)m+2}, ..., \hat{b}_{2(N+1)m+2Nn})$. Since $C_i\overline{z} = \overline{b}_i$ for all $i \in T_0$, we get

$$I(\overline{z}, \overline{b}) = T_0 \cup T_1(\overline{z}, \overline{b}), \tag{3.4}$$

where

$$T_1(\overline{z}, \overline{b}) = \{i \in T_1 | C_i\overline{z} = (\hat{b})_i\}.$$

For every subset $I \subset T$, we put $\overline{I} = T \backslash I$ and let C_I (resp.,$C_{\overline{I}}$) be the matrix composed by the rows $C_i, i \in I$, of C (resp., the rows $C_i, i \in \overline{I}$).

The following proposition gives formulas of the normal cone and tangent cone to a convex polyhedron (2.5). Its proof can be found in [16, Lemma 3.1].

Proposition 3.1 (Cf. [16, Lemma 3.1])**.** *Let $K_1(b)$ be defined by* (2.5)*, $z \in K_1(b)$ and $I(z, b)$ be defined by* (3.3)*. Then one has the following representations:*
(i)

$$N(z; K_1(b)) = \big\{y \in Z : y \in pos\big\{C_i^T : i \in I(z, b)\big\}\big\}, \tag{3.5}$$

where

$$pos\big\{C_i^T : i \in I(z, b)\big\} = \big\{\sum_{i\in I(z,b)} \lambda_i C_i^T, \lambda_i \geq 0\big\};$$

(ii)

$$T(z; K_1(b)) = \big\{v \in Z : C_i v \leq 0, \forall i \in I(z, b)\big\}. \tag{3.6}$$

Let us define a mapping $F_2 : Z \times \Pi \to 2^Z$ by

$$F_2(z, b) = N(z; K_1(b))$$

and assume that Ω_2 is the graph of F_2. The following lemmas give formulas computing the prenormal cone to Ω_2 at a given point.

Lemma 3.2 ([20, Lemma 4.1])**.** *If $(z^*, b^*, v^*) \in \hat{N}((z, b, v); \Omega_2)$ then*

$$(z^*, v^*) \in \big(T(z; K_1(b)) \cap v^\perp\big)^* \times \big(T(z; K_1(b)) \cap v^\perp\big), \tag{3.7}$$

$$z^* = -C_I^T b_I^* \tag{3.8}$$

and

$$b_{\overline{I}}^* = 0, \tag{3.9}$$

where $I = I(z, b)$ and $v^\perp = \big\{z \in Z | \langle v, z\rangle = 0\big\}$.

Recall that, a set Q is called a closed face of a cone H if and only if there exists $\overline{v} \in H^*$ such that $Q = \big\{z \in H | \langle\overline{v}, z\rangle = 0\big\}$. We now give an upper estimate for the Mordukhovich normal cone to Ω_2 at a given point.

Theorem 3.3. *Suppose that* $(\overline{z},\overline{b},\overline{v})\in\Omega_2$ *and* $(z^*,b^*,v^*)\in N\big((\overline{z},\overline{b},\overline{v});\Omega_2\big)$. *Then there exists an index set* $I'\subset I(\overline{z},\overline{b}):=T_0\cup T_1(\overline{z},\overline{b})$ *and a closed face* Q *of the polyhedral convex cone* $T(\mathcal{F}_{I'};K_1(\overline{b}))\cap\overline{v}^{\perp}$ *such that*

$$(z^*,v^*)\in Q^*\times Q, \tag{3.10}$$

$$\overline{v}\in pos\{C_i^T : i\in I'\}, \tag{3.11}$$

$$z^*=-C_{I'}^T b_{I'}^* \tag{3.12}$$

and

$$b_{\overline{I}'}^*=0. \tag{3.13}$$

where

$$\mathcal{F}_{I'}=\big\{z=(x,u)|C_{I'}z=b_{I'},C_{\overline{I}'}z<b_{\overline{I}'}\}$$

with $\overline{I'}=T_1(\overline{z},\overline{b})\setminus I'$.

