Journal of Nonlinear and Convex Analysis
Volume 12, Number 3, 2011, 577–597

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

THE MODIFIED MANN ITERATION METHODS FOR AN ASYMPTOTICALLY STRICT PSEUDO-CONTRACTIVE FAMILY IN HILBERT SPACES

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Abstract. In this paper, we first propose a modified Mann's iteration method for an asymptotically strict pseudo-contractive family of self-mappings defined on a closed convex subset in a Hilbert space. Next we study the weak and strong convergence of modified Mann's iteration method for such an asymptotically strict pseudo-contractive family. And some applications for the parallel algorithm and the cyclic algorithm relating to our main results are added, which extend and improve the corresponding ones due to Lopez Acedo and Xu [Nonlinear Analysis TMA, 67 (2007), 2258–2271] for a finite family of strict pseudo-contractions.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a mapping. We use $Fix(T)$ to denote the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$. Recall that $T : C \to C$ is said to be a *strict pseudo-contraction* [1] if there exists a constant $0 \le \kappa < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \tag{1.1}$$

for all $x, y \in C$. For such a case, T is said to be a κ-strict pseudo-contraction. A 0-strict pseudo-contraction T is nonexpansive; that is, T is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$.

Recall also that $T : C \to C$ is said to be an *asymptotically strict pseudo-contraction* [18] if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ of non-negative real numbers with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\|T^n x - T^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 + \kappa\|(I - T^n)x - (I - T^n)y\|^2 \tag{1.2}$$

for all $x, y \in C$ and $n \ge 1$; see also [7] or [16]. When (1.2) holds, T is afterward said to be an asymptotically κ-strict pseudo-contraction (with respect to the sequence $\{\gamma_n\}$ in case a distinction is needed). Note that if $\kappa = 0$, then T is asymptotically nonexpansive [4], that is,

$$\|T^n x - T^n y\| \le k_n\|x - y\|$$

2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 65J15.
Key words and phrases. Asymptotically strict pseudo-contractive family, modified Mann's iteration method, weak (strong) convergence, parallel algorithm, cyclic algorithm.
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for all $x, y \in C$ and $n \geq 1$, where $k_n := \sqrt{1+\gamma_n} \to 1$. It is also known [17] that the class of κ-strict pseudo-contractions and the class of asymptotically κ-strict pseudo-contractions are independent.

Iterative methods are often used to solve the fixed point equation $Tx = x$. The most well-known method is perhaps the Picard successive iteration method when T is a contraction. Picard's method generates a sequence $\{x_n\}$ successively as $x_n = Tx_{n-1}$ for $n \geq 2$ with $x_1 := x$ arbitrary, and this sequence converges in norm to the unique fixed point of T. However, if T is not a contraction (for instance, if T is nonexpansive), then Picard's successive iteration fails, in general, to converge. Instead, Mann's iteration method [11] prevails, which, an averaged process in nature, generates a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.3}$$

where the initial guess $x_0 \in C$ is arbitrarily chosen and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ lies in the interval $[0,1]$.

The Mann's algorithm for nonexpansive mappings has been extensively investigated; see [1, 5, 9, 19, 25, 26, 27, 28] and the references therein. One of the well known results is proven by Reich [19] for a nonexpansive mapping $T : C \to C$, which asserts the weak convergence of the sequence $\{x_n\}$ generated by (1.3) in a uniformly convex Banach space with a Frechet differentiable norm under the control condition $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. However iterative methods for strict pseudo-contractions are far less developed though Browder and Petryshyn [1] initiated their work in 1967. Recently, Marino and Xu [12] developed and extended Reich's result to strict pseudo-contractions in the Hilbert space setting. More precisely, they proved the weak convergence of the Mann's iteration process (1.3) for a κ-strict pseudo-contraction $T : C \to C$. Subsequently, an analogue of this result was investigated for the class of asymptotically κ-strict pseudo-contractions by Kim and Xu [7] with the following modified Mann's algorithm:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \quad n \geq 0, \tag{1.4}$$

where the initial guess $x_0 \in C$ is arbitrarily chosen and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ lies in the interval $[0,1]$; see also [6, 8, 20, 21, 24] and the references therein for convergence of the modified Mann iteration process (1.4) for asymptotically nonexpansive mappings.

It is known that the Mann iteration method (1.3) is in general not strongly convergent [3] for either nonexpansive mappings or strict pseudo-contractions. In 2003, a method (called hybrid method) to modify the Mann iteration method (1.3) so that strong convergence is guaranteed has been proposed by Nakajo and Takahashi [15] for a single nonexpansive mapping T with $Fix(T) \neq \emptyset$ in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0, \end{cases} \tag{1.5}$$

where P_K denotes the metric projection from H onto a nonempty closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{Fix(T)}x_0$. This result has been extended to the class of asymptotically nonexpansive mappings by Kim and Xu [6], and subsequently to the one of κ-strict pseudo-contractions by Marino and Xu [13] as follows.

Theorem MX (see Theorem 4.1 of [13]) *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a κ-strict pseudo-contraction for some $0 \le \kappa < 1$ and assume that the fixed point set $Fix(T)$ of T is nonempty. Let $\{x_n\}$ be the sequence generated by the following hybrid algorithm:*

$$(1.6)\quad \begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \ge 0. \end{cases}$$

Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\alpha_n < 1$ for all $n \ge 0$. Then $\{x_n\}$ converges strongly to $P_{Fix(T)}x_0$.

Quite recently, Kim and Xu [7] gave an analogue of Theorem MX for the class of asymptotically κ-strict pseudo-contractions.

Theorem KX (see Theorem 4.1 of [7]) *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically κ-strict pseudo-contraction for some $0 \le \kappa < 1$. Assume that the fixed point set $Fix(T)$ of T is nonempty and bounded. Let $\{x_n\}$ be the sequence generated by the following hybrid algorithm:*

$$(1.7)\quad \begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)(\kappa - \alpha_n) \\ \qquad\qquad \|x_n - T^n x_n\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \ge 0 \end{cases}$$

where

$$\theta_n = \triangle_n^2 (1-\alpha_n)\gamma_n \to 0 \text{ as } n \to \infty, \quad \triangle_n = \sup\{\|x_n - z\|^2 : z \in Fix(T)\} < \infty.$$

Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{Fix(T)}x_0$.

From now on, motivated by definition of (1.2), we say that a family $\Im = \{S_n : C \to C,\, n \ge 0\}$ of self-mappings of C is *asymptotically κ-strict pseudo-contractive* on C if there exist a constant $\kappa \in [0,1)$ and a sequence $\{\gamma_n\}_{n=0}^{\infty}$ of nonnegative real numbers with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$(1.8)\quad \|S_n x - S_n y\|^2 \le (1+\gamma_n)\|x-y\|^2 + \kappa\|(I-S_n)x - (I-S_n)y\|^2$$

for all $x, y \in C$ and all integers $n \ge 0$. When (1.8) holds, $\Im$ is often said to be an asymptotically κ-strict pseudo-contractive family. Especially, when $\kappa = 0$ in (1.8), the family $\Im$ is said to be *asymptotically nonexpansive.* Notice also that the asymptotically strict pseudo-contractive family $\Im = \{S_n : C \to C,\, n \ge 0\}$ obviously includes the class of strict pseudo-contractions and the class of asymptotically strict pseudo-contractions, simply by taking $S_n := T$ (or T^n), $n \geq 0$, for a strict pseudo-contraction (or asymptotically strict pseudo-contraction) $T : C \to C$, respectively.

