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WEAK AND STRONG CONVERGENCE THEOREMS FOR EXTENDED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. Let C be a closed convex subset of a real Hilbert space H. A mapping $U : C \to H$ is called extended hybrid if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned}\alpha(1+\gamma)\|Ux-Uy\|^2 + (1-\alpha(1+\gamma))\|x-Uy\|^2 \\ \leq (\beta+\alpha\gamma)\|Ux-y\|^2 + (1-(\beta+\alpha\gamma))\|x-y\|^2 \\ -(\alpha-\beta)\gamma\|x-Ux\|^2 - \gamma\|y-Uy\|^2\end{aligned}$$

for all $x, y \in C$. In this paper, we first show that the class of extended hybrid mappings contains the class of strict pseudo-contractions defined by Browder and Petryshyn [6]. We also obtain some important properties for extended hybrid mappings and strict pseudo-contractions in a Hilbert space. Using these results, we prove weak convergence theorems of Baillon's type [3] and of Mann's type [19] for extended hybrid mappings in a Hilbert space. Finally, we get strong convergence theorems of Halpern's type [9] and of the hybrid methods [22] and [30] for these mappings.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping $T : C \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A mapping $T : C \to H$ is said to be *a strict pseudo-contraction* [6] if there exists a real number k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2 \tag{1.1}$$

for all $x, y \in C$. We also call such a mapping T a *k-strict pseudo-contraction.* A k-strict pseudo-contraction $T : C \to H$ is nonexpansive if $k = 0$. A mapping $T : C \to H$ is said to be *nonspreading* [17] and *hybrid* [28] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.2}$$

and

$$3\|Tx - Ty\|^2 \leq \|x-y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.3}$$

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for all $x, y \in C$, respectively; see also [11], [12], [13] and [16]. We know from [28] that a nonexpansive mapping, a nonspreading mapping and a hybrid mapping are deduced from a firmly nonexpansive mapping. A mapping $T : C \to H$ is said to be *firmly nonexpansive* [5], [8] if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$$

for all $x, y \in C$. A firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space; see, for instance, [4] and [7]. Recently, Kocourek, Takahashi and Yao [15] considered a broad class of nonlinear mappings in a Hilbert space which contains the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings: A mapping $T : C \to H$ is called *generalized hybrid* [15] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.4}$$

for all $x, y \in C$. We call such a mapping an $(\alpha,\, \beta)$-*generalized hybrid* mapping. For example, an $(\alpha,\, \beta)$-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. Hojo, Takahashi and Yao [10] also introduced a class of nonlinear mappings in a Hilbert space which contains the class of generalized hybrid mappings: A mapping $U : C \to H$ is called *extended hybrid* if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} \alpha(1 + \gamma)\|Ux - Uy\|^2 &+ (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \\ &\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ &\quad - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned} \tag{1.5}$$

for all $x, y \in C$.

In this paper, we first show that the class of extended hybrid mappings contains the class of strict pseudo-contractions in a Hilbert space. We also obtain some important properties for extended hybrid mappings and strict pseudo-contractions in a Hilbert space. Using these results, we prove weak convergence theorems of Baillon's type [3] and of Mann's type [19] for extended hybrid mappings in a Hilbert space. Finally, we get strong convergence theorems of Halpern's type [9] and of the hybrid methods [22], [30] for these mappings.

2. Preliminaries

Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot\ \rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [27], we know the following basic equality: For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.1}$$

Furthermore, we know that for $x, y, u, v \in H$,

$$2\,\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.2}$$

Let C be a nonempty closed convex subset of H and let T be a mapping from C into H. Then, we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [14]. It is not so difficult to show this fact in a Hilbert space. In fact, to show that $F(T)$ is closed, let us take a sequence $\{z_n\} \subset F(T)$ such that $z_n \to z_0$. Since C is closed and convex, C is weakly closed and hence $z_0 \in C$. We also have

$$\|z_0 - Tz_0\| \leq \|z_0 - z_n\| + \|z_n - Tz_0\| \leq 2\|z_0 - z_n\|$$

for $n \in \mathbb{N}$. Tending $n \to \infty$, we have that $z_0 \in F(T)$ and hence $F(T)$ is closed. To show that $F(T)$ is convex, let us take $z_1, z_2 \in F(T)$ and $\lambda \in [0,1]$, and put $z_0 = \lambda z_1 + (1-\lambda)z_2$. Then we have from (2.1) that

$$\begin{aligned}
\|z_0 - Tz_0\|^2 &= \|\lambda z_1 + (1-\lambda)z_2 - Tz_0\|^2 \\
&= \|\lambda(z_1 - Tz_0) + (1-\lambda)(z_2 - Tz_0)\|^2 \\
&= \lambda\|z_1 - Tz_0\|^2 + (1-\lambda)\|z_2 - Tz_0\|^2 - \lambda(1-\lambda)\|z_1 - z_2\|^2 \\
&\leq \lambda\|z_1 - z_0\|^2 + (1-\lambda)\|z_2 - z_0\|^2 - \lambda(1-\lambda)\|z_1 - z_2\|^2 \\
&= \lambda(1-\lambda)^2\|z_1 - z_2\|^2 + \lambda^2(1-\lambda)\|z_1 - z_2\|^2 - \lambda(1-\lambda)\|z_1 - z_2\|^2 \\
&= \lambda(1-\lambda)(1-\lambda+\lambda-1)\|z_1 - z_2\|^2 = 0
\end{aligned}$$

and hence $z_0 \in F(T)$. So, $F(T)$ is convex.

Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \geq 0$$

for all $x \in H$ and $u \in C$. Furthermore, we know that

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle \tag{2.3}$$

for all $x, y \in H$; see [27] for more details. The following lemma was proved by Takahashi and Toyoda [31].

Lemma 2.1. *Let D be a nonempty closed convex subset of a real Hilbert space H. Let P be the metric projection of H onto D and let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in D$ and $n \in \mathbb{N}$, then $\{Px_n\}$ converges strongly.*

Let C be a nonempty closed convex subset of H. Then, we know that a mapping $T : C \to H$ is called *generalized hybrid* [15] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{2.4}$$

for all $x, y \in C$. We can show that if $x = Tx$, then for any $y \in C$,

$$\alpha\|x - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|x - y\|^2 + (1-\beta)\|x - y\|^2$$

and hence

$$\|x - Ty\| \leq \|x - y\|. \tag{2.5}$$

This means that an $(\alpha,\ \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive. A mapping $S : C \to H$ is *super hybrid* [15, 33] if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\|Sx - Sy\|^2 &+ (1 - \alpha + \gamma)\|x - Sy\|^2 \\ &\leq \big(\beta + (\beta - \alpha)\gamma\big)\|Sx - y\|^2 + \big(1 - \beta - (\beta - \alpha - 1)\gamma\big)\|x - y\|^2 \\ &+ (\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2\end{aligned}$$

for all $x, y \in C$. We call such a mapping an $(\alpha,\ \beta,\ \gamma)$-*super hybrid* mapping. An $(\alpha,\ \beta,\ 0)$-super hybrid mapping is $(\alpha,\ \beta)$-generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings. Kocourek, Takahashi and Yao [15] also proved the following fixed point theorem for super hybrid mappings in a Hilbert space.

