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NEW EXPONENTIAL ESTIMATE FOR ROBUST STABILITY OF NONLINEAR NEUTRAL TIME-DELAY SYSTEMS WITH CONVEX POLYTOPIC UNCERTAINTIES

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Dedicated to the 70th birth day of Professor P.H. Sach

Abstract. This paper deals with the problem of robust stability for a class of nonlinear neutral time delay systems with convex polytopic uncertainties. The time-delay is assumed to be a time-varying non-differentiable function belonging to a given interval. By constructing a set of improved Lyapunov-Krasovskii parameter-dependent functionals and using linear matrix inequality (LMI) technique, new exponential estimate for the robust stability of the system is established. The delay-dependent sufficient conditions for the robust stability of the systems are presented in terms of LMIs. A numerical example is given to show the effectiveness of the results.

1. Introductions

Time-delay occurs in most of practical models, such as, aircraft stabilization, chemical engineering systems, inferred grinding model, manual control, neural network, nuclear reactor, population dynamic model, ship stabilization, and systems with lossless transmission lines. The existence of this time-delay may be the source for instability and bad performance of the system. Hence, the problem of stability analysis for time-delay systems has received much attention of many researchers in recent years, see [2, 4, 6, 10, 14] and the references therein. In practice, the system model can be described by functional differential equations of neutral type, which depends on both state and state derivatives. Neutral system examples include distributed networks, heat exchanges, and processes involving steam. Recently, the stability analysis of neutral systems has been widely investigated by many researchers, see [9, 11, 13]. Theoretically, the linear neutral system with time delays is much more complicated, especially for the case where the system matrices belong to some convex polytope. Based on parameter-dependent Lyapunov functionals, sufficient conditions conditions for asymptotic stability of linear polytopic systems have been proposed in [12]. Some delay-dependent conditions for asymptotic stability have been derived in [5, 7, 15, 16], which improve the estimate of the stability domain.

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However, some conservatism still remain since common matrix variable required to satisfy the whole sets of LMIs and the time delays are assumed to be either constants or differentiable and upper bounded. To the best of our knowledge, so far, the result on the stability for nonlinear neutral polytopic systems with interval time-varying delays has not been fully studied. The result of [12] for linear neutral polytopic system with constant delays. Paper [13] deals with asymptotic stability of nonlinear neutral systems with time-varying delays, but without polytopic uncertainties and the time-varying delay function is assumed to be differentiable.

In this paper, we revisit the robust exponential stability problem for nonlinear neutral systems with both convex polytopic uncertainties and interval time-varying delays. The novel feature of the results obtained in this paper is twofold. First, the system considered in this paper is nonlinear neutral subjected to interval, non-differentiable delay, which means that the lower and upper bounds for the time-varying delay are available, but the delay function is bounded but not necessary to be differentiable. This allows the time-delay to be a fast time-varying function and the lower bound is not restricted to being zero. Second, by constructing a set of new parameter-dependent Lyapunov Krasovskii functionals, novel delay-dependent sufficient conditions for the exponential stability of the system are obtained in terms of LMI conditions.

The paper is organized as follows. Section 2 presents notations, definitions and some well-known technical propositions needed for the proof of the main result. Delay-dependent exponential stability conditions of the system are presented in Section 3. An numerical example illustrated effectiveness of the conditions is given in Section 4.

2. Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all nonnegative real numbers; R^n denotes the $n-$dimensional Euclidean space with the norm $\|.\|$ and scalar product $x^{\mathsf{T}}y$ of two vectors x, y; $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$, resp.) denotes the maximal (the minimal, resp.) number of the real part of eigenvalues of A; A^{T} denotes the transpose of the matrix A and I denote the identity matrix; $\mathbf{0}_n$ denote the zero matrix in R^n. $Q \geq 0$ ($Q > 0$, resp.) means that Q is semi-positive definite (positive definite, resp.) i.e. $\langle Qx, x\rangle \geq 0$ for all $x \in R^n$ (resp. $\langle Qx, x\rangle > 0$ for all $x \neq 0$); $A \geq B$ means $A - B \geq 0$; $\mathcal{C}^1([a, b], R^n)$ denotes the set of all continuously differentiable functions on $[a, b]$. The segment of the trajectory $x(t)$ is denoted by $x_t = \{x(t+s) : s \in [-\bar{h}, 0]\}$.

