Journal of Nonlinear and Convex Analysis
Volume 12, Number 3, 2011, 611–624

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL


# PRIMAL-DUAL INTERIOR POINT METHOD BASED ON A NEW BARRIER FUNCTION

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Abstract. In this paper, we introduce a new barrier function. Using the barrier function we propose a new search direction for primal-dual interior point method(IPM) for solving linear optimization(LO). We show that the new large-update method has $\mathcal{O}(n^{5/8}\log(n/\epsilon))$ iteration bound which improves the iteration bound with a factor $n^{3/8}$ when compare with the method based on the classical logarithmic barrier function. For small-update methods the iteration bound is $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$ which is currently the best known bound.

## 1. Introduction

In this paper we propose a new primal-dual IPM for the following standard LO problem

$$\min\{c^T x : Ax = b,\ x \geq 0\}, \tag{1.1}$$

where $A \in \mathbf{R}^{m\times n}$ with rank$(A) = m$, $c$, $x \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, and its dual problem

$$\max\{b^T y : A^T y + s = c,\ s \geq 0\}, \tag{1.2}$$

where $y \in \mathbf{R}^m$ and $s \in \mathbf{R}^n$.

Since Karmarkar's paper ([12]) in 1984, the interior point methods(IPMs) have shown their efficiency in solving large-scale linear programming problems with a wide variety of successful applications. It is generally agreed that the iteration complexity of the algorithm is an appropriate measure for its efficiency and we regard polynomial-time algorithm as a sign that the problem has been solved satisfactorily.([8]) Most of polynomial-time interior point algorithms for LO are based on the logarithmic barrier function([3, 10, 11, 18]). Peng et al.([14, 15]) proposed new variants of IPMs based on self-regular barrier functions and proved the new large-update IPMs have polynomial iteration bounds. Each barrier function is defined by a univariate rational function which is called a kernel function. Roos et al.([4, 5, 6, 7, 8, 9]) proposed new primal-dual IPMs for LO problems based on exponential barrier functions and proved the polynomial complexity of the algorithm. Recently, Amini et al.([1, 2]) proposed the primal-dual IPM based on the generalized version of the barrier function in [5].

Motivated by their works, we introduce a new rational barrier function and propose a new primal-dual IPM for LO based on this barrier function. Since the barrier function in this paper does not belong to the family of self-regular functions, a large

2010 *Mathematics Subject Classification.* 90C05, 90C51 .
*Key words and phrases.* Primal-dual interior point method, kernel function, complexity, polynomial algorithm, linear optimization problem.
This work was supported by 2010 Dongseo Frontier Project of Dongseo University.

part of the analysis is different from the ones in [14, 15]. And we analyze the complexity for large-update and small-update methods based on three conditions on the kernel function which is defined in [5]. The complexity bounds obtained by the algorithm are $\mathcal{O}(n^{\frac{5}{8}} \log \frac{n}{\epsilon})$ and $\mathcal{O}(\sqrt{n} \log \frac{n}{\epsilon})$, for large-update methods and for small-update methods, respectively.

The paper is organized as follows. In Section 2 we recall the generic IPM and the motivation of the new algorithm. In Section 3 we define a new barrier function and give its properties which are essential for complexity analysis. In Section 4 we derive the complexity result for the algorithm.

We use the following notations throughout the paper. $\mathbf{R}^{\mathbf{n}}_{+}$ and $\mathbf{R}^{\mathbf{n}}_{++}$ denote the set of $n$-dimensional nonnegative vectors and positive vectors, respectively. For $x, s \in \mathbf{R}^{\mathbf{n}}$, $x_{min}$ and $xs$ denote the smallest component of the vector $x$ and the componentwise product of the vectors $x$ and $s$, respectively. $\mathbf{e}$ denotes the $n$-dimensional vector of ones. For $a \in \mathbf{R}$, $\lfloor a \rfloor := \max\{m \in \mathbf{Z} \mid m \leq a\}$ and $\lceil a \rceil := \min\{n \in \mathbf{Z} \mid n \geq a\}$. For $f(t),\ g(t) : \mathbf{R}_{++} \to \mathbf{R}_{++}$, $f(t) = \mathcal{O}(g(t))$ if $f(t) \leq c_1 g(t)$ for some positive constant $c_1$ and $f(t) = \Theta(g(t))$ if $c_2 g(t) \leq f(t) \leq c_3 g(t)$ for some positive constants $c_2$ and $c_3$.