Theorem 2.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be an $(\alpha,\ \beta,\ \gamma)$-super hybrid mapping. Then, S has a fixed point in C. In particular, if $S : C \to C$ is an $(\alpha,\ \beta)$-generalized hybrid mapping, then S has a fixed point in C.*

We also know a fixed point theorem [10] for generalized hybrid non-self mappings in a Hilbert space.

Theorem 2.3. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let α and β be real numbers. Let T be an (α, β)-generalized hybrid mapping of C into H with $\alpha - \beta \geq 0$. Suppose that there exists $m > 1$ such that for any $x \in C$, $Tx = x + t(y - x)$ for some $y \in C$ and t with $1 \leq t \leq m$. Then, T has a fixed point in C.*

To prove one of our main results, we need the following lemma [2]:

Lemma 2.4. *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [25].

For a sequence $\{C_n\}$ of nonempty closed convex subsets of a Hilbert space H, define s-Li$_n C_n$ and w-Ls$_n C_n$ as follows: $x \in$s-Li$_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$w-Ls$_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies that

$$C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n, \tag{2.6}$$

we say that $\{C_n\}$ converges to C_0 in the sense of Mosco [21] and we write $C_0 =$M-$\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [21]. We know the following theorem [34].

Theorem 2.5. *Let H be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. If $C_0 =M$-$\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in H$, $P_{C_n}x$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of H onto C_n and C_0, respectively.*

3. Extended hybrid mappings

Let H be a Hilbert space and let C be a nonempty closed convex subset of H. We recall that a mapping $U : C \to H$ is called *extended hybrid* [10] if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} \alpha(1+\gamma)\|Ux - Uy\|^2 &+ (1-\alpha(1+\gamma))\|x - Uy\|^2 \\ &\le (\beta+\alpha\gamma)\|Ux - y\|^2 + (1-(\beta+\alpha\gamma))\|x-y\|^2 \\ &\quad - (\alpha-\beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned} \tag{3.1}$$

for all $x, y \in C$ and such a mapping U is called $(\alpha,\ \beta,\ \gamma)$-extended hybrid. In [10], the authors derived a relation between the class of generalized hybrid mappings and the class of extended hybrid mappings in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let T and U be mappings of C into H such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$, where $Ix = x$ for all $x \in H$. Then, for $1+\gamma > 0$, $T : C \to H$ is an (α, β)-generalized hybrid mapping if and only if $U : C \to H$ is an $(\alpha, \beta,\ \gamma)$-extended hybrid mapping. In this case, $F(T) = F(U)$.*

In this section, we first prove a fixed point theorem for strict pseudo-contractions in a Hilbert space.

Theorem 3.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \le k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. Then, U is a (1,0,-k)-extended hybrid mapping and $F(U)$ is closed and convex. If, in addition, C is bounded and U is a mapping of C into itself, then $F(U)$ is nonempty.*

Proof. Let $U : C \to H$ be a k-strict pseudo-contraction. Then, $0 \le k < 1$ and

$$\|Ux - Uy\|^2 \le \|x - y\|^2 + k\|(I-U)x - (I-U)y\|^2 \tag{3.2}$$

for all $x, y \in C$. So, we have from (2.2) that for all $x, y \in C$,

$$\begin{aligned}\|Ux - Uy\|^2 &\le \|x-y\|^2 + k\|(I-U)x - (I-U)y\|^2 \\ &= \|x-y\|^2 + k\|x - y - (Ux - Uy)\|^2 \\ &= \|x-y\|^2 + k(\|x-y\|^2 + \|Ux-Uy\|^2 - 2\langle x-y, Ux-Uy\rangle) \\ &= \|x-y\|^2 + k(\|x-y\|^2 + \|Ux-Uy\|^2 \\ &\quad - \|x-Uy\|^2 - \|y-Ux\|^2 + \|x-Ux\|^2 + \|y-Uy\|^2)\end{aligned}$$

and hence

$$\begin{aligned}(1-k)\|Ux - Uy\|^2 + k\|x - Uy\|^2 &\le -k\|Ux - y\|^2 \\ &\quad + (1+k)\|x-y\|^2 + k\|x-Ux\|^2 + k\|y-Uy\|^2. \qquad (3.3)\end{aligned}$$

Putting $\alpha = 1$, $\beta = 0$ and $\gamma = -k$ in (3.1), we get (3.3). Then, U is a (1,0,-k)-extended hybrid mapping. Furthermore, putting $T = (1-k)U + kI$, where $Ix = x$ for all $x \in H$, we have that

$$U = \frac{1}{1-k}T + \frac{-k}{1-k}I.$$

Using $1 + \gamma = 1 - k > 0$ and Theorem 3.1, we have that T is a (1,0)-generalized hybrid mapping, i.e., a nonexpansive mapping. So, $F(T)$ is closed and convex. From $F(T) = F(U)$, $F(U)$ is also closed and convex. Since C is a bounded closed convex set and T is a nonexpansive mapping of C into itself, $F(T)$ is nonempty; see [27]. Hence $F(U)$ is nonempty. □

In general, we have the following fixed point theorem for extended hybrid mappings in a Hilbert space.

Theorem 3.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β, γ be real numbers. Let $U : C \to H$ be an (α, β, γ)-extended hybrid mapping with $1 + \gamma > 0$. Then $F(U)$ is closed and convex. If, in addition, C is bounded, $0 \le -\gamma < 1$ and U is a mapping of C into itself, then $F(U) \ne \emptyset$.*

Proof. Let $U : C \to H$ be an (α, β, γ)-extended hybrid mapping with $1 + \gamma > 0$. Putting $T = (1+\gamma)U - \gamma I$, we have

$$U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I.$$

From Theorem 3.1, we have that T is an (α, β)-generalized hybrid mapping of C into H. If $F(U) \ne \emptyset$, then $F(T) \ne \emptyset$ from $F(U) = F(T)$. Then we have from (2.5) that $T : C \to H$ is quasi-nonexpansive. So, we have that $F(T)$ is closed and convex and hence $F(U)$ is closed and convex. If $F(U) = \emptyset$, it is obvious that $F(U)$ is closed and convex. Let $U : C \to C$ be an (α, β, γ)-extended hybrid mapping with $0 \le -\gamma < 1$. We note that if $0 \le -\gamma < 1$, then $1 + \gamma > 0$. Since $0 \le -\gamma < 1$ and $T = (1+\gamma)U - \gamma I$, we have from Theorem 3.1 that T is an (α, β)-generalized hybrid mapping of C into itself. Using Theorem 2.2, we have $F(T) \ne \emptyset$. So, $F(U) \ne \emptyset$. □

Using Theorem 3.3, we have the following fixed point theorem.

Theorem 3.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \leq k < 1$. Let $U : C \to H$ be a mapping such that*

$$2\|Ux - Uy\|^2 \leq \|x - Uy\|^2 + \|Ux - y\|^2 + k(\|(I-U)x - (I-U)y\|^2 - 2\langle x - Ux, y - Uy\rangle) \tag{3.4}$$

for all $x, y \in C$. Then, $F(U)$ is closed and convex. In addition, if C is bounded and U is a mapping C into itself, then $F(U) \neq \emptyset$.