Consider a nonlinear neutral system with interval state-delay and convex polytopic uncertainties of the form

$$
\begin{cases} \dot{x}(t) - D(\xi)\dot{x}(t-\tau) = A_0(\xi)x(t) + A_1(\xi)x(t-h(t)) + f_\xi(t, x(t), x(t-h(t))),\ t \geq 0, \\ x(t) = \phi(t),\ t \in [-\bar{h}, 0], \end{cases} \tag{2.1}
$$

where $x(t) \in R^n$ is the system state; time-varying delay function $h(t)$ satisfies the conditions

$$0 \leq h_m \leq h(t) \leq h_M,$$

and $\bar{h} = \max\{h_m, h_M\}$. The state-space data are subject to uncertainties and belong to the polytope Ω given by

$$\Omega = \left\{ [A_0, A_1, D, f](\xi) := \sum_{i=1}^{p} \xi_i [A_{0i}, A_{1i}, D_i, f_i],\ \xi_i \geq 0, \sum_{i=1}^{p} \xi_i = 1 \right\},$$

where $A_{0i}, A_{1i}, D_i, i = 1, ..., p$, are given constant matrices with appropriate dimensions and $f_i := f_i(t, .)$ are given vector functions satisfying

$$\|f_i(t,x,y)\|^2 \leq a_{0i}^2 \|x\|^2 + a_{1i}^2 \|y\|^2, \quad i = 1, 2, \ldots, p, \forall (x,y), t \geq 0. \tag{2.2}$$

The initial function $\phi \in \mathcal{C}^1([-\bar{h}, 0], R^n)$ with the norm

$$\|\phi\| = \sup_{-\bar{h} \leq t \leq 0} \sqrt{\|\phi(t)\|^2 + \|\dot{\phi}(t)\|^2}.$$

Definition 2.1. Given $\alpha > 0$. System (2.1) is $\alpha-$exponentially stable if every solution $x(t, \phi)$ of the system satisfies the following condition:

$$\exists \gamma > 0: \quad \|x(t,\phi)\| \leq \gamma \|\phi\| e^{-\alpha t}, \quad \forall t \geq 0.$$

We introduce the following technical well-known propositions, which will be used in the proof of our results.

Proposition 2.2 (Schur complement Lemma [1]). *For given matrices X, Y, Z with appropriate dimensions satisfying $X = X^\top, Y^\top = Y > 0$. Then $X + Z^\top Y^{-1} Z < 0$ if and only if*

$$\begin{bmatrix} X & Z^\top \\ Z & -Y \end{bmatrix} < 0 \quad \text{or} \begin{bmatrix} -Y & Z \\ Z^\top & X \end{bmatrix} < 0.$$

Proposition 2.3. *Let S be a symmetric positive definite matrix. Then, for any $x, y \in R^n$ and matrix F, we have*

$$2\langle Fy, x\rangle - \langle Sy, y\rangle \leq \langle FS^{-1}F^\top x, x\rangle.$$

The proof of the above proposition is easily derived from completing the square:

$$\langle S(y - S^{-1}F^\top x), y - S^{-1}F^\top x\rangle \geq 0.$$

Proposition 2.4 ([3]). *For any symmetric positive definite matrix W, scalar $\nu > 0$ and vector function $\omega : [0, \nu] \longrightarrow R^n$ such that the concerned integrals are well defined, then*

$$\left[\int_0^\nu w(s)ds\right]^\top W \left[\int_0^\nu w(s)ds\right] \leq \nu \int_0^\nu w^\top(s) W w(s) ds.$$