In this paper we first propose either the following modified Mann iteration method for an asymptotically κ-strict pseudo-contractive family $\Im = \{S_n : C \to C, n \geq 0\}$:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \quad n \geq 0, \tag{1.9}$$

where the initial guess $x_0 \in C$ is arbitrarily chosen, or the following hybrid iteration method

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1 - \alpha_n)[\theta_n \\ \qquad\qquad + (\kappa - \alpha_n)\|x_n - S_n x_n\|^2]\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0, \end{cases} \tag{1.10}$$

where

$$\theta_n = \gamma_n \cdot \sup\{\|x_n - z\|^2 : z \in F := \cap_{n=0}^{\infty} Fix(S_n)\},$$

and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ lies in the interval $[0, 1]$.

Motivated and inspired by the research works in [12], [7] and [10], we next study the weak convergence of the above algorithm (1.9) and strong convergence of the hybrid algorithm (1.10), respectively, for such an asymptotically strict pseudo-contractive family $\Im = \{S_n : C \to C, n \geq 0\}$. Also, some applications for the parallel algorithm (4.3) and the cyclic algorithm (4.16) relating to our main results are added, which extend and improve the corresponding ones due to Lopez Acedo and Xu [10] for a finite family $\{T_i\}_{i=0}^{N-1}$ of κ_i-strict pseudo-contractions.

2. Preliminaries

Let H be a real Hilbert space with the duality product $\langle \cdot, \cdot \rangle$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We also denote the weak ω-limit set of $\{x_n\}$ by

$$\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}.$$

We now need some facts and tools in a real Hilbert space H which are listed as lemmas below (see [14] for necessary proofs of Lemmas 2.2 and 2.5).

Lemma 2.1. *Let H be a real Hilbert space. There hold the following identities (which will be used in the various places in the proofs of the results of this paper).*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \quad x, y \in H.$

(ii) *For all $\lambda_i \in [0, 1]$ with $\sum_{i=0}^{N-1} \lambda_i = 1$, and $x, y \in H$, the following equality holds:*

$$\|\sum_{i=0}^{N-1} \lambda_i x_i\|^2 = \sum_{i=0}^{N-1} \lambda_i \|x_i\|^2 - \sum_{i<j}^{N-1} \lambda_i \lambda_j \|x_i - x_j\|^2. \tag{2.1}$$

In particular, for $n = 2$ we have

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \quad t \in [0, 1]. \tag{2.2}$$

Lemma 2.2 ([14]). *Let H be a real Hilbert space. Given a nonempty closed convex subset $C \subset H$ and points $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set*

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Recall that given a nonempty closed convex subset K of a real Hilbert space H, the nearest point projection P_K from H onto K assigns to each $x \in H$ its nearest point denoted $P_K x$ in K from x to K; that is, $P_K x$ is the unique point in K with the property

$$\|x - P_K x\| \leq \|x - y\|, \quad y \in K.$$

Lemma 2.3. *Let K be a nonempty closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0, \quad y \in K.$$

Lemma 2.4 ([10]). *Let K be a nonempty closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume*

(i) *The weak ω-limit set $\omega_w(x_n) \subset K$.*
(ii) *For each $z \in K$, $\lim_{n\to\infty} \|x_n - z\|$ exists.*

Then $\{x_n\}$ is weakly convergent to a point in K.

Lemma 2.5 ([14]). *Let K be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $x_0 \in H$. Let $q = P_K x_0$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\|x_n - x_0\| \leq \|q - x_0\|, \quad n \geq 1. \tag{2.3}$$

Then $x_n \to q$.

We also need the following lemmas.

Lemma 2.6 ([23]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 + \gamma_n) a_n, \quad n \geq n_0$$

for some positive integer n_0, where $\{\gamma_n\}$ is a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

Proposition 2.7 ([7]). *Assume C is a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically κ-strict pseudo-contraction.*

(i) *For each $n \geq 1$, T^n satisfies the Lipschitz condition:*

$$\|T^n x - T^n y\| \leq L_n \|x - y\|, \quad x, y \in C, \tag{2.4}$$

where $L_n = \frac{\kappa + \sqrt{1 + \gamma_n(1-\kappa)}}{1-\kappa}$ (later L_n is called the Lipschitz constant of T^n).

(ii) *The demiclosedness principle holds for $I - T$ in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x_0$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x_0 = 0$. In particular,*

$$x_n \rightharpoonup x_0 \quad \text{and} \quad (I - T)x_n \to 0 \quad \Rightarrow \quad (I - T)x_0 = 0.$$

(iii) *The fixed point set $Fix(T)$ of T is closed and convex so that the projection $P_{Fix(T)}$ is well-defined.*

3. Convergence theorems

We begin with the following lemmas which are useful in our further discussion.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let a family $\Im = \{S_n : C \to C,\, n \geq 0\}$ be asymptotically κ-strict pseudo-contractive on C. Then, for each $n \geq 0$, S_n satisfies the Lipschitz condition, namely,*

$$\|S_n x - S_n y\| \leq L_n \|x - y\|, \tag{3.1}$$

where $L_n := \frac{\kappa + \sqrt{1+\gamma_n(1-\kappa)}}{1-\kappa} \to \frac{1+\kappa}{1-\kappa}$ as $n \to \infty$.

Proof. Similarly, (3.1) can be derived by replacing T^n in the proof of Proposition 2.6 (i) in [7] with S_n. □

Remark 3.2. Note that the common fixed point set $F := \cap_{n=0}^{\infty} Fix(S_n)$ is closed, but we don't know whether it is convex or not. However, it is not hard to see that F is convex provided the family $\Im = \{S_n : C \to C,\, n \geq 0\}$ satisfies the following *continuity* condition:

$$\forall\, v \in C, \;\; \|S_n v - v\| \to 0 \;\; \Rightarrow \;\; v \in F. \tag{3.2}$$

Indeed, let $p, q \in F$ and $v := \lambda p + (1-\lambda) q \in C$ with $\lambda \in (0,1)$. To show the convexity of F, we must show that $\|S_n v - v\| \to 0$. Now use (ii) of Lemma 2.1 and (1.8) to get

$$\begin{aligned}
\|S_n v - v\|^2 &= \|\lambda(S_n v - p) + (1-\lambda)(S_n v - q)\|^2 \\
&= \lambda\|S_n v - S_n p\|^2 + (1-\lambda)\|S_n v - S_n q\|^2 - \lambda(1-\lambda)\|p-q\|^2 \\
&\leq \lambda[(1+\gamma_n)\|v-p\|^2 + \kappa\|v - S_n v\|^2] + \\
&\quad (1-\lambda)[(1+\gamma_n)\|v-q\|^2 + \kappa\|v - S_n v\|^2] - \lambda(1-\lambda)\|p-q\|^2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
(1-\kappa)\|S_n v - v\|^2 &\leq \lambda(1-\lambda)(1+\gamma_n)\|p-q\|^2 - \lambda(1-\lambda)\|p-q\|^2 \\
&= \lambda(1-\lambda)\gamma_n\|p-q\|^2 \to 0
\end{aligned}$$

because $\gamma_n \to 0$ as $n \to \infty$. So, we obtain that $\|S_n v - v\| \to 0$.

Lemma 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let a family $\Im = \{S_n : C \to C,\, n \geq 0\}$ be asymptotically κ-strict pseudo-contractive on C. Assume that $F \neq \emptyset$ and the control sequences $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ are chosen so that*

(i) $\sum_{n=0}^{\infty} \gamma_n < \infty$, *and*
(ii) $\kappa + \epsilon \leq \alpha_n \leq 1 - \epsilon$, *where $\epsilon \in (0,1)$ is a small enough constant.*

Starting from an arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by the algorithm (1.9). Then there hold the following properties.