Proof. Using (2.2), we have that the inequality (3.4) is equivalent to

$$\begin{aligned} 2(1-k)\|Ux - Uy\|^2 + (-1+2k)\|x - Uy\|^2 &\leq (1-2k)\|Ux - y\|^2 + 2k\|x - y\|^2 \\ &\quad + k\|x - Ux\|^2 + k\|y - Uy\|^2. \end{aligned} \tag{3.5}$$

On the other hand, putting $\alpha = 2$, $\beta = 1$ and $\gamma = -k$ in (3.1), we get this inequality (3.5). So, U is a (2,1,-k)-extended hybrid mapping. Using $0 \leq k < 1$ and Theorem 3.3, we have the desired result. □

For example, taking $k = \frac{1}{2}$ in (3.4), we obtain that

$$2\|Ux - Uy\|^2 \leq 2\|x - y\|^2 + \|x - Ux\|^2 + \|y - Uy\|^2$$

for all $x, y \in C$. Using Theorem 3.4, we have that such a mapping U has a fixed point in C if C is bounded, closed and convex. Furthermore, $F(U)$ is closed and convex.

We also have the following important result for extended hybrid mappings in a Hilbert space.

Theorem 3.5. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β, γ be real numbers and let $U : C \to H$ be an (α, β, γ)-extended hybrid mapping with $1 + \gamma > 0$. Then, $I - U$ is demiclosed, i.e., $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$ imply $z \in F(U)$.*

Proof. Since $U : C \to H$ is extended hybrid, there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} \alpha(1+\gamma)\|Ux - Uy\|^2 + (1 - \alpha(1+\gamma))\|x - Uy\|^2 &\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ &\quad - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned}$$

for all $x, y \in C$. Suppose $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$. Let us consider

$$\begin{aligned} \alpha(1+\gamma)\|Ux_n - Uz\|^2 + (1 - \alpha(1+\gamma))\|x_n - Uz\|^2 &\leq (\beta + \alpha\gamma)\|Ux_n - z\|^2 + (1 - (\beta + \alpha\gamma))\|x_n - z\|^2 \\ &\quad - (\alpha - \beta)\gamma\|x_n - Ux_n\|^2 - \gamma\|z - Uz\|^2. \end{aligned}$$

From this inequality, we have

$$\begin{aligned} \alpha(1+\gamma)\|Ux_n - x_n + x_n - Uz\|^2 + (1 - \alpha(1+\gamma))\|x_n - Uz\|^2 \\ \leq (\beta + \alpha\gamma)\|Ux_n - x_n + x_n - z\|^2 + (1 - (\beta + \alpha\gamma))\|x_n - z\|^2 \end{aligned}$$

$$- (\alpha - \beta)\gamma\|x_n - Ux_n\|^2 - \gamma\|z - Uz\|^2.$$

We apply a Banach limit μ to both sides of the inequality. Then, we have

$$\begin{aligned}\alpha(1+\gamma)\mu_n\|Ux_n - x_n + x_n - Uz\|^2 + (1-\alpha(1+\gamma))\mu_n\|x_n - Uz\|^2 \\ \le (\beta+\alpha\gamma)\mu_n\|Ux_n - x_n + x_n - z\|^2 + (1-(\beta+\alpha\gamma))\mu_n\|x_n - z\|^2 \\ - (\alpha-\beta)\gamma\mu_n\|x_n - Ux_n\|^2 - \gamma\mu_n\|z - Uz\|^2.\end{aligned}$$

We know from the properties of μ that

$$\begin{aligned}\mu_n\|Ux_n - x_n + x_n - Uz\|^2 \\ &= \mu_n(\|Ux_n - x_n\|^2 + \|x_n - Uz\|^2 + 2\langle Ux_n - x_n, x_n - Uz\rangle) \\ &= \mu_n\|Ux_n - x_n\|^2 + \mu_n\|x_n - Uz\|^2 + 2\mu_n\langle Ux_n - x_n, x_n - Uz\rangle \\ &= \mu_n\|x_n - Uz\|^2\end{aligned}$$

and $\mu_n\|Ux_n - x_n + x_n - z\|^2 = \mu_n\|x_n - z\|^2$. So, we have

$$\begin{aligned}\alpha(1+\gamma)\mu_n\|x_n - Uz\|^2 + (1-\alpha(1+\gamma))\mu_n\|x_n - Uz\|^2 \\ \le (\beta+\alpha\gamma)\mu_n\|x_n - z\|^2 + (1-(\beta+\alpha\gamma))\mu_n\|x_n - z\|^2 \\ - \gamma\|z - Uz\|^2\end{aligned}$$

and hence

$$\mu_n\|x_n - Uz\|^2 \le \mu_n\|x_n - z\|^2 - \gamma\|z - Uz\|^2.$$

From $\mu_n\|x_n - Uz\|^2 = \mu_n\|x_n - z + z - Uz\|^2 = \mu_n\|x_n - z\|^2 + \|z - Uz\|^2$, we also have

$$\mu_n\|x_n - z\|^2 + \|z - Uz\|^2 \le \mu_n\|x_n - z\|^2 - \gamma\|z - Uz\|^2.$$

Hence, we obtain $(1+\gamma)\|z - Uz\|^2 \le 0$. Since $1+\gamma > 0$, we have $\|z - Uz\|^2 \le 0$. Then, $Uz = z$. This implies that $I - U$ is demiclosed. □

Using Theorems 3.2 and 3.6, we have the following result obtained by Marino and Xu [20]; see also [1].

Corollary 3.6. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \le k < 1$ and $U : C \to H$ be a k-strict pseudo-contraction. Then, $I - U$ is demiclosed, i.e., $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$ imply $z \in F(U)$.*

Proof. We know from Theorem 3.2 that a k-strict pseudo-contraction $U : C \to H$ is (1,0,-k)-entended hybrid. Furthermore, $0 \le k < 1$ implies $1 + \gamma = 1 - k > 0$. So, we have the desired result from Theorem 3.6. □

4. Nonlinear ergodic theorem

In this section, using the technique developed in [24], [29] and [32], we prove a nonlinear ergodic theorem of Baillon's type [3] for extended hybrid mappings in a Hilbert space. For proving it, we need the following two lemmas proved by Takahashi and Yao and Kocourek [33] and Hojo, Takahashi and Yao [10].

Lemma 4.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to H$ be a generalized hybrid mapping. Suppose that there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Then, $z \in F(T)$.*

Lemma 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a generalized hybrid mapping from C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Define $S_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x$. Then, $\lim_{n\to\infty} \|S_n x - TS_n x\| = 0$. In particular, if C is bounded, then*

$$\lim_{n\to\infty} \sup_{x\in C} \|S_n x - TS_n x\| = 0.$$

Theorem 4.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β and γ be real numbers and let $U : C \to C$ be an (α, β, γ)-extended hybrid mapping such that $0 \le -\gamma < 1$ and $F(U) \neq \emptyset$. Let P be the mertic projection of H onto $F(U)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=1}^{n}((1+\gamma)U - \gamma I)^k x$$

converges weakly to $z \in F(U)$, where $z = \lim_{n\to\infty} PT^n x$ and $T = (1+\gamma)U - \gamma I$.