3. Main result

Let $U_{ki}, (k = 1, \ldots, 7, i = 1, \ldots, p), M$ be $n \times n$ matrices, $P_i, Q_i, R_i, S_i, T_i, Z_i, (i = 1, \ldots, p)$, be symmetric positive definite matrices and constants $\alpha > 0, \epsilon > 0$, we denote

$$\Xi_i(P_j, Q_j, R_j, S_j, T_j, Z_j, \mathcal{U}_j) =$$

$$= \begin{bmatrix} \Xi_{11} & A_{0i}^{\mathsf{T}}U_{2j} & \Xi_{13} & A_{0i}^{\mathsf{T}}U_{4j} & -U_{1j}^{\mathsf{T}} + A_{0i}^{\mathsf{T}}U_{5j} & \Xi_{16} & \Xi_{17} \\ * & -R_j & U_{2j}^{\mathsf{T}}A_{1i} & 0 & -U_{2j}^{\mathsf{T}} & U_{2j}^{\mathsf{T}}D_i & U_{2j}^{\mathsf{T}} \\ * & * & \Xi_{33} & Z_j + A_{1i}^{\mathsf{T}}U_{4j} & -U_{3j}^{\mathsf{T}} + A_{1i}^{\mathsf{T}}U_{5j} & U_{3j}^{\mathsf{T}}D_i + A_{1i}^{\mathsf{T}}U_{6j} & U_{3j}^{\mathsf{T}} + A_{1i}^{\mathsf{T}}U_{7j} \\ * & * & * & -Q_j - Z_j & -U_{4j}^{\mathsf{T}} & U_{4j}^{\mathsf{T}}D_i & U_{4j}^{\mathsf{T}} \\ * & * & * & * & \Xi_{55} & U_{5j}^{\mathsf{T}}D_i - U_{6j} & U_{5j}^{\mathsf{T}} - U_{7j} \\ * & * & * & * & * & \Xi_{66} & U_{6j}^{\mathsf{T}} + D_i^{\mathsf{T}}U_{7j} \\ * & * & * & * & * & * & \Xi_{77} \end{bmatrix},$$

where

$$\Xi_{11} = A_{0i}^{\mathsf{T}}(P_j + U_{1j}) + (P_j + U_{1j})A_{0i} + \epsilon a_{0j}^2 I + Q_j + R_j - T_j;$$
$$\Xi_{13} = P_jA_{1i} + U_{1j}^{\mathsf{T}}A_{1i} + A_{0i}^{\mathsf{T}}U_{3j} + T_j;$$
$$\Xi_{16} = P_jD_i + U_{1j}^{\mathsf{T}}D_i + A_{0i}^{\mathsf{T}}U_{6j};$$
$$\Xi_{17} = A_{0i}^{\mathsf{T}}U_{7j} + P_j + U_{1j}^{\mathsf{T}};$$
$$\Xi_{33} = -T_j - Z_j + A_{1i}^{\mathsf{T}}U_{3j} + U_{3j}^{\mathsf{T}}A_{1i} + \epsilon a_{1j}^2 I;$$
$$\Xi_{55} = -U_{5j} - U_{5j}^{\mathsf{T}} + S_j + h_M^2 T_j + (h_M - h_m)^2 Z_j;$$
$$\Xi_{66} = -S_j + D_i^{\mathsf{T}}U_{6j} + U_{6j}^{\mathsf{T}}D_i;$$
$$\Xi_{77} = -\epsilon I + U_{7j}^{\mathsf{T}} + U_{7j},$$

