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CONTROLLABILITY FOR NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY TERMS

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Abstract. In this paper, we deal with the approximate controllability for the nonlinear functional differential equations with time delay and establish a variation of constant formula for solutions of the given equations. We replace the compactness of fundamental operators by the compactness of a given Gelfand triple spaces, and we need the inequality constraint on the range condition of the controller.

1. Introduction

Let H and V be two complex Hilbert spaces forming a Gelfand triple $V \subset H \subset V^*$ with a pivot space H.

In this paper we investigate the approximate controllability for the following nonlinear functional differential equation on H:

$$(1.1) \quad \begin{cases} x'(t) + Ax(t) = \int_{-h}^{0} g(t,s,x(t),x(t+s))\mu(ds) + (Bu)(t), & 0 < t \le T, \\ x(0) = g^0, \quad x(s) = g^1(s) \quad s \in [-h,0). \end{cases}$$

Here, the principal operator A is given as a single valued, monotone operator, which is hemicontinuous and coercive from V to V^*. Here V^* stands for the dual space of V. If the nonlinear term belongs to $L^2(0,T;V^*)$, the basic assumption made in these investigations is taken from the regularity result for the quasi-autonomous differential equation(see Theorem 2.6 of Chapter III in [3]).

Most studies have been devoted to the semilinear system without time delay, and the paper treating the nonlinear system with delay are not many. The regularity of solution of the semilinear functional differential equations with unbounded delays has been surveyed in Jeong, Kwun and Park [7] and Vrabie [10]. The approximate controllability for semilinear systems has been also studied in [7]. The existence of solutions for a class of nonlinear evolution equations with a nonlinear operator A were developed in many references [1, 2, 4, 6]. Ahmed and Xiang [2] gave some existence results for the initial value problem in case where the nonlinear term is not monotone, which improved Hirano's result [6].

Recently, as for the some considerations on the trajectory set of semilinear parabolic equations and that of its corresponding linear system(in case $h \equiv 0$), we

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refer to Naito[8] and [9, 12] and references therein. Carrasco and Lebia [5] discussed sufficient conditions for approximate controllability of parabolic equations with delay.

We will first establish a variation of constant formula for solutions of the given equation with a nonlinear operator A on $L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$ under a general condition of the Lipschitz continuity of the nonlinear operator, which is reasonable and widely used in case of the nonlinear system. The main research direction is to find conditions on the nonlinear term such that the regularity result of (1.1) is preserved under perturbation.

We briefly explain the contents of this paper. Section 2 presents the preliminaries and assumptions. In Section 3, we will obtain that almost all part of the regularity for quasi-autonomous differential equations can also applicable to (1.1) with nonlinear perturbations. The approach used here is similar to that developed in [3] on the general nonlinear evolution equations. Moreover in Section 4, we establish the approximate controllability of control system (1.1) with condition on compactness of the embedding $V \subset H$.

In order to prove the control problem, as in [7] we must assume the uniform boundedness of the nonlinear term, although we have some remark on this hypothesis. Since we apply the Leray-Schauder degree of mapping theorem in the proof of the main theorem, we need some compactness hypothesis. So we make the natural assumption that the embedding $V \subset H$ is compact. Then the embedding $L^2(0,T;V)\cap W^{1,2}(0,T;V^*) \subset L^2(0,T;H)$ is compact in view of Aubin's result [1], and we show that the mapping which maps a control u to the mild solution of (1.1) is a compact operator from $L^2(0,T;H)$ to itself.

2. Preliminaries and Assumptions

If H is identified with its dual space, we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norms on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. Thus, in terms of the intermediate theory we may assume that

$$(V,V^*)_{\frac{1}{2},2} = H,$$

where $(V,V^*)_{\frac{1}{2},2}$ denotes the real interpolation space between V and V^*. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1,v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$. For the sake of simplicity, we may consider

$$||u||_* \le |u| \le ||u||, \quad u \in V.$$

We note that a nonlinear operator A is said to be hemicontinuous on V if

$$w - \lim_{t\to 0} A(x+ty) = Ax$$

for every $x,\ y \in V$ where "$w - \lim$" indicates the weak convergence on V.

Let $A : V \longrightarrow V^*$ be given a monotone operator and hemicontinuous from V to V^* such that

$$\begin{cases} (Au - Av, u-v) \ge \omega_1||u-v||^2 - \omega_2|u-v|^2, \\ ||Au||_* \le \omega_3(||u||+1) \end{cases} \tag{2.1}$$

for every $u, v \in V$ where ω_2 is a real number and $\omega_1,\ \omega_3$ are some positive constants.