Proof. Put $T = (1+\gamma)U - \gamma I$. Since $0 \le -\gamma < 1$, we have from Theorem 3.1 that T is an (α, β)-generalized hybrid mapping of C into itself, i.e.,

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{4.1}$$

for all $x, y \in C$. Since T is a generalized hybrid mapping and $F(T) = F(U) \neq \emptyset$, T is quasi-nonexpansive. So, $F(T)$ is closed and convex. Let $x \in C$ and $u \in F(T)$. Then, we have $\|T^{n+1}x - u\| \le \|T^n x - u\|$. Putting $D = F(T)$ in Lemma 2.1, we have that $\lim_{n\to\infty} PT^n x$ converges strongly. Put $z = \lim_{n\to\infty} PT^n x$. Let us show $S_n x \rightharpoonup z$. Since $\{T^n x\}$ is bounded, so is $\{S_n x\}$. Let $\{S_{n_i} x\}$ be a subsequence of $\{S_n x\}$ such that $S_{n_i} x \rightharpoonup v$. By Lemma 4.2, we know $\lim_{n\to\infty} \|S_n x - TS_n x\| = 0$. Using Lemma 4.1, we have $v = Tv$. To show $S_n x \rightharpoonup z$, it is sufficient to prove $z = v$. From $v \in F(T)$, we have

$$\begin{aligned}\langle v - z, T^k x - PT^k x\rangle &= \langle v - PT^k x, T^k x - PT^k x\rangle + \langle PT^k x - z, T^k x - PT^k x\rangle \\ &\le \langle PT^k x - z, T^k x - PT^k x\rangle \\ &\le \|PT^k x - z\|\|T^k x - PT^k x\| \\ &\le \|PT^k x - z\|L\end{aligned}$$

for all $k \in \mathbb{N}$, where $L = \sup\{\|T^k x - PT^k x\| : k \in \mathbb{N}\}$. Summing these inequalities from $k = 1$ to n_i and dividing by n_i, we have

$$\left\langle v - z, S_{n_i} x - \frac{1}{n_i}\sum_{k=1}^{n_i} PT^k x\right\rangle \le \frac{1}{n_i}\sum_{k=1}^{n_i} \|PT^k x - z\|L.$$

Since $S_{n_i} x \rightharpoonup v$ as $i \to \infty$ and $PT^n x \to z$ as $n \to \infty$, we have $\langle v - z, v - z\rangle \le 0$. This implies $z = v$. Therefore, $\{S_n x\}$ converges weakly to $z \in F(T) = F(U)$, where $z = \lim_{n\to\infty} PT^n x$. So, we get the desired result. □

Using Theorem 4.3, we obtain the following corollary.

Corollary 4.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \leq k < 1$ and $U : C \to C$ be a k-strict pseudo-contraction and $F(U) \neq \emptyset$. Let P be the mertic projection of H onto $F(U)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{m=1}^{n} ((1-k)U + kI)^m x$$

converges weakly to $z \in F(U)$, where $z = \lim_{n\to\infty} PT^n x$ and $T = (1-k)U + kI$.

Proof. We know from Theorem 3.2 that a k-strict pseudo-contraction $U : C \to C$ is (1,0,-k)-entended hybrid. Furthermore, $0 \leq k < 1$ and $-\gamma = k$ imply $0 \leq -\gamma < 1$. So, we have the desired result from Theorem 4.3. □

5. Weak convergence theorem of Mann's type

In this section, we prove a weak convergence theorem of Mann's type [19] for extended hybrid mappings in a Hilbert space. Before proving the theorem, we need the following lemma proved by Takahashi, Yao and Kocourek [33].

Lemma 5.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let α, β and γ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an (α, β, γ)-super hybrid mapping with $F(S) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = P_C\{\alpha_n x_n + (1-\alpha_n)(\frac{1}{1+\gamma} S x_n + \frac{\gamma}{1+\gamma} x_n)\}, \quad n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element v of $F(S)$, where $v = \lim_{n\to\infty} P_{F(S)} x_n$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Theorem 5.2. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let α, β and γ be real numbers. Let $U : C \to H$ be an (α, β, γ)-extended hybrid mapping such that $1 + \gamma > 0$ and $F(U) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = P_C\{\alpha_n x_n + (1-\alpha_n)((1+\gamma)U x_n - \gamma x_n)\}, \quad n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element v of $F(U)$, where $v = \lim_{n\to\infty} P_{F(U)} x_n$ and $P_{F(U)}$ is the metric projection of H onto $F(U)$.

Proof. Put $T = (1+\gamma)U - \gamma I$. Then, we have from $1 + \gamma > 0$ and Theorem 3.1 that $T : C \to H$ is an (α, β)-generalized hybrid mapping and $F(U) = F(T) \neq \emptyset$. Furthermore, we have that

$$x_{n+1} = P_C\{\alpha_n x_n + (1-\alpha_n) T x_n\}, \quad n \in \mathbb{N}.$$

Using Lemma 5.1 with $\gamma = 0$, we have that $\{x_n\}$ converges weakly to an element v of $F(T)$, where $v = \lim_{n\to\infty} P_{F(T)} x_n$ and $P_{F(T)}$ is the metric projection of H onto $F(T) = F(U)$. □

As direct consequences of Theorem 5.2, we obtain the following results.

Corollary 5.3. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let γ be a real number with $1+\gamma>0$ and let $U : C \to H$ be an $(2,\, 1,\, \gamma)$-extended hybrid mapping, i.e.,*

$$\begin{aligned}2(1+\gamma)\|Ux-Uy\|^2 - (1+2\gamma)\|x-Uy\|^2 \\ \le (1+2\gamma)\|Ux-y\|^2 - 2\gamma\|x-y\|^2 \\ -\gamma\|x-Ux\|^2 - \gamma\|y-Uy\|^2\end{aligned}$$

for all $x, y \in C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C\big\{\alpha_n x_n + (1-\alpha_n)((1+\gamma)Ux_n - \gamma x_n)\big\}, \quad n \in \mathbb{N}.$$

If $F(U) \ne \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element v of $F(U)$, where $v = \lim_{n\to\infty} P_{F(U)}x_n$ and $P_{F(U)}$ is the metric projection of H onto $F(U)$.

Corollary 5.4. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let γ be a real number with $1+\gamma>0$ and let $U : C \to H$ be an $(\frac{3}{2},\, \frac{1}{2},\, \gamma)$-extended hybrid mapping, i.e.,*

$$\begin{aligned}3(1+\gamma)\|Ux-Uy\|^2 - (1+3\gamma))\|x-Uy\|^2 \\ \le (1+3\gamma)\|Ux-y\|^2 + (1-3\gamma))\|x-y\|^2 \\ -2\gamma\|x-Ux\|^2 - 2\gamma\|y-Uy\|^2\end{aligned}$$

for all $x, y \in C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C\big(\alpha_n x_n + (1-\alpha_n)((1+\gamma)Ux_n - \gamma x_n)\big), \quad n \in \mathbb{N}.$$

If $F(U) \ne \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element v of $F(U)$, where $v = \lim_{n\to\infty} P_{F(U)}x_n$ and $P_{F(U)}$ is the metric projection of H onto $F(U)$.