$\mathbb{M} = \mathrm{diag}\{M, \mathbf{0}_{6n}\}$

$\lambda_1 = \min_{1\le j\le p} \lambda_{\min}(P_j), \lambda_P = \max_{1\le j\le p} \lambda_{\max}(P_j), \lambda_Q = \max_j \lambda_{\max}(Q_j), \lambda_R = \max_j \lambda_{\max}(R_j),$
$\lambda_S = \max_j \lambda_{\max}(S_j), \lambda_T = \max_j \lambda_{\max}(T_j), \lambda_Z = \max_j \lambda_{\max}(Z_j),$
$\lambda_2 = \lambda_P + h_m\lambda_Q + \tau(\lambda_R + \lambda_S) + \frac{1}{2}e^{2\alpha h_M}\Big(h_M^3\lambda_T + (h_M - h_m)^2(h_M + h_m)\lambda_Z\Big).$

$$[P, Q, R, S, T, Z](\xi) = \sum_{j=1}^{p} \xi_j[P_j, Q_j, R_j, S_j, T_j, Z_j],$$

$$U_k(\xi) = \sum_{i=1}^{p} \xi_i U_{ki}, k = 1, \ldots, 7.$$

Theorem 3.1. *Assume that, for system* (2.1)*, there exist matrices* $U_{1i}, U_{2i}, U_{3i}, U_{4i}, U_{5i}, U_{6i}, U_{7i}$ $(i = 1, \ldots, p)$*, a symmetric semi-positive definite matrix* M*, symmetric positive definite matrices* $P_i, Q_i, R_i, S_i, T_i, Z_i, (i = 1, \ldots, p)$*, and a positive number* ϵ*, such that the following linear matrix inequalities hold:*

(3.1) $\quad \Xi_i(P_i, Q_i, R_i, S_i, T_i, Z_i, \mathcal{U}_i) + \mathbb{M} < 0, \quad i = 1, \ldots, p;$

(3.2) $\quad \Xi_i(P_j, Q_j, R_j, S_j, T_j, Z_j, \mathcal{U}_j) + \Xi_j(P_i, Q_i, R_i, S_i, T_i, Z_i, \mathcal{U}_i) - \frac{2}{p-1}\mathbb{M} < 0,$
$i = 1, \ldots, p-1, \ j = i+1, \ldots, p.$

Then exists a positive number α_* such that every solution $x(t,\phi)$ of the system satisfies

$$\|x(t,\phi)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|e^{-\alpha t}, \quad t \ge 0, \quad \forall \alpha \in (0,\alpha_*].$$

Proof. From the conditions (3.1), there exists $\delta > 0$ such that

$$\Xi_i(P_i, Q_i, R_i, S_i, T_i, Z_i, \mathcal{U}_i) + \mathbb{M} < -\delta I, \quad i = 1, 2, \ldots, p.$$

Consider the equation

$$\rho(\alpha) = 2\alpha\lambda_P + \Big(1 - e^{-2\alpha\tau}\Big)(\lambda_R + \lambda_S) + \Big(1 - e^{-2\alpha h_m}\Big)\lambda_Q$$
$$+ h_M^2\Big(e^{2\alpha h_M} - 1\Big)\lambda_T + \Big(h_M - h_m\Big)^2\Big(e^{2\alpha h_M} - 1\Big)\lambda_Z.$$

Note that, the function $\rho(\alpha)$ is continuous and strictly increasing in $\alpha \in [0,\infty)$, $\rho(0) = 0$, $\rho(\alpha) \to \infty$ as $\alpha \to \infty$. Hence, there is a unique positive solution α_* of the equation $\rho(\alpha) = \dfrac{\delta}{p}$ and $\rho(\alpha) < \dfrac{\delta}{p}$ for all $\alpha \in (0,\alpha_*]$. For any $\alpha \in (0,\alpha_*]$, consider the following Lyapunov-Krasovskii functional