(a) *For each $q \in \overline{co}(F)$, $\lim_{n\to\infty} \|x_n - q\|$ exists, where $\overline{co}(F)$ denotes the closed convex hull of F.*
(b) *$\|x_n - S_n x_n\| \to 0$ and, furthermore, $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$.*

Proof. (a) First we show that $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$. Indeed, let $p \in F$. By virtue of (1.8), we see

$$\|S_nx_n - p\|^2 = \|S_nx_n - S_np\|^2 \le (1+\gamma_n)\|x_n - p\|^2 + \kappa\|x_n - S_nx_n\|^2.$$

Then this jointed with the identity (2.2) and the hypothesis (ii) yields

$$\begin{aligned} &\|x_{n+1} - p\|^2 = \|\alpha_n(x_n - p) + (1-\alpha_n)(S_nx_n - p)\|^2 \\ =\;& \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|S_nx_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - S_nx_n\|^2 \\ \le\;& [1+\gamma_n(1-\alpha_n)]\|x_n - p\|^2 - (1-\alpha_n)(\alpha_n - \kappa)\|x_n - S_nx_n\|^2 \\ \le\;& (1+\gamma_n)\|x_n - p\|^2 - \epsilon^2\|x_n - S_nx_n\|^2, \end{aligned} \tag{3.3}$$

in particular,

$$\|x_{n+1} - p\|^2 \le (1+\gamma_n)\|x_n - p\|^2$$

and, since $\sum_{n=0}^{\infty}\gamma_n < \infty$ by (i), an application of Lemma 2.6 ensures that $\lim_{n\to\infty}\|x_n - p\|$ exists. Next, let $q \in co(F)$, that is,

$$q = \sum_{i=0}^{N-1}\lambda_i p_i,$$

where all $p_i \in F$ and $\lambda_i \in [0,1]$ such that $\sum_{i=0}^{N-1}\lambda_i = 1$. Using the identity (2.1), we have

$$\begin{aligned} \|x_n - q\|^2 &= \|\sum_{i=0}^{N-1}\lambda_i(x_n - p_i)\|^2 \\ &= \sum_{i=0}^{N-1}\lambda_i\|x_n - p_i\|^2 - \sum_{i<j}^{N-1}\lambda_i\lambda_j\|p_i - p_j\|^2 \end{aligned}$$

for all n. Since $\lim_{n\to\infty}\|x_n - p_i\|$ exists for $i = 0, 1, \ldots, N-1$, the above identity yields

$$\lim_{n\to\infty}\|x_n - q\|^2 = \sum_{i=0}^{N-1}\lambda_i \lim_{n\to\infty}\|x_n - p_i\|^2 - \sum_{i<j}^{N-1}\lambda_i\lambda_j\|p_i - p_j\|^2$$

and hence $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in co(F)$,which quickly gives (a).

(b) Since $\{x_n\}$ is bounded, so is $\{S_nx_n\}$. Now rewrite (3.3) in the form

$$\|x_n - S_nx_n\|^2 \le \frac{1}{\epsilon^2}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) + \frac{\gamma_n}{\epsilon^2}\|x_n - p\|.$$

As $\gamma_n \to 0$ and $\{x_n\}$ is bounded as $n \to \infty$, we get

$$\|x_n - S_nx_n\| \to 0. \tag{3.4}$$

From definition of x_{n+1}, it follows that

$$\|x_{n+1} - x_n\| = (1-\alpha_n)\|x_n - S_nx_n\| \to 0. \tag{3.5}$$

Hence (b) is obtained, which completes the proof. □

Lemma 3.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let a family $\Im = \{S_n : C \to C,\ n \geq 0\}$ be asymptotically κ-strict pseudo-contractive on C. Assume that F is a nonempty bounded subset of C, and also that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $0 \leq \alpha_n < 1$ for $n \geq 0$. Let $\{x_n\}$ be the sequence generated by the hybrid algorithm (1.10), starting from an arbitrarily given $x_0 \in C$. Then there hold the following properties.*

(a) $\|x_n - x_0\| \leq \|q - x_0\|$ *for all* $n \geq 1$, *where* $q := P_{\overline{co}(F)} x_0$.
(b) $\|x_n - x_{n+1}\| \to 0$ *and, furthermore,* $\|x_n - S_n x_n\| \to 0$ *as* $n \to \infty$.
(c) $\|y_n - x_n\| \to 0$ *as* $n \to \infty$.

Proof. First observe that C_n is closed convex by Lemma 2.2 and also that Q_n is closed convex for all $n \geq 0$. Next we show that $F \subset C_n$ for $n \geq 0$. Indeed, we have, for all $p \in F$, replacing x_{n+1} in (3.3) with y_n we have

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(S_n x_n - p)\|^2 \\ &\leq [1 + \gamma_n(1-\alpha_n)]\|x_n - p\|^2 - (1-\alpha_n)(\alpha_n - \kappa)\|x_n - S_n x_n\|^2 \\ &\leq \|x_n - p\|^2 + (1-\alpha_n)[\theta_n + (\kappa - \alpha_n)\|x_n - S_n x_n\|^2] \end{aligned}$$

and thus $p \in C_n$ for all $n \geq 0$. This shows $F \subset C_n$ for each $n \geq 0$.

Next we show that

$$F \subset Q_n, \quad n \geq 0. \tag{3.6}$$

We prove this by induction. For $n = 0$, we have $F \subset C = Q_0$. Assume that $F \subset Q_k$. Since x_{k+1} is the projection of x onto $C_k \cap Q_k$, by Lemma 2.3 we have

$$\langle x_{k+1} - z, x_0 - x_{k+1} \rangle \geq 0, \quad z \in C_k \cap Q_k.$$

As $F \subset C_k \cap Q_k$ by the induction assumption, the last inequality holds, in particular, for all $z \in F$. This together with the definition of Q_{k+1} implies that $F \subset Q_{k+1}$. Hence (3.6) holds for all $n \geq 0$, and x_n is well defined for all n. Furthermore, since $C_n \cap Q_n$ is closed and convex, it follows from $F \subset C_n \cap Q_n$ that

$$\overline{co}(F) \subset C_n \cap Q_n, \quad n \geq 0.$$

Notice that the definition of Q_n actually implies $x_n = P_{Q_n} x_0$. This together with the fact $\overline{co}(F) \subset Q_n$ further implies

$$\|x_n - x_0\| \leq \|p - x_0\|, \quad p \in \overline{co}(F).$$

In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|q - x_0\|, \quad \text{where } q := P_{\overline{co}(F)} x_0. \tag{3.7}$$

Hence (a) is fulfilled.