Taking $\gamma = -\frac{1}{2}$ in Corollaries 5.3 and 5.4, we obtain two mappings such that

$$2\|Ux-Uy\|^2 \le 2\|x-y\|^2 + \|x-Ux\|^2 + \|y-Uy\|^2$$

and

$$\begin{aligned}3\|Ux-Uy\|^2 + \|x-Uy\|^2 + \|y-Ux\|^2 \\ \le 5\|x-y\|^2 + 2\|x-Ux\|^2 + 2\|y-Uy\|^2\end{aligned}$$

for all $x, y \in C$, respectively. We can apply Corollaries 5.3 and 5.4 for such mappings and then obtain weak convergence theorems in a Hilbert space. Next, we prove a weak convergence theorem of Mann's type for a class of non-self mappings containing the class of nonexpansive mappings in a Hilbert space. For proving it, we state the following lemma proved by Takahashi, Yao and Kocourek [33].

Lemma 5.5. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let γ be a real number with $\gamma \ne -1$ and let $S : C \to H$ be a mapping such that*

$$\|Sx-Sy\|^2 + 2\gamma\langle x-y, Sx-Sy\rangle \le (1+2\gamma)\|x-y\|^2$$

for all $x, y \in C$. *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \leq \alpha_n \leq 1$ *and* $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. *Suppose* $\{x_n\}$ *is a sequence generated by* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)P_C\big(\frac{1}{1+\gamma}Sx_n + \frac{\gamma}{1+\gamma}x_n\big), \quad n = 1, 2, \ldots.$$

If $F(S) \neq \emptyset$, *then the sequence* $\{x_n\}$ *converges weakly to an element* v *of* $F(S)$, *where* $v = \lim_{n\to\infty} P_{F(S)}x_n$ *and* $P_{F(S)}$ *is the metric projection of* H *onto* $F(S)$.

Theorem 5.6. *Let* H *be a Hilbert space, let* C *be a nonempty closed convex subset of* H *and let* P_C *be the metric projection of* H *onto* C. *Let* α, β *and* γ *be real numbers. Let* γ *be a real number with* $1+\gamma > 0$ *and let* $U : C \to H$ *be a mapping with* $F(U) \neq \emptyset$ *such that*

$$\|Ux - Uy\|^2 \leq \|x-y\|^2 - \gamma\|(I-U)x - (I-U)y\|^2$$

for all $x, y \in C$. *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \leq \alpha_n \leq 1$ *and* $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. *Suppose* $\{x_n\}$ *is a sequence generated by* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)P_C\big((1+\gamma)Ux_n - \gamma x_n)\big), \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges weakly to an element* v *of* $F(U)$, *where* $v = \lim_{n\to\infty} P_{F(U)}x_n$ *and* $P_{F(U)}$ *is the metric projection of* H *onto* $F(U)$.

Proof. We have that for any $x, y \in C$,

$$\begin{aligned}
\|Ux - Uy\|^2 &\leq \|x-y\|^2 - \gamma(\|(I-U)x - (I-U)y\|^2 \\
&\iff \|Ux-Uy\|^2 \leq \|x-y\|^2 - \gamma(\|x-y\|^2 + \|Ux-Uy\|^2 \\
&\qquad - \|x-Uy\|^2 - \|Ux-y\|^2 + \|Ux-x\|^2 + \|y-Uy\|^2) \\
&\iff (1+\gamma)\|Ux-Uy\|^2 - \gamma\|x-Uy\|^2 \\
&\qquad \leq \gamma\|Ux-y\|^2 + (1-\gamma)\|x-y\|^2 - \gamma\|Ux-x\|^2 - \gamma\|y-Uy\|^2.
\end{aligned}$$

Thus, U is a $(1, 0, \gamma)$-extended hybrid mapping with $1+\gamma > 0$. Put $T = (1+\gamma)U - \gamma I$. Then, we have from Theorem 3.1 that $T : C \to H$ is an (1, 0)-generalized hybrid mapping, i.e., a nonexpansive mapping and $F(U) = F(T) \neq \emptyset$. Using Lemma 5.5 with $\gamma = 0$ or Reich's theorem [23], we have that $\{x_n\}$ converges weakly to an element v of $F(T)$, where $v = \lim_{n\to\infty} P_{F(T)}x_n$ and $P_{F(T)}$ is the metric projection of H onto $F(T) = F(U)$. □

As a direct consequence of Theorem 5.6, we have the following corollary.

Corollary 5.7. *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* k *be a real number with* $0 \leq k < 1$ *and* $U : C \to C$ *be a* k*-strict pseudo-contraction and* $F(U) \neq \emptyset$. *Let* P *be the mertic projection of* H *onto* $F(U)$. *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \leq \alpha_n \leq 1$ *and* $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. *Suppose* $\{x_n\}$ *is a sequence generated by* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\{(1-k)Ux_n + kx_n\}, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges weakly to an element* v *of* $F(U)$, *where* $v = \lim_{n\to\infty} P_{F(U)}x_n$ *and* $P_{F(U)}$ *is the metric projection of* H *onto* $F(U)$.

Proof. We know from Theorem 3.2 that a k-strict pseudo-contraction $U : C \to H$ is (1,0,-k)-entended hybrid. Furthermore, $0 \leq k < 1$ and $-\gamma = k$ imply $1 + \gamma > 0$. So, we have the desired result from Theorem 5.6. □

Using Corollary 5.7, we prove a weak convergence theorem of Mann's type for strict pseudo-contractions which was obtained by Marino and Xu [20]; see also [1].

Theorem 5.8. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \leq k < 1$ and $U : C \to C$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Let $\{\beta_n\}$ be a sequence of real numbers such that $k < \beta_n < 1$ and $\sum_{n=1}^{\infty}(\beta_n - k)(1 - \beta_n) = \infty$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)Ux_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges weakly to an element v of $F(U)$.

Proof. We have that for any $n \in \mathbb{N}$,

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n)Ux_n \\ &= \frac{\beta_n - k}{1 - k}x_n + (1 - \frac{\beta_n - k}{1 - k})\{(1 - k)Ux_n + kx_n\}. \end{aligned}$$

Putting $\alpha_n = \frac{\beta_n - k}{1-k}$, we have from $1 > \beta_n > k$ that $1 - k > \beta_n - k > 0$ and hence $1 > \frac{\beta_n - k}{1-k} = \alpha_n > 0$. Furthermore, we have that

$$\begin{aligned} &\sum_{n=1}^{\infty}(\beta_n - k)(1 - \beta_n) = \infty \\ \Longleftrightarrow &\sum_{n=1}^{\infty}(1 - k)\alpha_n(1 - k)(1 - \alpha_n) = \infty \\ \Longleftrightarrow &(1 - k)^2 \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty \\ \Longleftrightarrow &\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty. \end{aligned}$$

From Corollary 5.7, we have the desired result. □

6. Strong convergence theorems

In this section, we first prove a strong convergence theorem of Halpern's type [9] for extended hybrid mappings in a Hilbert space.