$$V(t, x_t) = \sum_{i=1}^{6} V_k, \tag{3.3}$$

where,

$$V_1 = x^{\mathsf{T}}(t)P(\xi)x(t),$$
$$V_2 = \int_{t-h_m}^{t} e^{2\alpha(s-t)}x^{\mathsf{T}}(s)Q(\xi)x(s)ds$$
$$V_3 = \int_{t-\tau}^{t} e^{2\alpha(s-t)}x^{\mathsf{T}}(s)R(\xi)x(s)ds,$$
$$V_4 = \int_{t-\tau}^{t} e^{2\alpha(s-t)}\dot{x}^{\mathsf{T}}(s)S(\xi)\dot{x}(s)ds,$$
$$V_5 = h_M\int_{t-h_M}^{t}\int_{s}^{t} e^{2\alpha(\theta-t+h_M)}\dot{x}^{\mathsf{T}}(\theta)T(\xi)\dot{x}(\theta)d\theta ds,$$
$$V_6 = (h_M - h_m)\int_{t-h_M}^{t-h_m}\int_{s}^{t} e^{2\alpha(\theta-t+h_M)}\dot{x}^{\mathsf{T}}(\theta)Z(\xi)\dot{x}(\theta)d\theta ds,$$

and $\alpha > 0$ will be defined.

It is easy to verify from (3.3) that

$$\lambda_1\|x(t)\|^2 \le V(x_t) \le \lambda_2\|x_t\|^2, \quad t \in R^+. \tag{3.4}$$

Taking derivative of V_1 along trajectories of system (2.1) we have

$$\dot{V}_1 = 2x^{\mathsf{T}}(t)P(\xi)\dot{x}(t)$$
$$= x^{\mathsf{T}}(t)\Big[P(\xi)A_0(\xi) + A_0(\xi)^{\mathsf{T}}P(\xi)\Big]x(t)$$
$$+ 2x^{\mathsf{T}}(t)P(\xi)\Big[A_1(\xi)x(t-h(t)) + D(\xi)\dot{x}(t-\tau) + f_\xi\Big],$$

where, for convenient, we denote $f_\xi := f_\xi(t, x(t), x(t-h(t))) = \sum_{j=1}^p \xi_j f_j$.

From (2.2) we obtain

$$a_{0j}^2 x^\mathsf{T}(t)x(t) + a_{1j}^2 x^\mathsf{T}(t-h(t))x(t-h(t)) - f_j^\mathsf{T} f_j \geq 0, \quad j = 1, \ldots, p,$$

and hence,

$$\sum_{j=1}^p \xi_j \Big[a_{0j}^2 x^\mathsf{T}(t)x(t) + a_{1j}^2 x^\mathsf{T}(t-h(t))x(t-h(t)) - f_j^\mathsf{T} f_j\Big] \geq 0. \tag{3.5}$$

By completing the square we have

$$\begin{aligned}
f_\xi^\mathsf{T} f_\xi &= \Big(\sum_{j=1}^p \xi_j f_j\Big)^\mathsf{T}\Big(\sum_{j=1}^p \xi_j f_j\Big) \\
&= \sum_{j=1}^p \xi_j^2 f_j^\mathsf{T} f_j + 2\sum_{i<j} \xi_j \xi_j f_i^\mathsf{T} f_j \\
&\leq \sum_{j=1}^p \xi_j^2 f_j^\mathsf{T} f_j + \sum_{i<j} \xi_i \xi_j (f_i^\mathsf{T} f_i + f_j^\mathsf{T} f_j) \\
&= \Big(\sum_{i=1}^p \xi_i\Big)\Big(\sum_{j=1}^p \xi_j f_j^\mathsf{T} f_j\Big) = \sum_{j=1}^p \xi_j f_j^\mathsf{T} f_j.
\end{aligned}$$