The fact $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0$. This together with Lemma 2.1 (i) implies

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{3.8}$$

This implies that the sequence $\{\|x_n - x_0\|\}$ is increasing. Since it is also bounded, we see that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Note that since $\{x_n\}$ is bounded, so is $\{S_n x_n\}$. Then it turns out from (3.8) that

$$\|x_{n+1} - x_n\| \to 0. \tag{3.9}$$

To prove the second part of (b), i.e., $\|x_n - S_n x_n\| \to 0$, use the fact $x_{n+1} \in C_n$ to get

$$\begin{aligned} &\|y_n - x_{n+1}\|^2 \\ \le\ &\|x_n - x_{n+1}\|^2 + (1-\alpha_n)[\theta_n + (\kappa - \alpha_n)\|x_n - S_n x_n\|^2]. \end{aligned} \tag{3.10}$$

On the other hand, by virtue of $y_n = \alpha_n x_n + (1-\alpha_n) S_n x_n$ and (2.2) in Lemma 2.1, we have

$$\begin{aligned} \|y_n - x_{n+1}\|^2 &= \|\alpha_n(x_n - x_{n+1}) + (1-\alpha_n)(S_n x_n - x_{n+1})\|^2 \\ &= \alpha_n\|x_n - x_{n+1}\|^2 + (1-\alpha_n)\|S_n x_n - x_{n+1}\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - S_n x_n\|^2. \end{aligned}$$

After substituting this equality into (3.10), by simplifying and dividing both sides by $(1-\alpha_n)$ (note that $\alpha_n < 1$ for all n), we arrive at

$$\|x_{n+1} - S_n x_n\|^2 \le \|x_{n+1} - x_n\|^2 + \theta_n + \kappa\|x_n - S_n x_n\|^2. \tag{3.11}$$

Also, since

$$\begin{aligned} &\|x_{n+1} - S_n x_n\|^2 = \|(x_{n+1} - x_n) + (x_n - S_n x_n)\|^2 \\ =\ &\|x_{n+1} - x_n\|^2 + \|x_n - S_n x_n\|^2 - 2\langle x_n - x_{n+1}, x_n - S_n x_n\rangle \end{aligned}$$

by the parallelogram law, substituting this equality into (3.11) and simplifying, we have

$$\begin{aligned} (1-\kappa)\|x_n - S_n x_n\|^2 &\le \theta_n + 2\langle x_n - x_{n+1}, x_n - S_n x_n\rangle \\ &\le \theta_n + 2\|x_n - x_{n+1}\|\,\|x_n - S_n x_n\|. \end{aligned}$$

Since $\theta_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ by (3.9), solving this quadratic inequality in $\|x_n - S_n x_n\|^2$ yields $\lim_{n\to\infty} \|x_n - S_n x_n\| = 0$. Hence (b) is proven. Finally, (c) can be immediately derived by combining (3.10) and (b). □

Now we present the weak convergence of the algorithm (1.9) and the strong convergence of the hybrid algorithm (1.10) for an asymptotically strict pseudo-contractive family $\Im = \{S_n : C \to C,\ n \ge 0\}$.

Theorem 3.5. *Under the same hypotheses with Lemma 3.3, assume, in addition, that $\omega_w(x_n) \subset F$ and $\Im$ satisfies the continuity condition (3.2). Then $\{x_n\}$ converges weakly to a point of F.*

Proof. Following Remark 3.2, we notice that the set F is closed and convex. By (a) of Lemma 3.3, $\lim_{n\to\infty} \|x_n - p\|$ exists for $p \in F$. Also, since $\omega_w(x_n) \subset F$ by the assumption, an application of Lemma 2.4 with $K := F$ ensures that $\{x_n\}$ converges weakly to a point in F. □

Theorem 3.6. *Under the same hypotheses with Lemma 3.4, assume, in addition, that $\omega_w(x_n) \subset F$ and $\Im$ satisfies the continuity condition (3.2). Then $x_n \to P_F x_0$.*

Proof. Obviously, F is closed and convex. Combined the assumption $\omega_w(x_n) \subset F$ with (a) of Lemma 3.4, an application of Lemma 2.5 (with $K := F$) ensures that $x_n \to q$, where $q = P_F x$. □

We here give an example of an asymptotically strict pseudo-contractive family of self-mappings which is not asymptotically nonexpansive.

Example 3.7. *Let $C = H = \ell^2$ and $t > 1$, $p \geq 1$. Then we can define*

$$S_n x = -\Big(t + \frac{1}{n^p}\Big)x, \qquad x \in C$$

for each $n \geq 1$ and let $S_0 = I$, the identity mapping on C. Then, $F := \cap_{n=0}^{\infty} Fix(S_n) = \{0\}$, the family $\Im = \{S_n : C \to C,\, n \geq 0\}$ is obviously not asymptotically nonexpansive, but asymptotically κ-strict pseudo-contractive on C for any $\kappa \in [\frac{t-1}{t+1}, 1)$. Indeed, let $x, y \in C$ and $\frac{t-1}{t+1} \leq \kappa < 1$. Since

$$\|S_n x - S_n y\|^2 = \Big(t + \frac{1}{n^p}\Big)^2 \|x - y\|^2,$$
$$\|(I - S_n)x - (I - S_n)y\|^2 = \Big(1 + t + \frac{1}{n^p}\Big)^2 \|x - y\|^2,$$

and also,

$$\begin{aligned} \Big(t + \frac{1}{n^p}\Big)^2 - \kappa\Big(1 + t + \frac{1}{n^p}\Big)^2 &\leq \Big(t + \frac{1}{n^p}\Big)^2 - \frac{t-1}{t+1}\Big(1 + t + \frac{1}{n^p}\Big)^2 \\ &= 1 + \frac{2}{n^p} + \frac{2}{t+1}\Big(\frac{1}{n^p}\Big)^2 \\ &< 1 + \frac{3}{n^p}, \end{aligned}$$

we have

$$\begin{aligned} \|S_n x - S_n y\|^2 &= \Big[\Big(t + \frac{1}{n}\Big)^2 - \kappa\Big(1 + t + \frac{1}{n}\Big)^2\Big]\|x - y\|^2 \\ &\quad + \kappa\Big(1 + t + \frac{1}{n}\Big)^2 \|x - y\|^2 \\ &\leq \Big(1 + \frac{3}{n^p}\Big)\|x - y\|^2 + \kappa\Big(1 + t + \frac{1}{n}\Big)^2 \|x - y\|^2 \\ &= (1 + \gamma_n)\|x - y\|^2 + \kappa\|(I - S_n)x - (I - S_n)y\|^2, \end{aligned}$$

where $\gamma_n := \frac{3}{n^p}$ (note that $\sum_{n=1}^{\infty} \gamma_n < \infty$ for $p > 1$; see (i) of Lemma 3.3). Therefore, $\Im = \{S_n : C \to C,\, n \geq 0\}$ is asymptotically κ-strict pseudo-contractive on C for any κ satisfying $\frac{t-1}{t+1} \leq \kappa < 1$.

Remark 3.8. Note that if the family $\Im = \{S_n : C \to C,\, n \geq 0\}$ is given as in Example 3.7 with $p > 1$, and $\{x_n\}$ is generated by the algorithm (1.9), then $\omega_w(x_n) = \{0\} = F$. In fact, assume without loss of generality that $x_n \rightharpoonup z \in C$. Since $\|x_n - x_{n+1}\| \to 0$ by (b) of Lemma 3.3, $x_{n+1} \rightharpoonup z$ too. Now choose a subsequence $\{n_k\}$ of $\{n\}$ such that $\alpha_{n_k} \to \alpha \in [\kappa + \epsilon, 1 - \epsilon]$ by (ii) of Lemma 3.3 as $k \to \infty$. Then the algorithm (1.9) yields

$$x_{n_k+1} = \alpha_{n_k} x_{n_k} - (1 - \alpha_{n_k})\Big(t + \frac{1}{n_k}\Big)x_{n_k}$$

and also

$$\langle x_{n_k+1}, y\rangle = [\alpha_{n_k} - (1-\alpha_{n_k})(t+\frac{1}{n_k})]\langle x_{n_k}, y\rangle, \qquad y \in H.$$

Now taking the limit on both sides as $k \to \infty$, we get

$$\langle z, y\rangle = [\alpha - (1-\alpha)t]\langle z, y\rangle \quad \Leftrightarrow \quad (1-\alpha)(1+t)\langle z, y\rangle = 0$$

for all $y \in H$. In particular, choosing $y = z$ yields $z = 0$. Therefore $\omega_w(x_n) \subset \{0\}$. For the converse inclusion, since $\{x_n\}$ is bounded; hence $\omega_w(x_n) \neq \emptyset$, the same argumentation as above gives

$$x_n \rightharpoonup 0 \in \omega_w(x_n),$$

which concludes that $\omega_w(x_n) = \{0\}$.