Theorem 6.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let γ be a real number with $1+\gamma > 0$ and let $U : C \to H$ be a mapping such that*

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 - \gamma\|(I - U)x - (I - U)y\|^2$$

for all $x, y \in C$. *Let* $\{\alpha_n\} \subset [0,1]$ *be a sequence of real numbers such that* $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. *Suppose* $\{x_n\}$ *is a sequence generated by* $x_1 = x \in C$, $u \in C$ *and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) P_C\big\{(1+\gamma)Ux_n - \gamma x_n\big\}, \quad n \in \mathbb{N}.$$

If $F(U) \neq \emptyset$, *then the sequence* $\{x_n\}$ *converges strongly to an element* v *of* $F(U)$, *where* $v = P_{F(U)}u$ *and* $P_{F(U)}$ *is the metric projection of* H *onto* $F(U)$.

Proof. As in the proof of Theorem 5.6, we have that U is a $(1,\ 0,\ \gamma)$-extended hybrid mapping of C into H. Put $T = (1+\gamma)U - \gamma I$. Then, we have from Theorem 3.1 that T is a $(1,\ 0)$-generalized hybrid mapping of C into H, i.e., T is a nonexpansive mapping of C into H. Furthermore, we have $F(U) = F(T)$. From Wittmann's theorem [35], we obtain $x_n \to P_{F(P_C T)}u$; see also Takahashi [26]. Let us show $F(P_C T) = F(T) = F(U)$. We know $F(T) = F(U)$. It is obvious that $F(T) \subset F(P_C T)$. We show $F(P_C T) \subset F(T)$. If $P_C T v = v$, we have from the property of P_C that for $u \in F(T)$,

$$\begin{aligned} 2\|v-u\|^2 &= 2\|P_C Tv - u\|^2 \\ &\leq 2\langle Tv - u, P_C Tv - u\rangle \\ &= \|Tv-u\|^2 + \|P_C Tv - u\|^2 - \|Tv - P_C Tv\|^2 \end{aligned}$$

and hence

$$2\|v-u\|^2 \leq \|v-u\|^2 + \|v-u\|^2 - \|Tv - v\|^2.$$

Then, we have $0 \leq -\|Tv - v\|^2$ and hence $Tv = v$. This completes the proof. □

As a direct consequence of Theorem 6.1, we have the following corollary.

Corollary 6.2. *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* k *be a real number with* $0 \leq k < 1$ *and* $U : C \to C$ *be a* k*-strict pseudo-contraction with* $F(U) \neq \emptyset$. *Let* P *be the mertic projection of* H *onto* $F(U)$. *Let* $\{\alpha_n\} \subset [0,1]$ *be a sequence of real numbers such that* $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. *Suppose* $\{x_n\}$ *is a sequence generated by* $x_1 = x \in C$, $u \in C$ *and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\big\{(1-k)Ux_n + kx_n\big\}, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges strongly to an element* v *of* $F(U)$, *where* $v = P_{F(U)}u$ *and* $P_{F(U)}$ *is the metric projection of* H *onto* $F(U)$.

Next, using an idea of mean convergence and the method of the proof in [18], we prove a strong convergence theorem of Halpern's type for extended hybrid mappings in a Hilbert space.

Theorem 6.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* α, β *and* k *be real numbers. Let* $U : C \to C$ *be a* $(\alpha,\ \beta,\ -k)$*-extended hybrid mapping such that* $0 \leq k < 1$ *and* $F(U) \neq \emptyset$ *and let* P *be the metric projection of* H *onto* $F(U)$. *Suppose* $\{x_n\}$ *is a sequence generated by* $x_1 = x \in C$, $u \in C$ *and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{m=1}^{n} ((1-k)U + kI)^m x_n \end{cases}$$

for all $n = 1, 2, \ldots,$ *where* $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then* $\{x_n\}$ *converges strongly to* Pu.

Proof. For an $(\alpha,\ \beta,\ -k)$-extended hybrid mapping $U : C \to C$, define

$$T = (1-k)U + kI.$$

Then, we have from Theorem 3.1 that $T : C \to C$ is an $(\alpha,\ \beta)$-generalized hybrid mapping such that $F(T) = F(U)$. Since $F(T) = F(U)$ is nonempty, we take $q \in F(T)$. Put $r = \|u - q\|$. We define

$$D = \{y \in H : \|y - q\| \le r\} \cap C.$$

Then D is a nonempty bounded closed convex subset of C. Furthermore, D is T-invariant and contains u. Thus we may assume that C is bounded without loss of generality. Since T is quasi-nonexpansive, we have that for all $q \in F(T)$ and $n = 1, 2, 3, \ldots,$

$$\begin{aligned} \|z_n - q\| &= \left\| \frac{1}{n} \sum_{m=1}^{n} T^m x_n - q \right\| \le \frac{1}{n} \sum_{m=1}^{n} \|T^m x_n - q\| \\ &\le \frac{1}{n} \sum_{m=1}^{n} \|x_n - q\| = \|x_n - q\|. \end{aligned} \tag{6.1}$$

Let us show $\limsup_{n\to\infty} \langle u - Pu, z_n - Pu\rangle \le 0$. Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ with $z_{n_i} \rightharpoonup v$. We may assume without loss of generality

$$\limsup_{n\to\infty} \langle u - Pu, z_n - Pu\rangle = \lim_{i\to\infty} \langle u - Pu, z_{n_i} - Pu\rangle.$$

By Lemma 4.2, we have $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. Using Lemma 4.1, we have $v \in F(T)$. Since P is the metric projection of H onto $F(T)$, we have

$$\lim_{i\to\infty} \langle u - Pu, z_{n_i} - Pu\rangle = \langle u - Pu, v - Pu\rangle \le 0.$$

This implies

$$\limsup_{n\to\infty} \langle u - Pu, z_n - Pu\rangle \le 0. \tag{6.2}$$

Since $x_{n+1} - Pu = (1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)$, from (6.1) we have

$$\begin{aligned} \|x_{n+1} - Pu\|^2 &= \|(1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)\|^2 \\ &\le (1-\alpha_n)^2 \|z_n - Pu\|^2 + 2\alpha_n \langle u - Pu, x_{n+1} - Pu\rangle \\ &\le (1-\alpha_n) \|x_n - Pu\|^2 + 2\alpha_n \langle u - Pu, x_{n+1} - Pu\rangle. \end{aligned}$$

Putting $s_n = \|x_n - Pu\|^2$, $\beta_n = 0$ and $\gamma_n = 2\langle u - Pu, x_{n+1} - Pu\rangle$ in Lemma 2.4, from $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (6.2) we have

$$\lim_{n\to\infty} \|x_n - Pu\| = 0.$$

This completes the proof. □

7. Strong convergence theorems by hybrid methods

In this section, using the hybrid method by Nakajo and Takahashi [22], we first prove a strong convergence theorem for extended hybrid non-self mappings in a Hilbert space. The method of the proof is due to Nakajo and Takahashi [22] and Marino and Xu [20].