## 2. Preliminaries

In this section we recall the basic concepts and the generic IPM. Without loss of generality, we assume that both (1.1) and (1.2) satisfy the interior-point condition (IPC)([16]), i.e., there exists $(x^0, y^0, s^0)$ such that

$$Ax^0 = b,\ x^0 > 0,\ A^T y^0 + s^0 = c,\ s^0 > 0.$$

By the duality theorem (Theorem II.2 in [16]), finding an optimal solution of (1.1) and (1.2) is equivalent to solving the following system:

$$\begin{cases} Ax = b, \quad x \geq 0, \\ A^T y + s = c, \quad s \geq 0, \\ xs = 0. \end{cases} \tag{2.1}$$

The basic idea of primal-dual IPMs is to replace the third equation in (2.1) by the parameterized equation $xs = \mu\mathbf{e}$ with $\mu > 0$. Now we consider the following system:

$$\begin{cases} Ax = b, \quad x > 0, \\ A^T y + s = c, \quad s > 0, \\ xs = \mu\mathbf{e}. \end{cases} \tag{2.2}$$

If the IPC holds, then the system (2.2) has a unique solution for each $\mu > 0$([13]). We denote this solution as $(x(\mu), y(\mu), s(\mu))$ and call $x(\mu)$ the $\mu$-center of (1.1) and $(y(\mu), s(\mu))$ the $\mu$-center of (1.2). The set of $\mu$-centers ($\mu > 0$) is the central path of (1.1) and (1.2) ([17]). The limit of the central path (as $\mu$ goes to zero) exists and since the limit point satisfies (2.1), it naturally yields optimal solutions for (1.1) and (1.2) ([16]). Primal-dual IPMs follow the central path approximately and approach the solution of (1.1) and (1.2) as $\mu$ goes to zero.

For given $(x, y, s) := (x^0, y^0, s^0)$ by applying Newton method to the system (2.2) we have the following Newton system

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = \mu\mathbf{e} - xs. \end{cases} \tag{2.3}$$

Since $A$ has full row rank, the system (2.3) has a unique solution $(\Delta x, \Delta y, \Delta s)$ which is called the search direction ([16]). By taking a step along the search direction $(\Delta x, \Delta y, \Delta s)$, one constructs a new positive $(x_+, y_+, s_+)$ with

$$x_+ = x + \alpha\Delta x,\ y_+ = y + \alpha\Delta y,\ s_+ = s + \alpha\Delta s,$$

for some $\alpha > 0$.

For the motivation of the new algorithm we define the following scaled vectors for $x,\ s \in \mathbf{R}^{\mathbf{n}}_{++}$:

$$v := \sqrt{\frac{xs}{\mu}},\ d_x := \frac{v\Delta x}{x},\ d_s := \frac{v\Delta s}{s}. \tag{2.4}$$

Using (2.4), we can rewrite the system (2.3) as follows:

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^T\Delta y + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{2.5}$$

where $\bar{A} := \frac{1}{\mu}AV^{-1}X$, $V := \mathrm{diag}(v)$, and $X := \mathrm{diag}(x)$. Note that the right side of the third equation in (2.5) equals the negative gradient of the logarithmic barrier function $\Psi_l(v)$, i.e.

$$d_x + d_s = -\nabla\Psi_l(v), \tag{2.6}$$

where

$$\Psi_l(v) := \sum_{i=1}^{n} \psi_l(v_i) = \sum_{i=1}^{n} \left( \frac{v_i^2 - 1}{2} - \log\ v_i \right).$$

We call $\psi_l$ the kernel function of the logarithmic barrier function $\Psi_l(v)$. In this paper we replace $\Psi_l(v)$ with a new barrier function $\Psi(v)$ which is defined in section 3 and assume that $\tau \geq 1$.

The generic interior point algorithm works as follows. Assume that we are given a strictly feasible point $(x, y, s)$ which is in a $\tau-$neighborhood of the given $\mu-$center. Then we decrease $\mu$ to $\mu_+ = (1-\theta)\mu$, for some fixed $\theta \in (0, 1)$ and then we solve the Newton system (2.3) to obtain the unique search direction. The positivity condition of a new iterate is ensured with the right choice of the step size $\alpha$ which is defined by some line search rule. This procedure is repeated until we find a new iterate $(x_+, y_+, s_+)$ that is in a $\tau-$neighborhood of the $\mu_+-$center and then we let $\mu := \mu_+$ and $(x, y, s) := (x_+, y_+, s_+)$. Then $\mu$ is again reduced by the factor $1 - \theta$ and we solve the Newton system targeting at the new $\mu_+$-center, and so on. This process is repeated until $\mu$ is small enough, say until $n\mu < \varepsilon$.

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**Primal-Dual Algorithm for LO**

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Input:
  a threshold parameter $\tau > 0$;
  an accuracy parameter $\varepsilon > 0$;
  a fixed barrier update parameter $\theta,\ 0 < \theta < 1$;
  $(x^0, s^0)$ and $\mu^0 := 1$ such that $\Psi_l(x^0, s^0, \mu^0) \leq \tau$.
begin
  $x := x^0;\ s := s^0;\ \mu := \mu^0;$
  while $n\mu \geq \varepsilon$ do
  begin
    $\mu := (1-\theta)\mu;$
    while $\Psi_l(v) > \tau$ do
    begin
      Solve the system (2.3) for $\Delta x, \Delta y, \Delta s$,
      Determine a step size $\alpha$;
      $x := x + \alpha\Delta x;$
      $s := s + \alpha\Delta s;$
      $y := y + \alpha\Delta y;$
      $v := \sqrt{\frac{xs}{\mu}};$
    end
  end
end

---

If $\theta$ is a constant independent of the dimension of the problem $n$, e.g. $\theta = \frac{1}{2}$, then we call the algorithm a large-update method. If $\theta$ depends on n, e.g. $\theta = \frac{1}{\sqrt{n}}$, then the algorithm is called a small-update method.