On the other hand, if the sequence $\{x_n\}$ is defined by the hybrid algorithm (1.10) and $\limsup_{n\to\infty} \alpha_n < 1$, then $\omega_w(x_n) = \{0\}$ can be similarly proven by virtue of (c) of Lemma 3.4. Consequently, our main results (without such restrictions) can be directly applied to this family, in other words, the sequence $\{x_n\}$ defined by (1.9) converges weakly to zero under assumption (ii) of Lemma 3.3, while the sequence x $\{x_n\}$ generated by (1.10) converges strongly to zero under the condition $\limsup_{n\to\infty} \alpha_n < 1$.

4. Applications to the parallel algorithm and the cyclic algorithm

Let C be a nonempty closed convex subset of a Hilbert space H. Unless other specified throughout this section, we always assume that

- (c_1) for each $i = 0, 1, \ldots, N-1$, $T_i : C \to C$ be an asymptotically κ_i-strict pseudo-contraction with respect to the sequence $\{\gamma_n^{(i)}\}_{n=0}^{\infty}$ for some $0 \leq \kappa_i < 1$,
- (c_2) for each $n \geq 0$, $\{\lambda_i^{(n)}\}$ is a finite sequence of positive numbers such that $\sum_{i=0}^{N-1} \lambda_i^{(n)} = 1$ for all n, and $\bar{\lambda}_i := \inf\{\lambda_i^{(n)} : n \geq 0\} > 0$ for $i = 0, 1, \ldots, N-1$.

Recently, Lopez Acedo and Xu [10] considered the problem of finding a point x such that

$$x \in F_N := \cap_{i=0}^{N-1} Fix(T_i),$$

where $\{T_i\}_{i=0}^{N-1}$ are κ_i-strict pseudo-contractions defined on C under the condition (c_2). As $F_N \neq \emptyset$, they investigated the weak convergence of the sequence $\{x_n\}$ generated explicitly by the following parallel algorithm:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i=0}^{N-1} \lambda_i^{(n)} T_i x_n, \quad n \geq 0, \tag{4.1}$$

and strong convergence of the following hybrid parallel algorithm:

$$(4.2)\qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=0}^{N-1} \lambda_i^{(n)} T_i x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)(\kappa - \alpha_n) \\ \qquad\qquad \|x_n - \sum_{i=0}^{N-1} \lambda_i^{(n)} T_i x_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \ge 0, \end{cases}$$

where the initial guess $x_0 \in C$ is arbitrarily chosen and $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1)$.

Throughout this section, let $\{T_i\}_{i=0}^{N-1}$ be a finite family of *asymptotically* κ_i-strict pseudo-contractions defined on C. Then we consider either the following modified parallel algorithm of (4.1):

$$(4.3)\qquad x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i=0}^{N-1} \lambda_i^{(n)} T_i^n x_n, \quad n \ge 1.$$

or the modified hybrid parallel algorithm of (4.2):

$$(4.4)\qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=0}^{N-1} \lambda_i^{(n)} T_i^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)[\theta_n \\ \qquad\qquad +(\kappa - \alpha_n)\|x_n - \sum_{i=0}^{N-1} \lambda_i^{(n)} T_i^n x_n\|^2]\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \ge 0, \end{cases}$$

where

$$\theta_n = \gamma_n \cdot \sup\{\|x_n - z\|^2 : z \in F_N\}.$$

For each $n \ge 0$, let a mapping $S_n : C \to C$ defined by

$$(4.5)\qquad S_n x = \sum_{i=0}^{N-1} \lambda_i^{(n)} T_i^n x$$

for all $x \in C$, where $T_i^0 = I$ for $i = 0, 1, \ldots, N-1$, Then parallel algorithms (4.3) and (4.4) can be written compactly as (1.9) and (1.10), respectively, noticing the fact

$$F_N = F := \cap_{n=0}^{\infty} Fix(S_n)$$

by the property (iii) of the following lemma 4.1.

Put $\gamma_n := \max\{\gamma_n^{(i)} : 1 \le i \le N\}$ for $n \ge 0$ and $\kappa := \max\{\kappa_i : 1 \le i \le N\}$. Obviously, $\gamma_n \to 0$ and $0 \le \kappa < 1$ and we therefore obtain the following properties of the mapping S_n.

Lemma 4.1. *Let $x, y \in C$ and $i = 0, 1, \ldots, N-1$. Then the following properties are satisfied.*

(i) $\|T_i^n x - T_i^n y\|^2 \le (1+\gamma_n)\|x-y\|^2 + \kappa\|(I - T_i^n)x - (I - T_i^n)y\|^2$.

(ii) $\|S_n x - S_n y\|^2 \le (1+\gamma_n)\|x-y\|^2 + \kappa\|(I - S_n)x - (I - S_n)y\|^2$. *In other words, the family $\Im = \{S_n : C \to C, n \ge 0\}$ is asymptotically κ-strict pseudo-contractive on C.*

(iii) *If $F_N \neq \emptyset$, then $F_N = F$; hence F is closed convex so that the projection P_F is well defined.*

Proof. (i) is obvious from the definition of asymptotically strict pseudo-contraction. To prove (ii), use the identity (2.1) of Lemma 2.1 to derive

$$\|(I-S_n)x-(I-S_n)y\|^2 = \Big\|\sum_{i=0}^{N-1}\lambda_i^{(n)}[(I-T_i^n)x-(I-T_i^n)y]\Big\|^2$$
$$= \sum_{i=0}^{N-1}\lambda_i^{(n)}\|(I-T_i^n)x-(I-T_i^n)y\|^2 - \sum_{i<j}^{N-1}\lambda_i^{(n)}\lambda_j^{(n)}\|(T_i^n x - T_i^n y)-(T_j^n x - T_j^n y)\|^2.$$

This yields a simple form:

$$\sum_{i=0}^{N-1}\lambda_i^{(n)}\|(I-T_i^n)x-(I-T_i^n)y\|^2 = \|(I-S_n)x-(I-S_n)y\|^2 + J, \tag{4.6}$$

where $J := \sum_{i<j}^{N-1}\lambda_i^{(n)}\lambda_j^{(n)}\|(T_i^n x - T_i^n y)-(T_j^n x - T_j^n y)\|^2 \geq 0$. Use (2.1), (i) and (4.6) in turn to get

$$\begin{aligned}
\|S_n x - S_n y\|^2 &= \Big\|\sum_{i=0}^{N-1}\lambda_i^{(n)}(T_i^n x - T_i^n y)\Big\|^2\\
&= \sum_{i=0}^{N-1}\lambda_i^{(n)}\|T_i^n x - T_i^n y\|^2 - J\\
&\leq \sum_{i=0}^{N-1}\lambda_i^{(n)}\{(1+\gamma_n)\|x-y\|^2 + \kappa\|(I-T_i^n)x-(I-T_i^n)y\|^2\} - J\\
&= (1+\gamma_n)\|x-y\|^2 + \kappa\sum_{i=0}^{N-1}\lambda_i^{(n)}\|(I-T_i^n)x-(I-T_i^n)y\|^2 - J\\
&= (1+\gamma_n)\|x-y\|^2 + \kappa\|(I-S_n)x-(I-S_n)y\|^2 - (1-\kappa)J\\
&\leq (1+\gamma_n)\|x-y\|^2 + \kappa\|(I-S_n)x-(I-S_n)y\|^2.
\end{aligned}$$

Hence (ii) is proven.