It is well known that A is maximal monotone and $R(A) = V^*$ where $R(A)$ denotes the range of A.

Let $\mathcal{L}$ and $\mathcal{B}$ be the Lebesgue σ-field on $[0, \infty)$ and the Borel σ-field on $[-h, 0]$, respectively. Let μ be a Borel measure on $[-h, 0]$ and $g : [0, \infty)\times[-h, 0]\times V \times V \to H$ be a nonlinear mapping satisfying the following:
(i) For any $x, y \in V$ the mapping $g(\cdot, \cdot, x, y)$ is strongly $\mathcal{L} \times \mathcal{B}$-measurable;
(ii) There exist positive constants L_0, L_1, L_2 such that

$$\text{(2.2)} \qquad \begin{cases} |g(t, s, x, y) - g(t, s, \hat{x}, \hat{y})| \le L_1||x - \hat{x}|| + L_2||y - \hat{y}||, \\ |g(t, s, 0, 0)| \le L_0 \end{cases}$$

for all $(t, s) \in [0, \infty) \times [-h, 0]$ and $x, \hat{x}, y, \hat{y} \in V$.

Remark 2.1. The above operator g is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [11].

For $x \in L^2(-h, T; V),\ T > 0$ we set

$$\text{(2.3)} \qquad G(t, x) = \int_{-h}^{0} g(t, s, x(t), x(t + s))\mu(ds).$$

Here, as in [11] we consider the Borel measurable corrections of $x(\cdot)$.

Let U be a Banach space and the controller operator B be a bounded linear operator from the Banach space $L^2(0, T; U)$ to $L^2(0, T; H)$.

Under assumptions mentioned above, we obtain the following result on the solvability of (1.1) by virtue of Theorem 3.1 of [7].

Proposition 2.2. *Let the assumptions (2.1) and (2.2) be satisfied. Then, for every $u \in L^2(0, T; U)$ and $(g^0, g^1) \in H \times L^2(0, T; V)$ the equation (1.1) has a unique solution*

$$x \in L^2(0, T; V) \cap C([0, T]; H) \cap W^{1,2}(0, T; V^*)$$

and there exists a constant $C_1 > 0$ depending on $T > 0$ such that

$$\text{(2.4)} \qquad ||x||_{L^2\cap C\cap W^{1,2}} \le C_1(1 + |g^0| + ||g^1||_{L^2(0,T;V)} + ||u||_{L^2(0,T;U)}).$$

3. Continuity of the solution mapping

Let us concern with the quasi-autonomous differential equation

$$\text{(3.1)} \qquad \begin{cases} \frac{dx(t)}{dt} + Ax(t) = k(t), \quad 0 < t \le T, \\ x(0) = g^0, \end{cases}$$

where A satisfies the hypotheses mentioned in Section 2. The following result is from Theorem 2.6 of Chapter III in [3].

Lemma 3.1. *Let $g^0 \in H$ and $k \in L^2(0, T; V^*)$. Then there exists a unique solution x of (3.1) belonging to*

$$C([0, T]; H) \cap L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$$

and satisfying

$$||x||_{C([0,T];H)\cap L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_2(|g^0| + ||k||_{L^2(0,T;V^*)}), \tag{3.2}$$

where C_2 *is a constant.*

Acting on both sides of (3.1) by $x(t)$, we have

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 + \omega_1||x(t)||^2 \le \omega_2|x(t)|^2 + (k(t), x(t)).$$

As is seen Theorem 2.6 in [3], integrating from 0 to t we can determine the constant $C_2 > 0$ in Lemma 3.1.

The following Lemma is from H. Brézis [[4]; Lemma A.5].

Lemma 3.2. *Let* $m \in L^1(0,T;\mathbb{R})$ *satisfying* $m(t) \ge 0$ *for all* $t \in (0,T)$ *and* $a \ge 0$ *be a constant. Let* b *be a continuous function on* $[0,T]$ *satisfying the following inequality:*

$$\frac{1}{2}b^2(t) \le \frac{1}{2}a^2 + \int_0^t m(s)b(s)ds, \quad t \in [0,T].$$

Then,

$$|b(t)| \le a + \int_0^t m(s)ds, \quad t \in [0,T].$$

Proof. Let

$$\beta_\epsilon(t) = \frac{1}{2}(a+\epsilon)^2 + \int_0^t m(s)b(s)ds, \quad \epsilon > 0.$$