## 3. The new barrier function

In this section we define a new barrier function and give its properties. We call $\psi : \mathbf{R}_{++} \to \mathbf{R}_{+}$ a kernel function if $\psi$ is twice differentiable and satisfies the following conditions:

$$\begin{aligned} &\psi'(1) = \psi(1) = 0, \\ &\psi''(t) > 0,\ t > 0, \\ &\lim_{t\to 0^+} \psi(t) = \lim_{t\to\infty} \psi(t) = \infty. \end{aligned} \tag{3.1}$$

Now we define a new kernel function $\psi(t)$ as follows:

$$\psi(t) := 8t^2 - 10t + \frac{2}{t^3},\ t > 0. \tag{3.2}$$

Then we have the following:

$$\psi'(t) = 16t - 10 - 6t^{-4},\ \psi''(t) = 16 + 24t^{-5},\ \psi'''(t) = -120t^{-6}. \tag{3.3}$$

From (3.3), $\psi(t)$ is clearly a kernel function and

$$\psi''(t) > 16, \; t > 0. \tag{3.4}$$

In this paper, we define the barrier function $\Psi(v) = \sum_{i=1}^{n} \psi(v_i)$, where $\psi(t)$ is defined in (3.2) and replace the function $\Psi_l(v)$ in (2.6) with the function $\Psi(v)$ as follows:

$$d_x + d_s = -\nabla\Psi(v). \tag{3.5}$$

Note that $d_x$ and $d_s$ are orthogonal because the vector $d_x$ belongs to null space and $d_s$ to the row space of the matrix $\bar{A}$. Since $d_x$ and $d_s$ are orthogonal, we have

$$d_x = d_s = 0 \;\Leftrightarrow\; \nabla\Psi(v) = 0 \Leftrightarrow\; v = \mathbf{e} \;\Leftrightarrow\; \Psi(v) = 0 \;\Leftrightarrow\; x = x(\mu), \; s = s(\mu).$$

We use $\Psi(v)$ as the proximity function. Also, we define the norm-based proximity measure $\delta(v)$ as follows:

$$\delta(v) := \frac{1}{2}||\nabla\Psi(v)|| = \frac{1}{2}||d_x + d_s||. \tag{3.6}$$

Consequently, in this paper we use the barrier function $\Psi(v)$ as the proximity function to find a search direction and to measure the proximity between the current iterate and the $\mu$-center. Hence the new search direction $(\Delta x, \Delta y, \Delta s)$ is obtained by solving the following modified Newton system:

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = -\mu v\nabla\Psi(v). \end{cases}$$

**Lemma 3.1.** *For $\psi(t)$ we have*
*(i) $\psi(t)$ is exponentially convex, $t > 0$,*
*(ii) $\psi''(t)$ is monotonically decreasing, $t > 0$,*
*(iii) $t\psi''(t) - \psi'(t) > 0$, $t > 0$.*

*Proof.* For $(i)$, by Lemma 2.1.2 in [15], it suffices to show that the function $\psi(t)$ satisfies $t\psi''(t) + \psi'(t) \geq 0$ for $t > 0$. Using (3.3), we have

$$t\psi''(t) + \psi'(t) = 32t - 10 + 18t^{-4}.$$

Let $g(t) = 32t - 10 + 18t^{-4}$. Then $g'(t) = 32 - 72t^{-5}$ and $g''(t) = 360t^{-6} > 0$, $t > 0$. Letting $g'(t) = 0$, we have $t = (\frac{9}{4})^{\frac{1}{5}}$. Since $g(t)$ is strictly convex and has a global minimum $g((\frac{9}{4})^{\frac{1}{5}}) > 37$. Hence we have the result.
For $(ii)$, from (3.3), $\psi'''(t) < 0$.
For $(iii)$, using (3.3), we have $t\psi''(t) - \psi'(t) = 10 + 30t^{-4} > 0$. This completes the proof. □

**Lemma 3.2.** *For $\psi(t)$ we have*
*(i) $8(t-1)^2 \;\leq\; \psi(t) \;\leq\; \frac{1}{32}(\psi'(t))^2, \; t > 0$,*
*(ii) $\psi(t) \;\leq\; 20(t-1)^2, \; t \geq 1$.*