Proof. The proof of the theorem is based on Lemma 3.2 and uses similar arguments as in the proof of [20, Theorem 4.3]. □

From the above theorem we obtain

Theorem 3.4. *Suppose that* $(\overline{z},\overline{b},\overline{v})\in\Omega_2$ *and* $v^*\in Z$. *If* $(z^*,b^*)\in D^*F_2(\overline{z},\overline{b},\overline{v})(v^*)$ *then there must exist an index set* $I'\subset I(\overline{z},\overline{b})$ *and a closed face* Q *of the polyhedral convex cone* $T(\mathcal{F}_{I'};K_1(\overline{b}))\cap\overline{v}^{\perp}$ *such that conditions* (3.11)-(3.13) *and condition*

$$(z^*,-v^*)\in Q^*\times Q \tag{3.14}$$

are satisfied.

The following theorem is a main result on the Aubin property of the solution map to problem (1.1)-(1.4).

Theorem 3.5. *Suppose that* $\overline{z}=(\overline{x},\overline{u})$ *is a solution of the problem* (1.1)-(1.4) *corresponding to parameter* $(\overline{\mu},\overline{w})$, *assumptions* $(H1)$ *and* $(H3)$ *are satisfied. If for any* $(z^*,b^*)\in R^{(N+1)m+Nn}\times R^{2Nm+2Nn}$, *one has* $(z^*,b^*)=(0,0)$ *whenever*

$$\big((\frac{\partial^2 f(\overline{z},\overline{\mu})}{\partial z^2})^T z^*,z^*\big)\in Q^*\times Q, \tag{3.15}$$

$$(\frac{\partial^2 f(\overline{z},\overline{\mu})}{\partial z^2})^T z^*=-C_{I'}^T b_{I'}^* \tag{3.16}$$

and

$$b_{\overline{I}'}^*=0 \tag{3.17}$$

for an index $I'\subset I(\overline{z},\overline{b})=T_0\cup T_1(\overline{z},\overline{b})$ *and a closed face* Q *of the polyhedral convex cone* $T(\mathcal{F}_{I'};K(\overline{w}))\cap(\nabla_z f(\overline{z},\overline{\mu}))^{\perp}$, *then the solution map* $M\times D\ni(\mu,w)\mapsto S(\mu,w)$ *has the Aubin property around* $(\overline{\mu},\overline{w},\overline{z})\in gph\ S$.

Proof. For the proof we need the following lemmas:

Lemma 3.6 ([13, Corollary 4.4])**.** *Let X, Y be finite dimensional Euclidean spaces, $\phi_1 : X \longrightarrow Y$ be strictly differentiable at $\overline{x} \in X$ and $\Phi_2 : X \rightrightarrows Y$ be a multifunction with closed graph. Then for any $\overline{y} \in \phi_1(\overline{x}) + \Phi_2(\overline{x})$ and $y^* \in Y$ one has*

$$D^*(\phi_1 + \Phi_2)(\overline{x}, \overline{y})(y^*) = (\bigtriangledown\phi_1(\overline{x}))^T y^* + D^*\Phi_2(\overline{x}, \overline{y} - \phi_1(\overline{x}))(y^*).$$

Lemma 3.7 ([7, Theorem 3.6])**.** *Let X, Y and Z be finite-dimensional Euclidean spaces, $F : X \times Y \rightrightarrows Z$ be a multifunction and G be an implicit multifunction which is defined by F, that is,*

$$G(y) = \big\{x \in X | 0 \in F(x, y)\big\}$$

and $(x_0, y_0) \in X \times Y$ be a pair such that $x_0 \in G(y_0)$. Assume that the following conditions hold:
(a) graph of F is locally closed around $\omega_0 = (x_0, y_0, 0_Z)$;
(b) the constraint qualification

$$KerD^*F(\omega_0) = \{0\}, \tag{3.18}$$

(c)

$$\bigcup_{z^* \in Z} \{y^* \in Y | (0, y^*) \in D^*F(\omega_0)(z^*)\} = \{0\}. \tag{3.19}$$

Then the implicit multifunction G has the Aubin property at (y_0, x_0), that is, there exist neighborhoods U of x_0, V of y_0 and a constant $l > 0$ such that

$$G(y') \cap U \subset G(y) + l||y' - y||B_X, \forall y, y' \in V.$$

Recall that $P = M \times \Pi = R^{Ns} \times R^{2(N+1)m+2Nn}$. Let us define mappings

$$\phi_1 : Z \times P \to Z,\ \Phi_2 : Z \times P \to 2^Z \text{ and } F : Z \times P \to 2^Z$$

by

$$\begin{aligned}\phi_1(z, \mu, b) &= \phi(z, \mu) = \nabla_z f(z, \mu)\\ \Phi_2(z, \mu, b) &= F_2(z, b) = N(z; K_1(b))\\ F(z, \mu, b) &= \phi_1(z, \mu, b) + \Phi_2(z, \mu, b).\end{aligned}$$