Finally to prove (iii), it suffices to show that $F \subset F_N$. Indeed, let $x = S_n x$ for all $n \geq 0$. Since $F_N \neq \emptyset$, for $p \in F_N$, use (2.1) and (i) to derive

$$\begin{aligned}
\|p-x\|^2 &= \|p-S_n x\|^2 = \Big\|\sum_{i=0}^{N-1}\lambda_i^{(n)}(p-T_i^n x)\Big\|^2\\
&= \sum_{i=0}^{N-1}\lambda_i^{(n)}\|p-T_i^n x\|^2 - \sum_{i<j}^{N-1}\lambda_i^{(n)}\lambda_j^{(n)}\|T_i^n x - T_j^n x\|^2\\
&\leq \sum_{i=0}^{N-1}\lambda_i^{(n)}\{(1+\gamma_n)\|p-x\|^2 + \kappa\|x-T_i^n x\|^2\} - \delta
\end{aligned}$$

$$= \quad (1+\gamma_n)\|p-x\|^2 + \kappa \sum_{i=0}^{N-1} \lambda_i^{(n)} \|x - T_i^n x\|^2 - \delta$$

where $\delta := \sum_{i<j}^{N-1} \lambda_i^{(n)} \lambda_j^{(n)} \|T_i^n x - T_j^n x\|^2$. Therefore, we have

$$\delta \le \gamma_n \|p-x\|^2 + \kappa \sum_{i=0}^{N-1} \lambda_i^{(n)} \|x - T_i^n x\|. \tag{4.7}$$

On the other hand, since $S_n x = x$ for all $n \ge 0$, it follows from (2.1) that

$$\begin{aligned} 0 &= \|S_n x - x\| = \Big\| \sum_{i=0}^{N-1} \lambda_i^{(n)} (T_i^n x - x) \Big\|^2 \\ &= \sum_{i=0}^{N-1} \lambda_i^{(n)} \|T_i^n x - x\|^2 - \delta. \end{aligned} \tag{4.8}$$

Substituting (4.8) into (4.7) and simplifying, we have

$$\begin{aligned} 0 &\le (1-\kappa) \sum_{i=0}^{N-1} \bar{\lambda}_i \|T_i^n x - x\|^2 \\ &\le (1-\kappa) \sum_{i=0}^{N-1} \lambda_i^{(n)} \|T_i^n x - x\|^2 \\ &\le \gamma_n \|p - x\|^2 \to 0 \end{aligned}$$

because $\gamma_n \to 0$. This implies that, for $i = 0, 1, \ldots, N-1$, $\lim_{n\to\infty} T_i^n x = x$ and so $x \in Fix(T_i)$ by continuity of T_i. Hence, $x \in F_N = \cap_{i=1}^N Fix(T_i)$, which proves $F \subset F_N$. Finally, by (iii) of Proposition 2.7, each $Fix(T_i)$ is closed convex for $i = 0, 1, \ldots, N-1$. Hence F_N is closed convex, and so is $F = F_N$. This completes the proof. □

Lemma 4.2. *Assume $F_N \neq \emptyset$. Let $x \in C$ and $p \in F_N$. Then,*

(i) $(1-\kappa) \sum_{i=0}^{N-1} \lambda_i^{(n)} \|x - T_i^n x\|^2 \le \|p - x\| (\gamma_n \|p - x\| + 2\|x - S_n x\|)$.
(ii) *Let $\{x_n\} \subset C$ such that $x_n \rightharpoonup z$ and $\|x_n - S_n x_n\| \to 0$. Assume, in addition, $\|x_n - x_{n+1}\| \to 0$. Then $z \in F_N$.*

Proof. Put $I := \sum_{i=0}^{N-1} \lambda_i^{(n)} \|x - T_i^n x\|^2$ and $J := \sum_{i<j}^{N-1} \lambda_i^{(n)} \lambda_j^{(n)} \|T_i^n x - T_j^n x\|^2$. Use (2.1) to get

$$\|x - S_n x\|^2 = \Big\| \sum_{i=0}^{N-1} \lambda_i^{(n)} (x - T_i^n x) \Big\|^2 = I - J.$$

Observe

$$\begin{aligned} \|p - S_n x\|^2 &= \|(p - x) + (x - S_n x)\|^2 \\ &= \|p - x\|^2 + \|x - S_n x\|^2 - 2\langle x - p, x - S_n x\rangle \\ &= \|p - x\|^2 + I - J - 2\langle x - p, x - S_n x\rangle \end{aligned} \tag{4.9}$$

by parallelogram law. Using (2.1) and (i) of Lemma 4.1 we have

$$\begin{aligned}\|p - S_n x\|^2 &= \Big\| \sum_{i=0}^{N-1} \lambda_i^{(n)} (p - T_i^n x) \Big\|^2 = \sum_{i=0}^{N-1} \lambda_i^{(n)} \|p - T_i^n x\|^2 - J \\ &\le \sum_{i=0}^{N-1} \lambda_i^{(n)} \{(1+\gamma_n)\|p - x\|^2 + \kappa \|x - T_i^n x\|^2\} - J \\ &\le (1+\gamma_n)\|p - x\|^2 + \kappa I - J. \qquad (4.10)\end{aligned}$$

Substituting (4.9) into (4.10) and simplifying we have

$$\begin{aligned}(1-\kappa) I &\le \gamma_n \|p - x\|^2 + 2\langle x - p, x - S_n x\rangle \\ &\le \|p - x\|(\gamma_n \|p - x\| + 2\|x - S_n x\|),\end{aligned}$$

which proves (i). To show (ii), replacing x with x_n in (i) gives

$$(1-\kappa) \sum_{i=0}^{N-1} \lambda_i^{(n)} \|x_n - T_i^n x_n\|^2 \le \|p - x_n\|(\gamma_n \|p - x_n\| + 2\|x - S_n x_n\|).$$

Since $\{x_n\}$ is bounded, $\gamma_n \to 0$ and $\|x_n - S_n x_n\| \to 0$, we can easily derive

$$\|x_n - T_i^n x_n\| \to 0, \quad i = 0, 1, \dots, N-1. \qquad (4.11)$$

On the other hand, by (i) of Proposition 2.7, for each $i = 0, 1, \dots, N-1$,

$$\|T_i^n x - T_i^n y\| \le L_n^{(i)} \|x - y\|, \quad x, y \in C,$$

where $L_n^{(i)}$ denotes the Lipschitz constant of T_i^n. Put $L_n := \max_{i=0}^{N-1} L_n^{(i)}$. Then

$$\|T_i^n x - T_i^n y\| \le L_n \|x - y\|, \quad x, y \in C, \ \ i = 0, 1, \dots, N-1. \qquad (4.12)$$

After using (4.12) in the following second inequality, apply (4.11) and the assumption $\|x_n - x_{n+1}\| \to 0$ to derive

$$\begin{aligned}\|T_i^n x_n - T_i^{n+1} x_n\| &\le \|T_i^n x_n - x_n\| + \|x_n - x_{n+1}\| \\ &\quad + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\| \\ &\le \|T_i^n x_n - x_n\| + (1 + L_{n+1})\|x_n - x_{n+1}\| \\ &\quad + \|x_{n+1} - T_i^{n+1} x_{n+1}\| \to 0. \qquad (4.13)\end{aligned}$$

For $i = 0, 1, \dots, N-1$, with the help of (4.11)-(4.13) we have

$$\begin{aligned}\|x_n - T_i x_n\| &\le \|x_n - T_i^n x_n\| + \|T_i^n x_n - T_i^{n+1} x_n\| + \|T_i^{n+1} x_n - T_i x_n\| \\ &\le (1 + L_1)\|x_n - T_i^n x_n\| + \|T_i^n x_n - T_i^{n+1} x_n\| \to 0. \qquad (4.14)\end{aligned}$$

Then the demiclosedness principle of $I - T_i$ (Proposition 2.7 (ii)) implies that $z \in Fix(T_i)$ for all $i = 0, 1, \dots, N-1$. Hence $z \in F_N = \cap_{i=0}^{N-1} Fix(T_i)$ and the proof is complete. □

As direct applications of Theorem 3.5 and 3.6, respectively, we obtain the following successive convergence problems of parallel algorithms for a finite family $\{T_i\}_{i=0}^{N-1}$ of N asymptotically κ_i-strict pseudo-contractions.