*Proof.* For $(i)$, using the first condition of (3.1) and (3.4), we have

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi \geq 16 \int_1^t \int_1^\xi d\zeta d\xi = 8(t-1)^2.$$

which proves the first inequality. The second inequality is obtained as follows:

$$\begin{aligned}\psi(t) &= \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi \le \frac{1}{16}\int_1^t \int_1^\xi \psi''(\xi)\psi''(\zeta)d\zeta d\xi \\ &= \frac{1}{16}\int_1^t \psi''(\xi)\psi'(\xi)d\xi = \frac{1}{16}\int_1^t \psi'(\xi)d\psi'(\xi) = \frac{1}{32}(\psi'(t))^2.\end{aligned}$$

For $(ii)$, using Taylor's Theorem, $\psi(1) = \psi^{'}(1) = 0$, $\psi^{'''} < 0$, and $\psi^{''}(1) = 40$, we have

$$\begin{aligned}\psi(t) &= \psi(1) + \psi^{'}(1)(t-1) + \frac{1}{2}\psi^{''}(1)(t-1)^2 + \frac{1}{3!}\psi^{'''}(\xi)(\xi-1)^3 \\ &= \frac{1}{2}\psi^{''}(1)(t-1)^2 + \frac{1}{3!}\psi^{'''}(\xi)(\xi-1)^3 \\ &< \frac{1}{2}\psi^{''}(1)(t-1)^2 = 20(t-1)^2,\end{aligned}$$

for some $\xi$, $1 \le \xi \le t$. This completes the proof. □

**Lemma 3.3** (Lemma 2.4 in [5]). *If $\psi(t)$ satisfies Lemma 3.1 $(ii)$ and $(iii)$, then $\psi(t)$ satisfies*

$$\psi^{''}(t)\psi^{'}(\beta t) - \beta\psi^{'}(t)\psi^{''}(\beta t) > 0,\ t > 1,\ \beta > 1.$$

Let $\varrho : [0,\infty) \to [1,\infty)$ be the inverse function of $\psi(t)$ for $t \ge 1$ and $\rho : [0,\infty) \to (0,1]$, the inverse function of $-\frac{1}{2}\psi^{'}(t)$ for $t \in (0,1]$. Then we have the following lemma.

**Lemma 3.4.** *For $\psi(t)$ we have*

$(i)$ $\sqrt{\frac{s}{8}+1} \le \varrho(s) \le 1 + \sqrt{\frac{s}{8}}$, $s \ge 0$,

$(ii)$ $\rho(z) \ge (\frac{3}{z+3})^{\frac{1}{4}}$, $z \ge 0$.

*Proof.* For $(i)$, let $s = \psi(t)$, $t \ge 1$, i.e. $\varrho(s) = t$, $t \ge 1$. By the definition of $\psi(t)$, $s = 8t^2 - 10t + \frac{2}{t^3}$. This implies that

$$8t^2 = s + 10t - \frac{2}{t^3} \ge s + 8$$

because $10t - \frac{2}{t^3}$ is monotone increasing with respect to $t$ and $t \ge 1$. Hence we have

$$t = \varrho(s) \ge \sqrt{\frac{s}{8}+1},\ s \ge 0.$$

Using Lemma 3.2 $(i)$, we have $s = \psi(t) \ge 8(t-1)^2$, $t > 0$. Then we have

$$t = \varrho(s) \le 1 + \sqrt{\frac{s}{8}},\ s \ge 0.$$

For $(ii)$, let $z = -\frac{1}{2}\psi'(t)$, $t \in (0,1]$. Then by the definition of $\rho$, $\rho(z) = t$, $t \in (0,1]$ and $2z = -\psi'(t)$. So we have $2z = -16t + 10 + 6t^{-4}$. Since $0 < t \le 1$,

$$6t^{-4} = 2z + 16t - 10 \le 2z + 6.$$

Hence we have

$$\rho(z) = t \ge (\frac{3}{z+3})^{\frac{1}{4}},\ z \ge 0.$$

□

Using Lemma 3.3, we have the following lemma. The reader can refer to Theorem 3.2 in [5] for the proof.

**Lemma 3.5.** *Let* $\varrho : [0, \infty) \to [1, \infty)$ *be the inverse function of* $\psi(t)$*,* $t \geq 1$*. Then we have*

$$\Psi(\beta v) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right), \quad v \in \mathbf{R}_{++}, \ \beta \geq 1.$$

In the following theorem we obtain an estimate for the effect of a $\mu$-update on the value of $\Psi(v)$.