Theorem 7.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β and k be real numbers and let $U : C \to H$ be an (α, β, $-k$)-extended hybrid mapping such that $k < 1$ and $F(U) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\{(1-k)Ux_n + kx_n\}, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|x_n - Ux_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset (-\infty, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = R_{F(U)}x$, where $P_{F(U)}$ is the metric projection of H onto $F(U)$.

Proof. Put $T = (1-k)U + kI$. We have $U = \frac{1}{1-k}T + \frac{-k}{1-k}I$. So, we have from Theorem 3.1 that T is an (α, β)-generalized hybrid mapping of C into H and $F(U) = F(T)$. Since $F(T)$ is closed and convex, $F(U)$ is closed and convex. So, there exists the mertic projection of H onto $F(U)$. Furthermore, we have

$$y_n = \alpha_n x_n + (1-\alpha_n)Tx_n$$

for all $n \in \mathbb{N}$. For any $z \in H$, the inequality

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|x_n - Ux_n\|^2$$

is equivalent to

$$2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|x_n - Ux_n\|^2.$$

So, we have that C_n, Q_n and $C_n \cap Q_n$ are closed and convex for all $n \in \mathbb{N}$. We next show that $C_n \cap Q_n$ is nonempty. Let $z \in F(T) = F(U)$. Since T is quasi-nonexpansive, we have that

$$\begin{aligned} \|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Tx_n - z\|^2 \\ &= \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|Tx_n - z\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \\ &= \|x_n - z\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|Ux_n - x_n\|^2. \end{aligned}$$

So, we have $z \in C_n$ and hence $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show by induction that $F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. From $F(T) \subset Q_1$, it follows that $F(T) \subset C_1 \cap Q_1$. Suppose that $F(T) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. From $x_{k+1} = P_{C_k\cap Q_k}x$, we have

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \geq 0, \quad \forall z \in C_k \cap Q_k.$$

Since $F(T) \subset C_k \cap Q_k$, we also have

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \geq 0, \quad \forall z \in F(T).$$

This implies $F(T) \subset Q_{k+1}$. So, we have $F(T) \subset C_{k+1} \cap Q_{k+1}$. By induction, we have $F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This means that $\{x_n\}$ is well-defined. Since $x_n \in C$ and $\langle x_n - x_n, x - x_n \rangle = 0$, we have $x_n \in Q_n$. Furthermore, from the definition of Q_n, we have $x_n = P_{Q_n} x$. Using $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \subset Q_n$, we have from (2.2) that

$$\begin{aligned} 0 &\le 2\langle x - x_n, x_n - x_{n+1} \rangle \\ &= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\ &\le \|x - x_{n+1}\|^2 - \|x - x_n\|^2. \end{aligned} \tag{7.1}$$

So, we get that

$$\|x - x_n\|^2 \le \|x - x_{n+1}\|^2. \tag{7.2}$$

Furthermore, since $x_n = P_{Q_n} x$ and $z \in F(T) \subset Q_n$, we have

$$\|x - x_n\|^2 \le \|x - z\|^2. \tag{7.3}$$

So, we have that $\lim_{n\to\infty} \|x - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{Tx_n\}$ is also bounded. From (7.1), we also have

$$\|x_n - x_{n+1}\|^2 \le \|x - x_{n+1}\|^2 - \|x - x_n\|^2$$

and hence

$$\|x_n - x_{n+1}\| \to 0. \tag{7.4}$$

From $x_{n+1} \in C_n$, we have that

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 - \alpha_n(1 - \alpha_n)\|x_n - Tx_n\|^2. \tag{7.5}$$

On the other hand, we know

$$\begin{aligned} \|y_n - x_{n+1}\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)Tx_n - x_{n+1}\|^2 \\ &= \alpha_n \|x_n - x_{n+1}\|^2 + (1 - \alpha_n)\|Tx_n - x_{n+1}\|^2 \\ &\quad - \alpha_n(1 - \alpha_n)\|x_n - Tx_n\|^2. \end{aligned} \tag{7.6}$$

From (7.5) and (7.6), we have

$$(1 - \alpha_n)\|Tx_n - x_{n+1}\|^2 \le (1 - \alpha_n)\|x_n - x_{n+1}\|^2.$$

Since $1 - \alpha_n > 0$, we have $\|Tx_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2$ and hence

$$\|Tx_n - x_{n+1}\| \to 0.$$

From

$$\|Tx_n - x_n\|^2 = \|Tx_n - x_{n+1}\|^2 + 2\langle Tx_n - x_{n+1}, x_{n+1} - x_n \rangle + \|x_{n+1} - x_n\|^2,$$

we also have

$$\|Tx_n - x_n\| \to 0. \tag{7.7}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup z^*$. From (7.7) and Lemma 4.1, we have $z^* \in F(T)$. Put $z_0 = P_{F(T)} x$. Since $z_0 = P_{F(T)} x \subset C_n \cap Q_n$ and $x_{n+1} = P_{C_n \cap Q_n} x$, we have that

$$\|x - x_{n+1}\|^2 \le \|x - z_0\|^2. \tag{7.8}$$

Since $\|\cdot\|^2$ is weakly lower semicontinuous, from $x_{n_i} \rightharpoonup z^*$ we have that

$$\begin{aligned}\|x - z^*\|^2 &= \|x\|^2 - 2\langle x, z^*\rangle + \|z^*\|^2 \\ &\leq \liminf_{i\to\infty}(\|x\|^2 - 2\langle x, x_{n_i}\rangle + \|x_{n_i}\|^2) \\ &= \liminf_{i\to\infty}\|x - x_{n_i}\|^2 \\ &\leq \|x - z_0\|^2.\end{aligned}$$

From the definition of z_0, we have $z^* = z_0$. So, we obtain $x_n \rightharpoonup z_0$. We finally show that $x_n \to z_0$. Since

$$\|z_0 - x_n\|^2 = \|z_0 - x\|^2 + \|x - x_n\|^2 + 2\langle z_0 - x, x - x_n\rangle, \quad \forall n \in \mathbb{N},$$

we have

$$\begin{aligned}\limsup_{n\to\infty}\|z_0 - x_n\|^2 &= \limsup_{n\to\infty}(\|z_0 - x\|^2 + \|x - x_n\|^2 + 2\langle z_0 - x, x - x_n\rangle) \\ &\leq \limsup_{n\to\infty}(\|z_0 - x\|^2 + \|x - z_0\|^2 + 2\langle z_0 - x, x - x_n\rangle) \\ &= \|z_0 - x\|^2 + \|x - z_0\|^2 + 2\langle z_0 - x, x - z_0\rangle \\ &= 0.\end{aligned}$$

So, we obtain $\lim_{n\to\infty}\|z_0 - x_n\| = 0$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. ☐

Using Theorem 7.1, we can prove the following theorem obtained by Marino and Xu [20].

Theorem 7.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \leq k < 1$ and let $U : C \to C$ be a k-strict pseudo contraction such that $F(U) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Ux_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (\beta_n - k)(1-\beta_n)\|x_n - Ux_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\beta_n\} \subset (-\infty, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = R_{F(U)}x$, where $P_{F(U)}$ is the metric projection of H onto $F(U)$.