Then

$$\frac{d\beta_\epsilon(t)}{dt} = m(t)b(t), \quad \tau \in (0,T),$$

and

$$\frac{1}{2}b^2(t) \le \beta_0(t) \le \beta_\epsilon(t), \quad t \in [0,T]. \tag{3.3}$$

Hence, we have

$$\frac{d\beta_\epsilon(t)}{dt} \le m(t)\sqrt{2}\sqrt{\beta_\epsilon(t)}.$$

Since $t \to \beta_\epsilon(t)$ is absolutely continuous and

$$\frac{d}{dt}\sqrt{\beta_\epsilon(t)} = \frac{1}{2\sqrt{\beta_\epsilon(t)}}\frac{d\beta_\epsilon(t)}{dt}$$

for all $t \in (0,T)$, it holds

$$\frac{d}{dt}\sqrt{\beta_\epsilon(t)} \le \frac{1}{\sqrt{2}}m(t),$$

that is,

$$\sqrt{\beta_\epsilon(t)} \le \sqrt{\beta_\epsilon(0)} + \frac{1}{\sqrt{2}}\int_0^t m(s)ds, \quad t \in (0,T).$$

Therefore, combining this with (3.3), we conclude that

$$|b(t)| \le \sqrt{2}\sqrt{\beta_\epsilon(t)} \le \sqrt{2}\sqrt{\beta_\epsilon(0)} + \int_0^t m(s)ds$$

$$= a + \epsilon + \int_0^t m(s)ds, \quad t \in [0, T]$$

for arbitrary $\epsilon > 0$. □

Lemma 3.3. *Let $x \in L^2(-h, T; V)$, $T > 0$. Then the nonlinear term $G(\cdot, x)$ defined by (2.3) belongs to $L^2(0, T; H)$ and*

$$\begin{aligned}(3.4)\qquad ||G(\cdot, x)||_{L^2(0,T;H)} &\le \mu([-h, 0])\{L_0\sqrt{T} \\ &+ (L_1 + L_2)||x||_{L^2(0,T;V)} + L_2||g^1||_{L^2(-h,0;V)}\}.\end{aligned}$$

Moreover, if $x_1,\ x_2 \in L^2(-h, T; V)$, then

$$\begin{aligned}(3.5)\qquad ||G(\cdot, x_1) - G(\cdot, x_2)||_{L^2(0,T;H)} &\le \mu([-h, 0]) \\ \times \{(L_1 + L_2)||x_1 - x_2||_{L^2(0,T;V)} &+ L_2||x_1 - x_2||_{L^2(-h,0;V)}\}.\end{aligned}$$

Proof. From (2,2) it is easily seen that

$$\begin{aligned}||G(\cdot, x)||_{L^2(0,T;H)} &\le \mu([-h, 0])\{L_0\sqrt{T} \\ &+ L_1||x||_{L^2(0,T;V)} + L_2||x||_{L^2(-h,T;V)}\} \\ &\le \mu([-h, 0])\{L_0\sqrt{T} + (L_1 + L_2)||x||_{L^2(0,T;V)} + L_2||x||_{L^2(-h,0;V)}\}.\end{aligned}$$

The proof of (3.5) is similar. □

Lemma 3.4. *Let $(g_i^0, g_i^1, k_i) \in H \times L^2(0, T; V) \times L^2(0, T; V^*)$ and x_i for $i = 1,\ 2$ be the solutions of the following equation:*

$$\begin{cases} \frac{dx_i(t)}{dt} + Ax_i(t) = G(t, x_i) + k_i(t), \quad 0 < t \le T, \\ x_i(0) = g_i^0, \quad x_i(s) = g_i^1(s) \quad s \in [-h, 0). \end{cases}$$

Then, we have that for $0 < c \le \omega_1$

$$\begin{aligned}(3.6)\qquad \frac{1}{2}|x_1(t) - x_2(t)|^2 &+ (\omega_1 - c)\int_0^t ||x_1(s) - x_2(s)||^2 ds \\ &\le \frac{e^{2\omega_2 t}}{2}\{|g_1^0 - g_2^0|^2 + \frac{1}{2c}\int_0^t ||k_1(s) - k_2(s)||_*^2 ds\} \\ &+ \int_0^t e^{2\omega_2(t-s)}|G(s, x_1) - G(s, x_2)|\,|x_1(s) - x_2(s)|ds,\end{aligned}$$

and

$$\begin{aligned}(3.7)\qquad e^{-\omega_2 t}|x_1(t) - x_2(t)| &\le |g_1^0 - g_2^0| + \frac{1}{2c}\int_0^t ||k_1(s) - k_2(s)||_* ds \\ &+ \int_0^t e^{-\omega_2 s}|G(s, x_1) - G(t, x_2)|ds.\end{aligned}$$