**Theorem 3.6.** *Let* $0 \leq \theta < 1$ *and* $v_+ = \frac{v}{\sqrt{1-\theta}}$*,* $v \in \mathbf{R}^{\mathbf{n}}_{++}$*. If* $\Psi(v) \leq \tau$*, then*

$$\Psi(v_+) \leq \frac{20}{1-\theta}\left(\sqrt{n}\theta + \sqrt{\frac{\tau}{8}}\right)^2.$$

*Proof.* Since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\varrho\left(\frac{\Psi(v)}{n}\right) \geq 1$, we have $\frac{\varrho\left(\frac{\Psi(v)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. Using Lemma 3.5 with $\beta = \frac{1}{\sqrt{1-\theta}}$, Lemma 3.2 $(ii)$, Lemma 3.4 $(i)$, and $\Psi(v) \leq \tau$, we have

$$\begin{aligned}
\Psi(v_+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(v)}{n}\right)\right)\\
&\leq 20n\left(\frac{\varrho\left(\frac{\Psi(v)}{n}\right)}{\sqrt{1-\theta}} - 1\right)^2 = 20n\left(\frac{\varrho\left(\frac{\Psi(v)}{n}\right) - \sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2\\
&\leq 20n\left(\frac{1 + \sqrt{\frac{\tau}{8n}} - \sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2\\
&\leq 20n\left(\frac{\theta + \sqrt{\frac{\tau}{8n}}}{\sqrt{1-\theta}}\right)^2 = \frac{20}{1-\theta}\left(\sqrt{n}\theta + \sqrt{\frac{\tau}{8}}\right)^2,
\end{aligned}$$

where the last inequality holds from $1 - \sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, $0 \leq \theta < 1$. This completes the proof. □

Denote

$$\tilde{\Psi}_0 := \frac{20}{1-\theta}\left(\sqrt{n}\theta + \sqrt{\frac{\tau}{8}}\right)^2. \tag{3.7}$$

Then $\tilde{\Psi}_0$ is an upper bound for $\Psi(v)$ during the process of the algorithm.

**Remark 3.7.** For large-update method with $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, $\tilde{\Psi}_0 = \mathcal{O}(n)$ and for small-update method with $\tau = \mathcal{O}(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, $\tilde{\Psi}_0 = \mathcal{O}(1)$.

## 4. Complexity results

In this section we compute a step size and the decrease of the proximity function during an inner iteration and give the complexity results of the algorithm. For fixed

$\mu$, if we take a step size $\alpha$, then we have new iterates $x_+ = x + \alpha\Delta x,\ s_+ = s + \alpha\Delta s$. Using (2.4), we have

$$x_+ = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}(v + \alpha d_x)$$

and

$$s_+ = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}(v + \alpha d_s).$$

Thus we have

$$v_+ := \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_s)}.$$

Define for $\alpha > 0$,

$$f(\alpha) = \Psi(v_+) - \Psi(v).$$

Then $f(\alpha)$ is the difference of proximities between a new iterate and a current iterate for fixed $\mu$. By Lemma 3.1 ($i$), we have

$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}\ ) \leq \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).$$

Hence we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)) - \Psi(v). \tag{4.1}$$

Obviously, we have

$$f(0) = f_1(0) = 0.$$

By taking the derivative of $f_1(\alpha)$ with respect to $\alpha$, we have

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi'(v_i + \alpha[d_x]_i)[d_x]_i + \psi'(v_i + \alpha[d_s]_i)[d_s]_i),$$

where $[d_x]_i$ and $[d_s]_i$ denote the $i$-th components of the vectors $d_x$ and $d_s$, respectively. Using (3.5) and (3.6), we have

$$f_1'(0) = \frac{1}{2}\nabla\Psi(v)^T(d_x + d_s) = -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2. \tag{4.2}$$

Differentiating $f_1'(\alpha)$ with respect to $\alpha$, we have

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi''(v_i + \alpha[d_x]_i)[d_x]_i^2 + \psi''(v_i + \alpha[d_s]_i)[d_s]_i^2). \tag{4.3}$$

Since $f_1''(\alpha) > 0$, $f_1(\alpha)$ is strictly convex in $\alpha$ unless $d_x = d_s = 0$.

**Lemma 4.1.** *Let $\delta(v)$ be as defined in* (3.6). *Then we have*

$$\delta(v) \geq 2\sqrt{2\Psi(v)}.$$

*Proof.* Using Lemma 3.2 ($i$) and (3.6), we have

$$\Psi(v) = \sum_{i=1}^{n}\psi(v_i) \leq \frac{1}{32}\sum_{i=1}^{n}(\psi'(v_i))^2 = \frac{1}{32}||\nabla\Psi(v)||^2 = \frac{\delta^2(v)}{8}.$$

Hence we have $\delta(v) \geq 2\sqrt{2\Psi(v)}$. □

**Remark 4.2.** Throughout the paper we assume that $\tau \geq 1$. Using Lemma 4.1 and the assumption $\Psi(v) \geq \tau$, we have

$$\delta(v) \geq 2\sqrt{2\Psi(v)} \geq 2\sqrt{2}. \tag{4.4}$$

For notational convenience we denote $\delta := \delta(v)$, and $\Psi := \Psi(v)$.