Proof. We first know that a (1,0,-k)-extended hybrid mapping with $0 \leq k < 1$ is a k-strict pseudo contraction. We also have that for any $n \in \mathbb{N}$,

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)Ux_n \\ &= \frac{\beta_n - k}{1-k}x_n + (1 - \frac{\beta_n - k}{1-k})\{(1-k)Ux_n + kx_n\}. \end{aligned}$$

Putting $\alpha_n = \frac{\beta_n - k}{1-k}$, we have from $1 > \beta_n$ that $1-k > \beta_n - k$ and hence $1 > \frac{\beta_n - k}{1-k} = \alpha_n$. Furthermore, we have that for any $n \in \mathbb{N}$ and $z \in C$,

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 - (\beta_n - k)(1-\beta_n)\|x_n - Ux_n\|^2$$

$$\Longleftrightarrow \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-k)\alpha_n(1-k)(1-\alpha_n)\|x_n - Ux_n\|^2$$
$$\Longleftrightarrow \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|x_n - Ux_n\|^2.$$

From Theorem 7.1, we have the desired result. □

Next, we prove a strong convergence theorem by the shrinking projection method [30] for extended hybrid non-self mappings in a Hilbert space.

Theorem 7.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β and k be real numbers and let $U : C \to H$ be an $(\alpha, \beta, -k)$-extended hybrid mapping such that $k < 1$ and $F(U) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\{(1-k)Ux_n + kx_n\}, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|Ux_n - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1}, and $\{\alpha_n\} \subset (-\infty, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(U)}x$, where $P_{F(U)}$ is the metric projection of H onto $F(U)$.

Proof. Put $T = (1-k)U + kI$. Then, we have from Theorem 3.1 that T is an (α, β)-generalized hybrid mapping of C into H and $F(U) = F(T)$. Since $F(T)$ is closed and convex, so is $F(U)$. Then, there exists the mertic projection of H onto $F(U)$. Furthermore, we have

$$y_n = \alpha_n x_n + (1-\alpha_n)Tx_n$$

for all $n \in \mathbb{N}$. We show that C_n are closed and convex, and $F(T) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from the assumption that $C_1 = C$ is closed and convex, and $F(T) \subset C_1$. Suppose that C_k is closed and convex, and $F(T) \subset C_k$ for some $k \in \mathbb{N}$. As in the proof of Theorem 7.1, we know that for $z \in C_k$, the inequality

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|x_n - Ux_n\|^2$$

is equivalent to

$$2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|x_n - Ux_n\|^2.$$

Since C_k is closed and convex, so is C_{k+1}. Take $z \in F(T) \subset C_k$. Then we have from (2.2) that

$$\begin{aligned} \|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Tx_n - z\|^2 \\ &= \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|Tx_n - z\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|Ux_n - x_n\|^2. \end{aligned}$$

Hence, we have $z \in C_{k+1}$ and hence $F(T) \subset C_{k+1}$. By induction, we have that C_n are closed and convex, and $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Since C_n is closed and convex, there exists the metric projection P_{C_n} of H onto C_n. Thus, $\{x_n\}$ is well-defined. Since $\{C_n\}$ is a nonincreasing sequence of nonempty closed convex subsets of H with respect to inclusion, it follows that

$$\emptyset \neq F(T) \subset \text{M-}\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n. \tag{7.9}$$

Put $C_0 = \bigcap_{n=1}^\infty C_n$. Then, by Theorem 2.5 we have that $\{P_{C_n}x\}$ converges strongly to $x_0 = P_{C_0}x$, i.e.,

$$x_n = P_{C_n}x \to x_0.$$

To complete the proof, it is sufficient to show that $x_0 = P_{F(T)}x$. Since $x_n = P_{C_n}x$ and $x_{n+1} = P_{C_{n+1}}x \in C_{n+1} \subset C_n$, we have from (2.2) that

$$\begin{aligned} 0 &\le 2\langle x - x_n, x_n - x_{n+1}\rangle \\ &= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\ &\le \|x - x_{n+1}\|^2 - \|x - x_n\|^2. \end{aligned} \tag{7.10}$$

Thus, we get that

$$\|x - x_n\|^2 \le \|x - x_{n+1}\|^2. \tag{7.11}$$

Furthermore, since $x_n = P_{C_n}x$ and $z \in F(T) \subset C_n$, we have

$$\|x - x_n\|^2 \le \|x - z\|^2, \tag{7.12}$$

from which it follows that $\lim_{n\to\infty} \|x - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{Tx_n\}$ are also bounded. From (7.10), we have

$$\|x_n - x_{n+1}\|^2 \le \|x - x_{n+1}\|^2 - \|x - x_n\|^2.$$

So, we have that

$$\|x_n - x_{n+1}\| \to 0. \tag{7.13}$$

From $x_{n+1} \in C_{n+1}$, we also have that

$$\begin{aligned} \|y_n - x_{n+1}\|^2 &\le \|x_n - x_{n+1}\|^2 - (1-k)^2\alpha_n(1-\alpha_n)\|x_n - Ux_n\|^2 \\ &= \|x_n - x_{n+1}\|^2 - \alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2. \end{aligned} \tag{7.14}$$

On the other hand, we have from (2.2) that

$$\begin{aligned} \|y_n - x_{n+1}\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Tx_n - x_{n+1}\|^2 \\ &= \alpha_n\|x_n - x_{n+1}\|^2 + (1-\alpha_n)\|Tx_n - x_{n+1}\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2. \end{aligned} \tag{7.15}$$

From (7.14) and (7.15), we have

$$(1-\alpha_n)\|Tx_n - x_{n+1}\|^2 \le (1-\alpha_n)\|x_n - x_{n+1}\|^2.$$

Since $1 - \alpha_n > 0$, we have $\|Tx_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2$ and hence

$$\|Tx_n - x_{n+1}\| \to 0.$$

Since

$$\|Tx_n - x_n\|^2 = \|Tx_n - x_{n+1}\|^2 + 2\langle Tx_n - x_{n+1}, x_{n+1} - x_n\rangle + \|x_{n+1} - x_n\|^2,$$

we also have

$$\|Tx_n - x_n\| \to 0. \tag{7.16}$$

From $x_n = P_{C_n}x \to x_0$, we have $x_n \rightharpoonup x_0$. Using (7.16) and Lemma 4.1 we have $x_0 \in F(T)$. Put $z_0 = P_{F(T)}x$. Since $z_0 = P_{F(T)}x \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}}x$, we have that

$$\|x - x_{n+1}\|^2 \leq \|x - z_0\|^2. \tag{7.17}$$

So, we have from $x_n = P_{C_n}x \to x_0$ that

$$\|x - x_0\|^2 = \lim_{n\to\infty} \|x - x_n\|^2 \leq \|x - z_0\|^2.$$

From the definition of z_0, we get $z_0 = x_0$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

Using Theorem 7.3 and the metod of proof in Theorem 7.2, we have the following strong convergence theorem for strict pseud-contractions in a Hilbert space.

Theorem 7.4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let k be a real number with $0 \leq k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Ux_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (\beta_n - k)(1-\beta_n)\|Ux_n - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1}, and $\{\beta_n\} \subset (-\infty, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(U)}x$, where $P_{F(U)}$ is the metric projection of H onto $F(U)$.

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