Proof. For $i = 1$, 2, we consider the following equation:

$$(3.8)\qquad \begin{cases} \frac{dx_i(t)}{dt} + Ax_i(t) = G(t, x_i) + k_i(t), \quad 0 < t \le T, \\ x_i(0) = g_i^0, \quad x_i(s) = g_i^1(s) \quad s \in [-h, 0). \end{cases}$$

Multiplying on (3.8) by $x_1(t) - x_2(t)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + \omega_1||x_1(t) - x_2(t)||^2\\ &\leq \omega_2|x_1(t) - x_2(t)|^2 + |G(t, x_1) - G(t, x_2)|\,|x_1(t) - x_2(t)|\\ &+ ||x_1(t) - x_2(t)||\,||k_1(t) - k_2(t)||_*.\end{aligned}$$

Put

$$H(t) = |G(t, x_1) - G(t, x_2)|\,|x_1(t) - x_2(t)|.$$

Noting that for $0 < c < \omega_1$

$$||x_1(t) - x_2(t)||\,||k_1(t) - k_2(t)||_* \leq c||x_1(t) - x_2(t)||^2 + \frac{1}{4c}||k_1(t) - k_2(t)||_*^2,$$

we have

$$\begin{aligned}&\frac{1}{2}|x_1(t) - x_2(t)|^2 + (\omega_1 - c)\int_0^t ||x_1(s) - x_2(s)||^2 ds\\ &\leq \frac{1}{2}|g_1^0 - g_2^0|^2 + \omega_2\int_0^t |x_1(s) - x_2(s)|^2 ds\\ &+ \frac{1}{4c}\int_0^t ||k_1(s) - k_2(s)||_*^2 ds + \int_0^t H(s)ds.\end{aligned} \tag{3.9}$$

From (3.9) it follows that

$$\begin{aligned}&\frac{d}{dt}\{e^{-2\omega_2 t}\int_0^t |x_1(s) - x_2(s)|^2 ds\}\\ &= 2e^{-2\omega_2 t}\{\frac{1}{2}|x_1(t) - x_2(t)|^2 - \omega_2\int_0^t |x_1(s) - x_2(s)|^2 ds\}\\ &\leq 2e^{-2\omega_2 t}\{\frac{1}{2}|g_1^0 - g_2^0|^2 + \frac{1}{4c}\int_0^t ||k_1(s) - k_2(s)||_*^2 ds + \int_0^t H(s)ds\}.\end{aligned} \tag{3.10}$$

Integrating (3.10) over $(0, t)$ we have

$$\begin{aligned}&e^{-2\omega_2 t}\int_0^t |x_1(s) - x_2(s)|^2 ds\\ &\leq \frac{1 - e^{-2\omega_2 t}}{2\omega_2}\{|g_1^0 - g_2^0|^2 + \frac{1}{2c}\int_0^t ||k_1(s) - k_2(s)||_*^2 ds\}\\ &\quad + 2\int_0^t \int_s^t e^{-2\omega_2 \tau} d\tau H(s)ds\\ &= \frac{1 - e^{-2\omega_2 t}}{2\omega_2}\{|g_1^0 - g_2^0|^2 + \frac{1}{2c}\int_0^t ||k_1(s) - k_2(s)||_*^2 ds\}\\ &\quad + 2\int_0^t \frac{e^{-2\omega_2 s} - e^{-2\omega_2 t}}{2\omega_2} H(s)ds\\ &= \frac{1 - e^{-2\omega_2 t}}{2\omega_2}\{|g_1^0 - g_2^0|^2 + \frac{1}{2c}\int_0^t ||k_1(s) - k_2(s)||_*^2 ds\}\\ &\quad + \frac{1}{\omega_2}\int_0^t (e^{-2\omega_2 s} - e^{-2\omega_2 t})H(s)ds,\end{aligned}$$

thus, we get

$$\omega_2 \int_0^t |x_1(s) - x_2(s)|^2 ds \le \frac{e^{2\omega_2 t} - 1}{2} \{|g_1^0 - g_2^0|^2 + \frac{1}{2c} \int_0^t ||k_1(s) - k_2(s)||_*^2 ds\} + \int_0^t (e^{2\omega_2(t-s)} - 1) H(s) ds.$$