**Lemma 4.3.** *Let $f_1(\alpha)$ be as defined in* (4.1) *and $\delta$ be as defined in* (3.6)*. Then we have*

$$f_1''(\alpha) \leq 2\delta^2 \psi''(v_{min} - 2\alpha\delta). \tag{4.5}$$

*Proof.* Since $d_x$ and $d_s$ are orthogonal, (3.5) and (3.6) imply that

$$\sqrt{||d_x||^2 + ||d_s||^2} = ||d_x + d_s|| = || - \nabla\Psi|| = 2\delta. \tag{4.6}$$

Hence, we have $||d_x|| \leq 2\delta$ and $||d_s|| \leq 2\delta$. Therefore, we have

$$v_i + \alpha[d_x]_i \geq v_{min} - 2\alpha\delta,\ v_i + \alpha[d_s]_i \geq v_{min} - 2\alpha\delta,\ 1 \leq i \leq n. \tag{4.7}$$

Using (4.3), Lemma 3.1 (*ii*), (4.7) and (4.6), we have

$$f_1''(\alpha) \leq \frac{1}{2}\psi''(v_{min} - 2\alpha\delta)\sum_{i=1}^{n}([d_x]_i^2 + [d_s]_i^2) = 2\delta^2\psi''(v_{min} - 2\alpha\delta).$$

This proves the lemma. □

**Lemma 4.4.** *If the step size $\alpha$ satisfies the inequality*

$$-\psi'(v_{min} - 2\alpha\delta) + \psi'(v_{min}) \leq 2\delta, \tag{4.8}$$

*then we have*

$$f_1'(\alpha) \leq 0.$$

*Proof.* Since $d(v_{min} - 2\zeta\delta) = -2\delta d\zeta$,

$$\begin{aligned} f_1'(\alpha) &= f_1'(0) + \int_0^\alpha f_1''(\zeta)d\zeta \\ &\leq -2\delta^2 + 2\delta^2\int_0^\alpha \psi''(v_{min} - 2\zeta\delta)d\zeta \\ &= -2\delta^2 - \delta\int_0^\alpha \psi''(v_{min} - 2\zeta\delta)d(v_{min} - 2\zeta\delta) \\ &= -2\delta^2 - \delta(\psi'(v_{min} - 2\alpha\delta) - \psi'(v_{min})) \\ &\leq -2\delta^2 + 2\delta^2 = 0, \end{aligned}$$

where the first inequality holds by (4.2) and (4.5) and the second inequality holds by the assumption. This proves the lemma. □

**Lemma 4.5.** *Let $\rho : [0, \infty) \to (0, 1]$ denote the inverse function of $-\frac{1}{2}\psi'(t)$ for $t \in (0, 1]$ and $\delta := \delta(v) \geq 0$, $v \in R^n_{++}$. Then, in the worst case, the largest step size $\hat{\alpha}$ satisfying* (4.8) *is given by*

$$\hat{\alpha} := \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)).$$

*Proof.* Given $\delta$, we want to find the largest possible $\alpha$ such that (4.8) holds. Since $\psi''(t)$ is monotone decreasing for $t > 0$, the derivative of the left side of (4.8) with respect to $v_{min}$ becomes

$$-\psi''(v_{min} - 2\alpha\delta) + \psi''(v_{min}) < 0.$$

Hence, the left side of (4.8) is monotone decreasing in $v_{min}$. For fixed $\delta$, if $v_{min}$ is smaller, then $\alpha$ will be smaller. Using (3.6), we have

$$\delta = \frac{1}{2}||\nabla\Psi(v)|| \geq \frac{1}{2}|\psi'(v_{min})| \geq -\frac{1}{2}\psi'(v_{min}).$$

Equality holds if and only if $v_{min}$ is the only coordinate in $v$ that differs from 1 and $v_{min} \leq 1$. Hence, the worst situation for the step size occurs when $v_{min}$ satisfies

$$-\frac{1}{2}\psi'(v_{min}) = \delta. \tag{4.9}$$

The derivative of the left side of (4.8) with respect to $\alpha$ equals $2\delta\psi''(v_{min} - 2\alpha\delta) \geq 0$ and hence the left side is increasing in $\alpha$. So the largest possible value of $\alpha$ satisfying (4.8) holds the following equality

$$-\frac{1}{2}\psi'(v_{min} - 2\alpha\delta) = 2\delta. \tag{4.10}$$

Due to the definition of $\rho$, (4.9) and (4.10) can be written as

$$v_{min} = \rho(\delta), \;\; v_{min} - 2\alpha\delta = \rho(2\delta).$$

This implies

$$\hat{\alpha} = \frac{1}{2\delta}(v_{min} - \rho(2\delta)) = \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)).$$