Then, it follows from (3.5) that

$$\epsilon\Big[a_0(\xi)^2 x^\mathsf{T}(t)x(t) + a_1(\xi)^2 x^\mathsf{T}(t-h(t))x(t-h(t)) - f_\xi^\mathsf{T} f_\xi\Big] \geq 0,$$

for any $\epsilon > 0$, where $a_0(\xi)^2 := \sum_{j=1}^p \xi_j a_{0j}^2$ and $a_1(\xi)^2 := \sum_{j=1}^p \xi_j a_{1j}^2$. Therefore, the derivative of V_1 satisfies

$$\begin{aligned}
\dot V_1 \leq\; & x^\mathsf{T}(t)\Big[P(\xi)A_0(\xi) + A_0(\xi)^\mathsf{T} P(\xi) + \epsilon a_0(\xi)^2 I\Big]x(t) \\
& + 2x^\mathsf{T}(t)P(\xi)\Big[A_1(\xi)x(t-h(t)) + D(\xi)\dot x(t-\tau) + f_\xi\Big] \\
& + \epsilon\Big[a_1(\xi)^2 x^\mathsf{T}(t-h(t))x(t-h(t)) - f_\xi^\mathsf{T} f_\xi\Big].
\end{aligned} \tag{3.6}$$

Next, taking derivatives of $V_k, k = 2, \ldots, 6$, along trajectories of system (2.1) we obtain

(3.7)
$$
\begin{aligned}
\dot V_2 &= x^{\mathsf T}(t)Q(\xi)x(t) - e^{-2\alpha h_m}x^{\mathsf T}(t-h_m)Q(\xi)x(t-h_m) - 2\alpha V_2;\\
\dot V_3 &= x^{\mathsf T}(t)R(\xi)x(t) - e^{-2\alpha\tau}x^{\mathsf T}(t-\tau)R(\xi)x(t-\tau) - 2\alpha V_3;\\
\dot V_4 &= \dot x^{\mathsf T}(t)S(\xi)\dot x(t) - e^{-2\alpha\tau}\dot x^{\mathsf T}(t-\tau)S(\xi)\dot x(t-\tau) - 2\alpha V_4;\\
\dot V_5 &= h_M^2 e^{2\alpha h_M}\dot x^{\mathsf T}(t)T(\xi)\dot x(t) - h_M\int_{t-h_M}^{t} e^{2\alpha(s-t+h_M)}\dot x^{\mathsf T}(s)T(\xi)\dot x(s)ds - 2\alpha V_5\\
&\le h_M^2 e^{2\alpha h_M}\dot x^{\mathsf T}(t)T(\xi)\dot x(t) - h_M\int_{t-h_M}^{t}\dot x^{\mathsf T}(s)T(\xi)\dot x(s)ds - 2\alpha V_5;\\
\dot V_6 &= (h_M-h_m)^2 e^{2\alpha h_m}\dot x^{\mathsf T}(t)Z(\xi)\dot x(t)\\
&\quad - (h_M-h_m)\int_{t-h_M}^{t-h_m} e^{2\alpha(s-t+h_M)}\dot x^{\mathsf T}(s)Z(\xi)\dot x(s)ds - 2\alpha V_6\\
&\le (h_M-h_m)^2 e^{2\alpha h_m}\dot x^{\mathsf T}(t)Z(\xi)\dot x(t)\\
&\quad - (h_M-h_m)\int_{t-h_M}^{t-h_m}\dot x^{\mathsf T}(s)Z(\xi)\dot x(s)ds - 2\alpha V_6.
\end{aligned}
$$

Furthermore, by applying Proposition 2.4 and the Leibniz-Newton formula, we have

$$
\begin{aligned}
-h_M\int_{t-h_M}^{t}\dot x^{\mathsf T}(s)T(\xi)\dot x(s)ds &\le -h(t)\int_{t-h(t)}^{t}\dot x^{\mathsf T}(s)T(\xi)\dot x(s)ds\\
&\le -\left[\int_{t-h(t)}^{t}\dot x(s)ds\right]^{\mathsf T} T(\xi)\left[\int_{t-h(t)}^{t}\dot x(s)ds\right]\\
&\le -\Big[x(t)-x(t-h(t))\Big]^{\mathsf T} T(\xi)\Big[x(t)-x(t-h(t))\Big];
\end{aligned}
\tag{3.8}
$$