Combining which with (3.9) it holds that

$$\begin{aligned} &\frac{1}{2}|x_1(t) - x_2(t)|^2 + \omega_1 \int_0^t ||x_1(s) - x_2(s)||^2 ds \\ &\le \frac{e^{2\omega_2 t}}{2} \{|g_1^0 - g_2^0|^2 + \frac{1}{2c} \int_0^t ||k_1(s) - k_2(s)||_*^2 ds\} + \int_0^t e^{2\omega_2(t-s)} H(s) ds \\ &= \frac{e^{2\omega_2 t}}{2} \{|g_1^0 - g_2^0|^2 + \frac{1}{2c} \int_0^t ||k_1(s) - k_2(s)||_*^2 ds\} \\ &+ \int_0^t e^{2\omega_2(t-s)} |G(s, x_1) - G(s, x_2)|\, |x_1(s) - x_2(s)| ds, \end{aligned}$$

which implies

$$\begin{aligned} &\frac{1}{2}(e^{-\omega_2 t}|x_1(t) - x_2(t)|)^2 + \omega_1 e^{-2\omega_2 t} \int_0^t ||x_1(s) - x_2(s)||^2 ds \\ &\le \frac{1}{2}\{|g_1^0 - g_2^0|^2 + \frac{1}{2c} \int_0^t ||k_1(s) - k_2(s)||_*^2 ds\} \\ &+ \int_0^t e^{-\omega_2 s} |G(s, x_1) - G(s, x_2)| e^{-\omega_2 s} |x_1(s) - x_2(s)| ds. \end{aligned}$$

Hence, we obtain (3.7) by using Lemma 3.2. □

Theorem 3.5. *Let the assumptions (2.1) and (2.2) be satisfied and* $(g^0, g^1, k) \in H \times L^2(0, T; V) \times L^2(0, T; V^*)$, *Then the solution* x *of the equation*

$$\text{(3.11)} \quad \begin{cases} x'(t) + Ax(t) = \int_{-h}^0 g(t, s, x(t), x(t+s))\mu(ds) + k(t), & 0 < t \le T, \\ x(0) = g^0, \quad x(s) = g^1(s) \quad s \in [-h, 0) \end{cases}$$

belongs to $L^2(0, T; V) \cap C([0, T]; H)$ *and the mapping*

$$H \times L^2(0, T; V) \times L^2(0, T; V^*) \ni (g^0, g^1, k) \mapsto x \in L^2(0, T; V) \cap C([0, T]; H)$$

is continuous.

Proof. From (3.6) and (3.7) in Lemma 3.4, it follows that

$$\text{(3.12)} \quad \begin{aligned} &\frac{1}{2}|x_1(t) - x_2(t)|^2 + \omega_1 \int_0^t ||x_1(s) - x_2(s)||^2 ds \\ &\le \frac{1}{2} e^{2\omega_2 t} \{|g_1^0 - g_2^0|^2 + \frac{1}{2c} \int_0^t ||k_1(s) - k_2(s)||_*^2\} \\ &+ \int_0^t e^{2\omega_2(t-s)} |G(s, x_1) - G(s, x_2)| e^{\omega_2 s} \{|g_1^0 - g_2^0| \end{aligned}$$

$$+\frac{1}{2c}\int_0^s ||k_1(\tau)-k_2(\tau)||_* d\tau\}ds$$
$$+\int_0^t e^{2\omega_2(t-s)}|G(s,x_1)-G(s,x_2)|\int_0^s e^{\omega_2(s-\tau)}|G(\tau,x_1)-G(\tau,x_2)|d\tau ds.$$

The third term of the right of (3.12) is estimated as

$$\frac{(e^{2\omega_2 t}-1)}{4\omega_2}\int_0^t |G(s,x_1)-G(s,x_2)|^2 ds. \tag{3.13}$$

We note that from (3.5) of Lemma 3.3 for $0<t<T$

$$\begin{aligned} ||G(\cdot,x_1)-G(\cdot,x_2)||_{L^2(0,t;H)} &\le \mu([-h,0]) \\ &\times\{(L_1+L_2)||x_1-x_2||_{L^2(0,t;V)}+L_2||g_1^1-g_2^2||_{L^2(-h,0;V)}\}. \end{aligned} \tag{3.14}$$