Theorem 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_0^{N-1}$ and $\{\lambda_i^{(n)}\}$ be as in (c$_1$) and (c$_2$), respectively. Let $\gamma_n := \max_{i=0}^{N-1} \gamma_n^{(i)}$ and $\kappa := \max_{i=0}^{N-1} \kappa_i$. Assume that $F_N \neq \emptyset$ and the control sequences $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ are chosen so that*

(i) $\sum_{n=0}^{\infty} \gamma_n < \infty$.
(ii) $\kappa + \epsilon \leq \alpha_n \leq 1 - \epsilon$, *where $\epsilon \in (0, 1)$ is a small enough constant.*

Starting from an arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by the parallel algorithm (4.3) or (1.9). Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=0}^{N-1}$.

Proof. By (ii) and (iii) of Lemma 4.1, it suffices to show that $\omega_w(x_n) \subset F$. This fact is directly derived from (ii) of Lemma 4.2 by reminding of (b) of Lemma 3.3. Then our conclusion is obtained by Theorem 3.5. □

Theorem 4.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_0^{N-1}$ and $\{\lambda_i^{(n)}\}$ be as in (c$_1$) and (c$_2$), respectively. Let $\gamma_n := \max_{i=0}^{N-1} \gamma_n^{(i)}$ and $\kappa := \max_{i=0}^{N-1} \kappa_i$. Assume that F_N is a nonempty bounded subset of C, and also that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $0 \leq \alpha_n < 1$ for $n \geq 0$. Let $\{x_n\}$ be the sequence generated by the modified hybrid parallel algorithm (4.4) or (1.10), starting from an arbitrarily given $x_0 \in C$. Then $x_n \to P_{F_N} x_0$.*

Proof. By (ii) and (iii) of Lemma 4.1, the family $\Im = \{S_n : C \to C, n \geq 0\}$ is asymptotically κ-strict pseudo-contractive on C and $F = F_N$ is closed convex. Immediately, the fact $\omega_w(x_n) \subset F$ is required from (ii) of Lemma 4.2 by reminding of (b) of Lemma 3.4. Then our conclusion is achieved by Theorem 3.6. □

Lopez Acedo and Xu [10] also investigated the convergence problems for the following cyclic algorithm:

$$\begin{aligned}
&x_0 \in C \text{ chosen arbitrarily},\\
&x_1 = \alpha_0 x_0 + (1 - \alpha_0) T_0 x_0,\\
&x_2 = \alpha_1 x_1 + (1 - \alpha_1) T_1 x_1,\\
&\quad\vdots\\
&x_N = \alpha_{N-1} x_{N-1} + (1 - \alpha_{N-1}) T_{N-1} x_{N-1},\\
&x_{N+1} = \alpha_N x_N + (1 - \alpha_N) T_0 x_N,\\
&\quad\vdots
\end{aligned}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $[0, 1]$. The above cyclic algorithm can be written in a more compact form as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{[n]} x_n, \quad n \geq 0, \tag{4.15}$$

where $T_{[k]} = T_{k \bmod N}$ for integer $k \geq 1$. The mod function takes values in the set $\{0, 1, 2, \cdots, N-1\}$ as

$$T_{[k]} = \begin{cases} T_0, & \text{if } q = 0; \\ T_q, & \text{if } 0 < q < N \end{cases}$$

for $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$.

For our argument for a finite family $\{T_i\}_{i=0}^{N-1}$ of *asymptotically* κ_i-strict pseudo-contractions defined on C, we consider the following modified cyclic algorithm instead of (4.15):

$$\begin{aligned} &x_0 \in C \text{ chosen arbitrarily,}\\ &x_1 = \alpha_0 x_0 + (1-\alpha_0)T_0 x_0,\\ &x_2 = \alpha_1 x_1 + (1-\alpha_1)T_1 x_1,\\ &\quad\vdots\\ &x_N = \alpha_{N-1}x_{N-1} + (1-\alpha_{N-1})T_{N-1}x_{N-1},\\ &x_{N+1} = \alpha_N x_N + (1-\alpha_N)T_0^2 x_N,\\ &\quad\vdots\\ &x_{2N} = \alpha_{2N-1}x_{2N-1} + (1-\alpha_{2N-1})T_{N-1}^2 x_{2N-1},\\ &x_{2N+1} = \alpha_{2N}x_{2N} + (1-\alpha_{2N})T_0^3 x_{2N},\\ &\quad\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]}^{k(n)} x_n, \quad n \geq 0, \tag{4.16}$$

where $k(n) := q+1$, as expressed with $n = qN + [n]$ for each $n \geq 0$. Then it is not hard to see that

$$k(n-N) = k(n) - 1 \ \text{ and } \ T_{[n-N]} = T_{[n]}, \quad n \geq N. \tag{4.17}$$

Observe that taking $T_i^n = T_i$ (then $\gamma_n^{(i)} = 0$) for all $n \geq 0$ in (4.16) reduces to (4.15). Also, it is not hard to see that the family $\Im = \{T_{[n]}^{k(n)} : C \to C, n \geq 0\}$ is asymptotically κ-strict pseudo-contraction. Indeed, use (i) of Lemma 4.1 to get

$$\|T_{[n]}^{k(n)}x - T_{[n]}^{k(n)}y\| \ \leq \ (1+\gamma_{k(n)})\|x-y\|^2 + \kappa\|(I - T_{[n]}^{k(n)})x - (I - T_{[n]}^{k(n)})y\|^2$$

for all $x, y \in C$, and $\gamma_{k(n)} \to 0$ because $k(n) \to \infty$ as $n \to \infty$.

Finally, as direct consequences of our main theorems, we also prove the following successive convergence problems of cyclic algorithms for a finite family $\{T_i\}_{i=0}^{N-1}$ of asymptotically κ_i-strict pseudo-contractions.

Theorem 4.5. *Under the same hypotheses with Theorem 4.3, the sequence $\{x_n\}$ generated by the cyclic algorithm (4.16) converges weakly to a common fixed point of $\{T_i\}_{i=0}^{N-1}$.*

Proof. Replacing all the S_n in the process of the proof of Lemma 3.3 with $T_{[n]}^{k(n)}$, we can immediately prove the following facts:

(1) $\lim_{n\to\infty}\|x_n - p\|$ exists for $p \in F_N$;

(2) $\|x_n - T_{[n]}^{k(n)}x_n\| \to 0$ (hence $\|x_n - x_{n+1}\| \to 0$) as $n \to \infty$.