Then we have

$$S(\mu, b) = \{z \in Z | 0 \in F(z, \mu, b)\}.$$

We first claim that F and Φ_2 has a closed graph. In fact, take any sequence $(z_k, \mu_k, b_k, z_k^*) \in \mathrm{gph}F$ and assume that $(z_k, \mu_k, b_k, z_k^*) \to (z, \mu, b, z^*)$ as $k \to \infty$. We have to show that $(z, \mu, b, z^*) \in \mathrm{gph}F$. Since $(z_k, \mu_k, b_k, z_k^*) \in \mathrm{gph}F$, we have

$$z_k^* \in \phi(z_k, \mu_k) + N(z_k; K_1(b_k)).$$

Hence we have

$$\langle \phi(z_k, \mu_k) - z_k^*, z' - z_k \rangle \geq 0, \forall z' \in K_1(b_k).$$

Fix any $z'' \in K_1(b)$. By Lemma 2.1, K_1 is Lipschitz continuous with Lipschitz constant l. Hence for each k, there exists $z_k'' \in K_1(b_k)$ such that $\|z'' - z_k''\| \leq l\|b - b_k\|$. This implies that $z_k'' \to z''$. Since

$$\langle \phi(z_k, \mu_k) - z_k^*, z_k'' - z_k \rangle \geq 0$$

and letting $k \to \infty$, we get

$$\langle \phi(z,\mu) - z^*, z'' - z\rangle \geq 0.$$

As $z'' \in K_1(b)$ is arbitrary, we obtain that $(z, \mu, b, z^*) \in \text{gph}F$. Hence F has a closed graph. Similarly, we can show that Φ_2 has a closed graph.

By changing order of components and using Lemma 3.6, we see that for all $z^* \in Z = R^{(N+1)m+Nn}$ one has

$$\begin{aligned} D^*F(\overline{z},\overline{\mu},\overline{b},0_Z)(z^*) &= \nabla\phi_1(\overline{z},\overline{\mu},\overline{b})^T(z^*) + D^*\Phi_2(\overline{z},\overline{\mu},\overline{b},-\phi_1(\overline{z},\overline{\mu},\overline{b}))\\ &= (\nabla_z\phi(\overline{z},\overline{\mu})^T z^*, \nabla_\mu\phi(\overline{z},\overline{\mu})^T z^*, 0_\Pi) + D^*\Phi_2(\overline{z},\overline{\mu},\overline{b},-\phi_1(\overline{z},\overline{\mu},\overline{b}))(z^*)\\ &= [(\nabla_z\phi(\overline{z},\overline{\mu})^T z^*, 0_\Pi) + D^*F_2(\overline{z},\overline{b},-\phi(\overline{z},\overline{\mu}))(z^*)] \times \{\nabla_\mu\phi(\overline{z},\overline{\mu})^T(z^*)\}. \end{aligned} \tag{3.20}$$

We now show that the implication

$$0 \in D^*F(\overline{z},\overline{\mu},\overline{b},0_Z)(z^*) \Rightarrow z^* = 0 \tag{3.21}$$

is valid, where $0_\Pi, 0_Z$ are zero elements in spaces Π and Z, respectively. Indeed, from (3.20) and condition $0 \in D^*F(\overline{z},\overline{\mu},\overline{b},0_Z)(z^*)$, we get

$$\nabla_\mu\phi(\overline{z},\overline{\mu})^T(z^*) = 0. \tag{3.22}$$

Since

$$\nabla_\mu\phi(\overline{z},\overline{\mu}) = \frac{\partial^2 f(\overline{z},\overline{\mu})}{\partial\mu\partial z} : Z \times M \to Z$$

is surjective, [11, Lemma 1.18] implies that the adjoint mapping

$$\nabla_\mu\phi(\overline{z},\overline{\mu})^T = (\frac{\partial^2 f(\overline{z},\overline{\mu})}{\partial\mu\partial z})^T : Z \to Z \times M$$

is injective. Hence we obtain from (3.22) that $z^* = 0$. Consequently, (3.21) is justified.