Let $T_1<T$ be such that

$$N:=\omega_1-(4\omega_2)^{-1}\mu([-h,0])^2(L_1+L_2)^2(e^{2\omega_2 T_1}-1)>0.$$

Then we can choose a constant $c_1>0$ such that

$$N-c_1\mu([-h,0])^2(L_1+L_2)^2e^{2\omega_2 T_1}>0$$

and

$$\begin{aligned} &|G(s,x_1)-G(s,x_2)|\{|g_1^0-g_2^0|+(2c)^{-1/2}||k_1-k_2||_{L^2(0,t;V^*)}\} \\ &\le \frac{1}{4c_1}\{|g_1^0-g_2^0|+(2c)^{-1/2}||k_1-k_2||_{L^2(0,t;V^*)}\}^2+c_1|G(s,x_1)-G(s,x_2)|^2. \end{aligned}$$

Thus, the second term of the right of (3.12) is estimated as

$$\begin{aligned} &\text{const.}\{|g_1^0-g_2^0|+(2c)^{-1/2}||k_1-k_2||_{L^2(0,T_1;V^*)}\}^2 \\ &+c_1e^{2\omega_2 T_1}\int_0^{T_1}|G(s,x_1)-G(s,x_2)|^2 ds. \end{aligned} \tag{3.15}$$

Hence, from (3.13), (3.14) and (3.15) it follows that there exists a constant $C>0$ such that

$$\begin{aligned} &||x_1-x_2||_{L^2(0,T;V)\cap C([0,T];H)} \\ &\le C(|g_1^0-g_2^0|+||g_1^1-g_2^2||_{L^2(-h,0;V)}+||k_1-k_2||_{L^2(0,T_1;V^*)}). \end{aligned} \tag{3.16}$$

Suppose $(g_n^0,g_n^1,k_n)\to(g^0,g^1,k)$ in $\in H\times L^2(0,T_1;V)\times L^2(0,T_1;V^*)$, and let x_n and x be the solutions (3.11) with (g_n^0,g_n^1,k_n) and (g^0,g^1,k), respectively. Then, by virtue of (3.16), we see that $x_n\to x$ in $L^2(0,T_1,V)\cap C([0,T_1];H)$. This implies that $x_n(T_1)\to x(T_1)$ in H. Therefore the same argument shows that $x_n\to x$ in

$$L^2(T_1,\min\{2T_1,T\};V)\cap C([T_1,\min\{2T_1,T\}];H).$$

Repeating this process, we conclude that $x_n\to x$ in $L^2(0,T;V)\cap C([0,T];H)$. □

4. Approximate controllability

We assume that the embedding $V \subset H$ is compact and A is a strongly continuous operator from $V$to V^* satisfying (2.1). In this section we are interested in the approximate controllability for the nonlinear functional control system (1.1) on H.

For $h \in L^2(0,T;H)$ and let x_h be the solution of the following equation with $B = I$:

$$\begin{cases} \frac{dx(t)}{dt} + Ax(t) = G(t,x) + h(t), \quad 0 < t, \\ x(0) = 0, \quad x(s) = 0 \quad -h \le s \le 0. \end{cases} \tag{4.1}$$

Theorem 4.1. *Let us define the solution mapping S from $L^2(0,T;V^*)$ to $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ by*

$$(Sh)(t) = x_h(t), \quad h \in L^2(0,T;V^*). \tag{4.2}$$

Then, the mapping $h \mapsto Sh = x_h$ is compact from $L^2(0,T;V^*)$ to $L^2(0,T;H)$.

Proof. With the aid of Lemma 3.1 and Proposition 2.2

$$\begin{aligned} &||Sh||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} = ||x_h|| \le C_2||G(\cdot,x_h) + h||_{L^2(0,T;V^*)} \\ &\le C_2\mu([-h,0])\{L_0\sqrt{T} + (L_1+L_2)||x||_{L^2(0,T;V)}\} + C_2||h||_{L^2(0,T;V^*)} \\ &\le C_2\mu([-h,0])[L_0\sqrt{T} + (L_1+L_2)\{C_1(1+||h||_{L^2(0,T;V^*)})\}] \\ &\ + C_2||h||_{L^2(0,T;V^*)}. \end{aligned}$$

Hence, if h is bounded in $L^2(0,T;V^*)$, then so is x_h in $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$. Since V is compactly embedded in H by assumption, the embedding $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset L^2(0,T;H)$ is compact in view of Theorem 2 of J. P. Aubin [1]. Hence, the mapping $h \mapsto Sh = x_h$ is compact from $L^2(0,T;V^*)$ to $L^2(0,T;H)$. □

Let $\mathcal{A}$ and $\mathcal{G}$ be the Nemitsky operators corresponding to the maps A and G, which are defined by $\mathcal{A}(x)(\cdot) = Ax(\cdot)$ and $\mathcal{G}(h)(\cdot) = G(\cdot, x_h)$, respectively. Then since the solution x belongs to $L^2(-h,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$, it is represented by

$$x_h(t) = \int_0^t ((I + \mathcal{G} - \mathcal{A}S)h)(s)ds, \tag{4.3}$$

From Theorem 4.1, it follows that $\mathcal{G}$ is a compact mapping from $L^2(0,T;V^*)$ to $L^2(0,T;H)$ and so is $\mathcal{A}S$ from $L^2(0,T;V^*)$ to itself. The solution of (1.1) is denoted by $x(T;g,u)$ associated with the nonlinear term g and a control u at the time T.