and

(3.9)
$$
\begin{aligned}
-(h_M-h_m)\int_{t-h_M}^{t-h_m}\dot x^{\mathsf T}(s)Z(\xi)\dot x(s)ds &\le -(h(t)-h_m)\int_{t-h(t)}^{t-h_m}\dot x^{\mathsf T}(s)Z(\xi)\dot x(s)ds\\
&\le -\left[\int_{t-h(t)}^{t-h_m}\dot x(s)ds\right]^{\mathsf T} Z(\xi)\left[\int_{t-h(t)}^{t-h_m}\dot x(s)ds\right]\\
&\le -\Big[x(t-h_m)-x(t-h(t))\Big]^{\mathsf T} Z(\xi)\Big[x(t-h_m)-x(t-h(t))\Big].
\end{aligned}
$$

By using the following identity

$$
-\dot x(t) + D(\xi)\dot x(t-\tau) + A_0(\xi)x(t) + A_1(\xi)x(t-h(t)) + f_\xi = 0,
$$

we have

$$
\begin{aligned}
&2\Big[x^{\mathsf T}(t)U_1(\xi)^{\mathsf T}+x^{\mathsf T}(t-\tau)U_2(\xi)^{\mathsf T}+x^{\mathsf T}(t-h(t))U_3(\xi)^{\mathsf T}\\
&\quad+x^{\mathsf T}(t-h_m)U_4(\xi)^{\mathsf T}+\dot x^{\mathsf T}(t)U_5(\xi)^{\mathsf T}+\dot x^{\mathsf T}(t-\tau)U_6(\xi)^{\mathsf T}+f_\xi^{\mathsf T}U_7(\xi)^{\mathsf T}\Big]\\
&\quad\times\Big[-\dot x(t)+D(\xi)\dot x(t-\tau)+A_0(\xi)x(t)+A_1(\xi)x(t-h(t))+f_\xi\Big]=0.
\end{aligned}
\tag{3.10}
$$

Combining from (3.6)-(3.10) we have

$$
\dot V(t,x_t)+2\alpha V(t,x_t)\le \eta^{\mathsf T}(t)\Phi(\xi)\eta(t),
\tag{3.11}
$$

where,
$\eta^{\mathsf T}(t)=\begin{bmatrix}x^{\mathsf T}(t) & x^{\mathsf T}(t-\tau) & x^{\mathsf T}(t-h(t)) & x^{\mathsf T}(t-h_m) & \dot x^{\mathsf T}(t) & \dot x^{\mathsf T}(t-\tau) & f_\xi^{\mathsf T}\end{bmatrix}$,
and

$$
\Phi(\xi)=\begin{bmatrix}
\Phi_{11}(\xi) & A_0(\xi)^{\mathsf T}U_2(\xi) & \Phi_{13}(\xi) & \Phi_{14}(\xi) & \Phi_{15}(\xi) & \Phi_{16}(\xi) & \Phi_{17}(\xi)\\
* & -e^{-2\alpha\tau}R(\xi) & U_2(\xi)^{\mathsf T}A_1(\xi) & 0 & -U_2(\xi)^{\mathsf T} & U_2(\xi)^{\mathsf T}D(\xi) & U_2(\xi)^{\mathsf T}\\
* & * & \Phi_{33}(\xi) & \Phi_{34}(\xi) & \Phi_{35}(\xi) & \Phi_{36}(\xi) & \Phi_{37}(\xi)\\
* & * & * & \Phi_{44}(\xi) & -U_4(\xi)^{\mathsf T} & U_4(\xi)^{\mathsf T}D(\xi) & U_4(\xi)^{\mathsf T}\\
* & * & * & * & \Phi_{55}(\xi) & \Phi_{56}(\xi) & \Phi_{57}(\xi)\\
* & * & * & * & * & \Phi_{66}(\xi) & \Phi_{67}(\xi)\\
* & * & * & * & * & * & \Phi_{77}(\xi)
\end{bmatrix},
$$