By (2), it is not hard to see that, for $i = 0, 1, \ldots, N-1$

$$\|x_n - x_{n+i}\| \to 0 \tag{4.18}$$

and

$$\|T_{[n]}^{k(n)}x_n - x_{n+i}\| \to 0. \tag{4.19}$$

Using (4.12) and (4.17), we have

$$\begin{aligned}
&\|x_n - T_{[n]}x_n\| = \|(x_n - T_{[n]}^{k(n)}x_n) + (T_{[n]}^{k(n)}x_n - T_{[n]}x_n)\| \\
\le\ & \|x_n - T_{[n]}^{k(n)}x_n\| + L_1\|T_{[n]}^{k(n)-1}x_n - x_n\| \\
\le\ & \|x_n - T_{[n]}^{k(n)}x_n\| + L_1(\|T_{[n]}^{k(n)-1}x_n - T_{[n-N]}^{k(n-N)}x_{n-N}\| + \|T_{[n-N]}^{k(n-N)}x_{n-N} - x_n\|) \\
\le\ & \|x_n - T_{[n]}^{k(n)}x_n\| + L_1(L_{k(n)-1}\|x_n - x_{n-N}\| + \|T_{[n-N]}^{k(n-N)}x_{n-N} - x_n\|)
\end{aligned}$$

for all $n \ge N$. Using (2), (4.18) and (4.19) with $n - N$ instead n, $i = 0$, the right hand side converges strongly to 0 and hence

$$\|x_n - T_{[n]}x_n\| \to 0. \tag{4.20}$$

For $i = 0, 1, \ldots, N-1$, use (4.12) to derive the following second inequality and also use (4.18) and (4.20) to get the convergence to 0 as

$$\begin{aligned}
\|x_n - T_{[n+i]}x_n\| \ &\le\ \|x_n - x_{n+i}\| + \|x_{n+i} - T_{[n+i]}x_{n+i}\| \\
&\qquad + \|T_{[n+i]}x_{n+i} - T_{[n+i]}x_n\| \\
&\le\ (1 + L_1)\|x_n - x_{n+i}\| + \|x_{n+i} - T_{[n+i]}x_{n+i}\| \to 0.
\end{aligned}$$

For simplicity, put $c_i^n := \|x_n - T_i x_n\|$ for $i = 0, 1, \ldots, N-1$ and $n \ge 0$. For the following enumeration (4.21) with N-rows, take, in turn, $i = 0, N-1, N-2, \cdots, 1$ in the set $\{\|x_n - T_{[n+i]}x_n\|\}$ for each row and enumerate each column for all $n \ge 0$.

$$\begin{array}{cccccccccccc}
c_0^0 & c_1^1 & c_2^2 & \cdots & c_{N-1}^{N-1} & c_0^N & c_1^{N+1} & \cdots & c_{N-1}^{2N-1} & c_0^{2N} & \cdots & \to 0 \\
c_{N-1}^0 & c_0^1 & c_1^2 & \cdots & c_{N-2}^{N-1} & c_{N-1}^N & c_0^{N+1} & \cdots & c_{N-2}^{2N-1} & c_{N-1}^{2N} & \cdots & \to 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \to 0 \\
c_1^0 & c_2^1 & c_3^2 & \cdots & c_0^{N-1} & c_1^N & c_2^{N+1} & \cdots & c_0^{2N-1} & c_1^{2N} & \cdots & \to 0
\end{array} \tag{4.21}$$

It is not hard to find a sequence $\{c_0^n\}$ positioned at each N-diagonal, repeatedly, such that $c_0^n = \|x_n - T_0 x_n\| \to 0$. Moving each row downwards once and the last row to the first cyclically, we can get the sequence $\{c_1^n\}$ appearing at the same position with $\{c_0^n\}$ such that $c_1^n = \|x_n - T_1 x_n\| \to 0$. Repeating these processes, we have

$$\|x_n - T_i x_n\| \to 0, \quad i = 0, 1, \ldots, N-1. \tag{4.22}$$

Finally to show $\omega_w(x_n) \subset F_N$, use the demiclosedness property of $I - T_i$ (see (ii) of Proposition 2.7). Then, use Lemma 2.4 (with $K = F_N$) to conclude that $\{x_n\}$ converges weakly to a point in F_N. □

Theorem 4.6. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_0^{N-1}$ and $\{\lambda_i^{(n)}\}_{n=0}^{\infty}$ be as in (c_1) and (c_2), respectively. Let $\gamma_n := \max_{i=0}^{N-1} \gamma_n^{(i)}$ and $\kappa := \max_{i=0}^{N-1} \kappa_i$. Assume that F_N is a nonempty bounded subset of C, and also that the control sequence $\{\alpha_n\}$ is chosen so that $0 \le \alpha_n < 1$ for all $n \ge 0$. Let $\{x_n\}$*

be the sequence generated by the following modified cyclic algorithm:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T_{[n]}^{k(n)} x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)[\theta_n \\ \qquad\qquad +(\kappa - \alpha_n)\|x_n - T_{[n]}^{k(n)} x_n\|^2]\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0, \end{cases}$$

where $\theta_n = \gamma_n \cdot \sup\{\|x_n - z\|^2 : z \in F_N\} \to 0$*. Then* $x_n \to P_{F_N} x_0$*.*

Proof. First, to claim the following observations (i)-(vi), simply replace S_n in the proof of Lemma 3.4 with $T_{[n]}^{k(n)}$.

(i) x_n is well defined for all $n \geq 1$.
(ii) $\|x_n - x_0\| \leq \|q - x_0\|$ for all n, where $q = P_{F_N} x_0$.
(iii) $\|x_{n+1} - x_n\| \to 0$.
(vi) $\|x_n - T_{[n]}^{k(n)} x_n\| \to 0$.

To derive $\omega_n(x_n) \subset F_N$, repeat the argument of (4.18)-(4.22) in the proof of Theorem 4.5. Finally use (ii) and Lemma 2.5 to arrive at the our conclusion. □

Remark 4.7. (a) The cyclic algorithm (4.15) was investigated by Xu and Ori [29] for the implicit iteration process for N nonexpansive mappings, later studied by Sun [22] for N quasi-nonexpansive mappings, and recently developed by Chang et al. [2] for the implicit iteration process with error for N asymptotically nonexpansive mappings.

(b) As taking $\gamma_n^{(i)} = 0$ (hence $\gamma_n = 0$) and $T_i^n = T_i$ for all n and $i = 0, 1, \ldots, N-1$ in (4.3) and (4.16), our results extend and improve the corresponding ones for a finite family $\{T_i\}_{i=0}^{N-1}$ of N κ_i-strict pseudo-contractions due to Lopez Acedo and Xu [10]; see Theorem 4.1 and 5.2 of [10].

(c) Note that Theorem 3.5 and 3.6 are also satisfied under the *weaker* assumption of $\Im = \{S_n : C \to C, n \geq 0\}$, more precisely, for an asymptotically κ-strict quasi-pseudo-contractive family; in view of (1.8), we say that $\Im = \{S_n : C \to C, n \geq 0\}$ is *asymptotically* κ*-strict quasi-pseudo-contractive* on C if $F := \cap_{n=1}^{\infty} Fix(S_n) \neq \emptyset$ and there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}_{n=0}^{\infty}$ of nonnegative real numbers with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\|S_n x - p\|^2 \leq (1 + \gamma_n)\|x - p\|^2 + \kappa\|x - S_n x\|^2$$

for all $x \in C$, $p \in F$ and all integers $n \geq 0$.

Acknowledgement

The authors thank the referee for his comments and suggestions which improved the presentation of this manuscript.

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Manuscript received June 23, 2010
revised April 14, 2011

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