Definition 4.2. The system (1.1) is said to be approximately controllable at time T if the reachable set $\{x(T;g,u) : u \in L^2(0,T;U)\}$ is dense in H, that is, $Cl\{x(T;g,u) : u \in L^2(0,T;U)\} = H$.

We assume that

(G) g is uniformly bounded: there exists a constant M_g such that

$$|g(t,s,x,y)| \le M_g,$$

for all $x,\ y \in V$.

Then it holds that

$$|G(t,x)| \le M_g \mu([-h,0]),$$

and hence

$$||\mathcal{G}(h)||_{L^2(0,T;H)} = ||G(\cdot, x_h)||_{L^2(0,T;H)} \le M_g \mu([-h,0])\sqrt{T}. \tag{4.4}$$

Theorem 4.3. *Let the assumption (G) hold and let us assume that*

$$\mathrm{Cl}\{y : y(t) = Bu(t),\ \ a.e.\ u \in L^2(0,T;U)\} = L^2(0,T;H). \tag{4.5}$$

Then

$$Cl\{(\mathcal{G} + I - \mathcal{A}S)h : h \in L^2(0,T;V^*)\} = L^2(0,T;V^*). \tag{4.6}$$

Thus, the system (1.1) is approximately controllable at time T.

Proof. Let us fix $T_0 > 0$ such that

$$\frac{\omega_3}{4\omega_1\omega_2}(e^{2\omega_2 T_0} - 1) > 0. \tag{4.7}$$

Let $z \in L^2(0,T_0;V^*)$ and r be a constant such that

$$z \in U_r = \{x \in L^2(0,T_0;V^*) : ||x||_{L^2(0,T_0;V^*)} < r\}.$$

Putting

$$N^2 := \frac{1}{4\omega_1\omega_2}(e^{2\omega_2 T_0} - 1),$$

Take a constant $d > 0$ such that

$$\{r + \omega_3 N M_g \mu([-h,0])\sqrt{T_0} + \omega_3\}(1 - \omega_3 N)^{-1} < d. \tag{4.8}$$

Noting that $L^2(0,T_0;H)$ is a dense subspace of $L^2(0,T_0;V^*)$, for every $h \in L^2(0,T_0;V^*)$, consider the following equation:

$$\begin{cases} x'(t) + Ax(t) = G(t,x) + h(t), \quad 0 < t, \\ x(0) = 0, \quad x(s) = 0 \quad -h \le s \le 0. \end{cases} \tag{4.9}$$

Taking scalar product on both sides of (4.9) by $x(t)$

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 + \omega_1||x(t)||^2 \le \omega_2|x(t)|^2 + |G(t,x) + h(t)|\,|x(t)|.$$

Integrating on $[0,t]$, by the similar process of the proof (3.6) and (3.7) we get

$$\frac{1}{2}|x(t)|^2 + \omega_1\int_0^t ||x(s)||^2 ds \le \int_0^t e^{2\omega_2(t-s)}|G(s,x) + h(s)|\,|x(s)|ds, \tag{4.10}$$

and

$$e^{-\omega_2 t}|x_1(t)| \le \int_0^t e^{-\omega_2 s}|G(s,x) + h(s)|ds. \tag{4.11}$$