It remains to check the condition

$$\bigcup_{z^*\in Z}\{(\mu^*, b^*) \in P^* | (0, \mu^*, b^*) \in D^*F(\overline{y})(z^*)\} = \{0\} \text{ with } \overline{y} = (\overline{z},\overline{\mu},\overline{b},0_Z). \tag{3.23}$$

In fact, assume that $(\mu^*, b^*) \in P$ satisfying $(0, \mu^*, b^*) \in D^*F(\overline{y})(z^*)$ for some $z^* \in Z$. By (3.20) we have

$$\mu^* = \nabla_\mu\phi(\overline{z},\overline{\mu})^T(z^*) \tag{3.24}$$

and

$$(0, b^*) \in (\nabla_z\phi(\overline{z},\overline{\mu})^T(z^*), 0_\Pi) + D^*F_2(\overline{z},\overline{b},-\phi(\overline{z},\overline{\mu}))(z^*). \tag{3.25}$$

The latter is equivalent to

$$(-\nabla_z\phi(\overline{z},\overline{\mu})^T(z^*), b^*) \in D^*F_2(\overline{z},\overline{b},-\phi(\overline{z},\overline{\mu}))(z^*).$$

By Theorem 3.4, there exists an index $I' \subset I(\overline{z},\overline{b}) = T_0 \cup T_1(\overline{z},\overline{b})$ and a closed face Q of the polyhedral convex cone

$$T(\digamma_{I'}; K_1(\overline{b})) \cap (\nabla_z f(\overline{z},\overline{\mu}))^\perp = T(\digamma_{I'}; K(\overline{w})) \cap (\nabla_z f(\overline{z},\overline{\mu}))^\perp$$

such that

$$\begin{cases} (-\nabla_z \phi(\overline{z}, \overline{\mu})^T(z^*), -z^*) \in Q^* \times Q \\ -\nabla_z \phi(\overline{z}, \overline{\mu})^T(z^*) = -C_{I'}^T b_{I'}^* \\ b_{\overline{I}'}^* = 0 \end{cases} \tag{3.26}$$

This is equivalent to

$$\begin{cases} \big((\frac{\partial^2 f(\overline{z}, \overline{\mu})}{\partial z^2})^T(-z^*), -z^* \big) \in Q^* \times Q \\ (\frac{\partial^2}{\partial z^2} f(\overline{z}, \overline{\mu}))^T(-z^*) = -C_{I'}^T b_{I'}^* \\ b_{\overline{I}'}^* = 0. \end{cases}$$

By assumptions of the theorem, it follows that $-z^* = 0$. Substituting $z^* = 0$ into (3.24) we have $\mu^* = 0$. Thus condition (3.23) is valid. Hence Lemma 3.7 is applicable. By Lemma 3.7, the map $M \times \Pi \ni (\mu, b) \mapsto S_1(\mu, b)$ has the Aubin property at $(\overline{\mu}, \overline{b}, \overline{x}, \overline{u})$, that is, there exist positive constants δ_1, δ_2, l and a neighborhood $V \subset X \times U$ of $(\overline{x}, \overline{u})$ such that

$$S_1(\mu', b') \cap V \subset S_1(\mu, b) + l(\|\mu' - \mu\| + \|b' - b\|)B_Z \tag{3.27}$$

for all $\mu, \mu' \in B(\overline{\mu}, \delta_1)$ and $b, b' \in B(\overline{b}, \delta_2) \cap D_1$.

Notice that for $b = b(w)$, $\overline{b} = b(\overline{w})$ one has $\|b(w) - b(w')\| = \sqrt{2}\|w - w'\|$ and $S(\mu, w) = S_1(\mu, b(w))$. Hence for all $\mu, \mu' \in B(\overline{\mu}, \frac{\delta_1}{\sqrt{2}})$ and $w, w' \in B(\overline{w}, \frac{\delta_2}{\sqrt{2}}) \cap D$ we obtain from (3.27) that

$$S(\mu', w') \cap V \subset S(\mu, w) + l\sqrt{2}(\|\mu' - \mu\| + \|w' - w\|)B_Z.$$

The proof of Theorem 3.5 is complete. □

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