This proves the lemma. □

**Lemma 4.6.** *Let $\rho$ and $\hat{\alpha}$ be as defined in Lemma 4.5. Then we have for $\delta > 0$,*

$$\hat{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}.$$

*Proof.* By the definition of $\rho$, we have

$$-\psi'(\rho(\delta)) = 2\delta.$$

If we differentiate the above equation with respect to $\delta$, we have $-\psi''(\rho(\delta))\rho'(\delta) = 2$. Since $\psi''(t) > 0$, for all $t > 0$, we have

$$\rho'(\delta) = -\frac{2}{\psi''(\rho(\delta))} < 0. \tag{4.11}$$

Hence, $\rho$ is monotonically decreasing with respect to $\delta$. Using Lemma 4.5 and (4.11), we have

$$\hat{\alpha} = \frac{1}{2\delta}\int_{2\delta}^{\delta} \rho'(\sigma)d\sigma = \frac{1}{\delta}\int_{\delta}^{2\delta} \frac{1}{\psi''(\rho(\sigma))}d\sigma. \tag{4.12}$$

By Lemma 3.1 $(ii)$, $\psi''(\rho(\sigma)) \leq \psi''(\rho(2\delta))$ for $\sigma \in [\delta, 2\delta]$, i.e. $\psi''(\rho(\sigma))$ is maximal when $\sigma = 2\delta$. From (4.12),

$$\hat{\alpha} = \frac{1}{\delta}\int_{\delta}^{2\delta} \frac{1}{\psi''(\rho(\sigma))}d\sigma \geq \frac{1}{\delta}\int_{\delta}^{2\delta} \frac{1}{\psi''(\rho(2\delta))}d\sigma = \frac{1}{\psi''(\rho(2\delta))}.$$

This proves the lemma. □

Define

$$\bar{\alpha} := \frac{1}{\psi''(\rho(2\delta))}. \tag{4.13}$$

Then we have $\bar{\alpha} \leq \hat{\alpha}$.

**Lemma 4.7.** *Let $\bar{\alpha}$ be as defined in* (4.13)*. If $\Psi(v) \geq \tau \geq 1$, then we have*

$$\bar{\alpha} \geq \frac{1}{(4\sqrt{2} + 32(\frac{4}{3})^{\frac{1}{4}})\delta^{\frac{5}{4}}}.$$

*Proof.* Using the definition of $\psi''(t)$, Lemma 3.4 (*ii*), and (4.4), we have

$$\begin{aligned}
\bar{\alpha} &= \frac{1}{\psi''(\rho(2\delta))} = \frac{1}{16 + 24(\rho(2\delta))^{-5}} \\
&\geq \frac{1}{16 + 24\left(\frac{2\delta+3}{3}\right)^{\frac{5}{4}}} = \frac{1}{16 + 16(\frac{2}{3})^{\frac{1}{4}}(\delta + \frac{3}{2})^{\frac{5}{4}}} \\
&\geq \frac{1}{16 + 32(\frac{4}{3})^{\frac{1}{4}}\delta^{\frac{5}{4}}} \geq \frac{1}{4\sqrt{2}\delta^{\frac{5}{4}} + 32(\frac{4}{3})^{\frac{1}{4}}\delta^{\frac{5}{4}}} \\
&= \frac{1}{(4\sqrt{2} + 32(\frac{4}{3})^{\frac{1}{4}})\delta^{\frac{5}{4}}}.
\end{aligned}$$

□

Define

$$\tilde{\alpha} = \frac{1}{(4\sqrt{2} + 32(\frac{4}{3})^{\frac{1}{4}})\delta^{\frac{5}{4}}}. \tag{4.14}$$

Then $\tilde{\alpha} \leq \bar{\alpha}$. We will use $\tilde{\alpha}$ as the default step size.

**Lemma 4.8** (Lemma 1.3.3 in [15])**.** *Suppose that $h(t)$ is a twice differentiable convex function with*

$$h(0) = 0, \;\; h'(0) < 0$$

*and $h(t)$ attains its (global) minimum at $t^* > 0$ and $h''(t)$ is increasing with respect to $t$. Then for any $t \in [0, t^*]$, we have*

$$h(t) \leq \frac{th'(0)}{2}.$$

**Lemma 4.9.** *If the step size $\alpha$ is such that $\alpha \leq \bar{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

*Proof.* Let the univariate function $h$ be such that

$$h(0) = f_1(0) = 0, \;\; h'(0) = f_1'(0) = -2\delta^2, \;\; h''(\alpha) = 2\delta^2\psi''(v_{min} - 2\alpha\delta).$$

Then $h(t)$ is twice differentiable, $h(0) = 0$, and $h'(0) < 0$. Since $h''(\alpha) > 0$, $h(t)$ is strictly convex and hence has a global minimum at some $\alpha^* > 0$. From (4.5), we have $f_1''(\alpha) \le h''(\alpha)$. As a result, we have $f_1'(\alpha) \le h'(\alpha)$ and $f_1(\alpha) \le h(\alpha)$. Taking $\alpha \le \bar{\alpha}$, we have

$$\begin{aligned} h'(\alpha) &= h'(0) + \int_0^\alpha h''(\zeta) d\zeta \\ &= -2\delta^2 + 2\delta^2 \int_0^\alpha \psi''(v_{min} - 2\xi\delta) d\xi \\ &= -2\delta^2 - \frac{2\delta^2}{2\delta} \int_0^\alpha \psi''(v_{min} - 2\xi\delta) d(v_{min} - \xi\delta) \\ &= -2\delta^2 + \delta(\psi'(v_{min}) - \psi'(v_{min} - 2\alpha\delta)) \\ &\le -2\delta^2 + 2\delta^2 = 0, \end{aligned}$$