Putting

$$H_1(s) = |G(s,x) + h(s)|$$

and combining (4.10) with (4.11) we obtain

$$
\begin{aligned}
(4.12)\qquad & \frac{1}{2}|x(t)|^2 + \omega_1 \int_0^t ||x(s)||^2 ds \leq \\
& \int_0^t e^{2\omega_2(t-s)} H_1(s) \int_0^s e^{\omega_2(s-\tau)} H_1(\tau) d\tau ds \\
& = e^{2\omega_2 t} \int_0^t e^{-\omega_2 s} H_1(s) \int_0^s e^{-\omega_2 \tau} H_1(\tau) d\tau ds \\
& = e^{2\omega_2 t} \int_0^t \frac{1}{2} \frac{d}{ds} \{ \int_0^s e^{-\omega_2 \tau} H_1(\tau) d\tau \}^2 ds = \frac{1}{2} e^{2\omega_2 t} \{ \int_0^t e^{-\omega_2 \tau} H_1(\tau) d\tau \}^2 \\
& \leq \frac{1}{2} e^{2\omega_2 t} \int_0^t e^{-2\omega_2 \tau} d\tau \int_0^t H_1(\tau)^2 d\tau = \frac{1}{4\omega_2} (e^{2\omega_2 t} - 1) \int_0^t H_1(s)^2 ds.
\end{aligned}
$$

Noting that

$$||G(\cdot, x)||_{L^2(0,T_0;H)} \leq M_g \mu([-h, 0]) \sqrt{T}_0,$$

it follows from (4.12) that

$$
\begin{aligned}
(4.13)\qquad & ||Sh||_{L^2(0,T_0;V)} = ||x||_{L^2(0,T_0;V)} \\
& \leq N(||h||_{L^2(0,T_0;V^*)} + M_g \mu([-h, 0]) \sqrt{T}_0).
\end{aligned}
$$

In order to prove (4.6) we will use the topological degree theory for the equation

$$(4.14)\qquad z = \lambda(\mathcal{G} - \mathcal{A}S)h + h, \quad 0 \leq \lambda \leq 1.$$

Let h be the solution of (4.14). Then since $z \in U_r$ and from (4.13), (2.1) we have

$$
\begin{aligned}
||h||_{L^2(0,T;V^*)} & \leq ||z|| + ||\mathcal{A}Sh|| + ||\mathcal{G}h|| \\
& \leq r + \omega_3(||Sh|| + 1) + M_g \mu([-h, 0]) \sqrt{T}_0 \\
& \leq r + \omega_3 N \{ ||h||_{L^2(0,T_0;V^*)} + M_g \mu([-h, 0]) \sqrt{T_0} \} + \omega_3 \\
& \quad + M_g \mu([-h, 0]) \sqrt{T_0},
\end{aligned}
$$

and hence,

$$||h|| \leq \{ r + \omega_3 N M_g \mu([-h, 0]) \sqrt{T}_0 + \omega_3 \} (1 - \omega_3 N)^{-1} < d.$$

It follows that $h \notin \partial U_d$ where ∂U_d stands for the boundary of U_d. Thus, from the homotopy property of topological degree theory, there exists $h \in U_d$ such that the equation

$$z = (\mathcal{G} + I - \mathcal{G}S)h$$

holds. By virtue of the assumption (4.5), there exists a sequence $\{u_n\}$ in $L^2(0, T_0; U)$ such that $Bu_n \mapsto h$ in $L^2(0, T_0; V^*)$. Then by Theorem 3.5 we have that $x(\cdot; Bu_n) \mapsto x_h$ in $L^2(0, T_0; V) \cap C([0, T_0]; H)$. Let $y \in H$. We can choose $g \in W^{1,2}(0, T_0; V^*)$ such that $g(0) = x_0$ and $g(T_0) = y$ and from the equation (4.14) there is $h \in L^2(0, T_0; V^*)$ such that $g' = (\mathcal{G} + I - \mathcal{A}S)h$. By the assumption (4.5) and Lemma 3.4 there exists $u \in L^2(0, T_0; U)$ such that

$$||x_h - x_{Bu}||_{L^2(0,T_0;V) \cap C([0,T_0];H)} \leq C_3 ||h - Bu||_{L^2(0,T_0;V^*)}$$

for some constant $C_3 > 0$. Thus, we have

$$|y - x_h(T)| = |\int_0^{T_0} \{((\mathcal{G} + I - \mathcal{A}S)h)(s) - ((\mathcal{G} + I - \mathcal{A}S)Bu)(s)\}ds|$$
$$\leq ||x_h - x_{Bu}||_{L^2(0,T_0;V)\cap C([0,T];H)} \leq C_1||h - Bu||_{L^2(0,T;V^*)}.$$

Therefore, the system (1.1) is approximately controllable at time T_0. Since the condition (4.7) is independent of initial values, we can solve the equation in $[T_0, 2T_0]$ with the initial value $x(T_0)$. By repeating this process, the approximate controllability for (1.1) can be extended the interval $[0, nT_0]$ for natural number n, i.e., for the initial $x(nT_0)$ in the interval $[nT_0, (n+1)T_0]$. □

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