where the inequality follows from (4.8). Since $h'''(\alpha) = -4\delta^3\psi'''(v_{min} - 2\alpha\delta) > 0$, $h''(\alpha)$ is monotonically increasing in $\alpha$. Thus, using Lemma 4.8, we have

$$f_1(\alpha) \le h(\alpha) \le \frac{1}{2}\alpha h'(0) = -\alpha\delta^2.$$

Since $f(\alpha) \le f_1(\alpha)$, the lemma is proved. □

**Theorem 4.10.** *Let $\tilde{\alpha}$ be as defined in* (4.14) *and* $\Psi(v) \ge 1$. *Then*

$$f(\tilde{\alpha}) \le -\frac{8^{\frac{3}{8}}\Psi(v)^{\frac{3}{8}}}{41}.$$

*Proof.* Using Lemma 4.9, (4.14), and Lemma 4.1 we have

$$f(\tilde{\alpha}) \le -\tilde{\alpha}\delta^2 = -\frac{\delta^{\frac{3}{4}}}{4\sqrt{2} + 32(\frac{4}{3})^{\frac{1}{4}}} \le -\frac{\delta^{\frac{3}{4}}}{41} \le -\frac{8^{\frac{3}{8}}\Psi(v)^{\frac{3}{8}}}{41}.$$

This completes the proof. □

**Lemma 4.11** (Lemma 1.3.2 in [15])**.** *Let $t_0, t_1, \ldots, t_{\hat{K}}$ be a sequence of positive numbers such that*

$$t_{k+1} \le t_k - \gamma t_k^{1-\tilde{\beta}}, \ k = 0, 1, \ldots, \hat{K} - 1,$$

*where $\gamma > 0$ and $0 < \tilde{\beta} \le 1$. Then* $\hat{K} \le \left\lfloor \frac{t_0^{\tilde{\beta}}}{\gamma\tilde{\beta}} \right\rfloor$.

We define the value of $\Psi(v)$ after the $\mu-$update as $\Psi_0$ and the subsequent values in the same outer iteration are denoted as $\Psi_k$, $k = 1, 2, \ldots$. Then we have

$$\Psi_0 \le \tilde{\Psi}_0,$$

where $\tilde{\Psi}_0$ is defined in (3.7). Let $K$ denote the total number of inner iterations per outer iteration. Then we have

$$\Psi_{K-1} > \tau, \ 0 \le \Psi_K \le \tau.$$

**Lemma 4.12.** *Let $\tilde{\Psi}_0$ be as defined in (3.7) and $K$ the total number of inner iterations in the outer iteration. Then we have*

$$K \le 31\tilde{\Psi}_0^{\frac{5}{8}}.$$

*Proof.* Using Theorem 4.10 and Lemma 4.11 with $\gamma := \frac{8^{\frac{3}{8}}}{41}$ and $\tilde{\beta} := \frac{5}{8}$, we have

$$K \;\leq\; \left(\frac{41}{8^{3/8}}\right)\left(\frac{8}{5}\right)\tilde{\Psi}_0^{\frac{5}{8}} \;\leq\; 31\tilde{\Psi}_0^{\frac{5}{8}}.$$

This completes the proof. □

**Theorem 4.13.** *Let a LO problem be given, $\tilde{\Psi}_0$ as defined in (3.7) and $\tau \geq 1$. Then the total number of iterations to have an approximate solution with $n\mu < \epsilon$ is bounded by*

$$\left\lceil \frac{31}{\theta}\tilde{\Psi}_0^{\frac{5}{8}} \log\frac{n}{\epsilon} \right\rceil.$$

*Proof.* If the central path parameter $\mu$ has the initial value $\mu^0 := 1$ and is updated by multiplying $1-\theta$ with $0 \leq \theta < 1$, then after at most

$$\left\lceil \frac{1}{\theta}\log\frac{n}{\epsilon} \right\rceil$$

iterations we have $n\mu < \epsilon$([16]). For the total number of iterations, we multiply the number of inner iterations by that of outer iterations. Hence the total number of iterations is bounded by

$$\left\lceil \frac{31}{\theta}\tilde{\Psi}_0^{\frac{5}{8}} \log\frac{n}{\epsilon} \right\rceil.$$

This completes the proof. □

**Remark 4.14.** By Remark 3.7, for large-update methods with $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, the algorithm has $\mathcal{O}(n^{\frac{5}{8}} \log\frac{n}{\epsilon})$ iteration complexity which improves the complexity for large-update IPMs based on the classical logarithmic barrier function. For small-update methods, we have $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$ iteration complexity which is the best complexity result so far.

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*Manuscript received July 28, 2010*
*revised March 4, 2011*

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