$$
\begin{aligned}
\Phi_{11}(\xi)&=\Big[A_0(\xi)+\alpha I\Big]^{\mathsf T}P(\xi)+P(\xi)\Big[A_0(\xi)+\alpha I\Big]+A_0(\xi)^{\mathsf T}U_1(\xi)+U_1(\xi)^{\mathsf T}A_0(\xi)\\
&\quad+\epsilon a_0(\xi)^2I+Q(\xi)+R(\xi)-T(\xi);\\
\Phi_{13}(\xi)&=P(\xi)A_1(\xi)+U_1(\xi)^{\mathsf T}A_1(\xi)+A_0(\xi)^{\mathsf T}U_3(\xi)+T(\xi);\\
\Phi_{14}(\xi)&=A_0(\xi)^{\mathsf T}U_4(\xi);\quad \Phi_{15}(\xi)=-U_1(\xi)^{\mathsf T}+A_0(\xi)^{\mathsf T}U_5(\xi);\\
\Phi_{16}(\xi)&=P(\xi)D(\xi)+U_1(\xi)^{\mathsf T}D(\xi)+A_0(\xi)^{\mathsf T}U_6(\xi);\\
\Phi_{17}(\xi)&=P(\xi)+U_1(\xi)^{\mathsf T}+A_0(\xi)^{\mathsf T}U_7(\xi);\\
\Phi_{33}(\xi)&=-T(\xi)-Z(\xi)+A_1(\xi)^{\mathsf T}U_3(\xi)+U_3(\xi)^{\mathsf T}A_1(\xi)+\epsilon a_1(\xi)^2I;\\
\Phi_{34}(\xi)&=Z(\xi)+A_1(\xi)^{\mathsf T}U_4(\xi);\quad \Phi_{35}(\xi)=-U_3(\xi)^{\mathsf T}+A_1(\xi)^{\mathsf T}U_5(\xi);\\
\Phi_{36}(\xi)&=U_3(\xi)^{\mathsf T}D(\xi)+A_1(\xi)^{\mathsf T}U_6(\xi);\quad \Phi_{37}(\xi)=U_3(\xi)^{\mathsf T}+A_1(\xi)^{\mathsf T}U_7(\xi);\\
\Phi_{44}(\xi)&=-e^{-2\alpha h_m}Q(\xi)-Z(\xi);\\
\Phi_{55}(\xi)&=S(\xi)+h_M^2e^{2\alpha h_M}T(\xi)+(h_M-h_m)^2e^{2\alpha h_M}Z(\xi)-U_5(\xi)-U_5(\xi)^{\mathsf T};\\
\Phi_{56}(\xi)&=U_5(\xi)^{\mathsf T}D(\xi)-U_6(\xi);\quad \Phi_{57}(\xi)=U_5(\xi)^{\mathsf T}-U_7(\xi);\\
\Phi_{66}(\xi)&=-e^{-2\alpha\tau}S(\xi)+U_6(\xi)^{\mathsf T}D(\xi)+D(\xi)^{\mathsf T}U_6(\xi);\quad \Phi_{67}(\xi)=U_6(\xi)^{\mathsf T}+D(\xi)^{\mathsf T}U_7(\xi);\\
\Phi_{77}(\xi)&=-\epsilon I+U_7(\xi)+U_7(\xi)^{\mathsf T}.
\end{aligned}
$$

Using property $\sum_{i=1}^{p}\xi_i=1$ we have

$$
\dot V(t,x_t)+2\alpha V(t,x_t)\le \eta^{\mathsf T}(t)\Big[\sum_{i=1}^{p}\xi_i^2\Xi_i(P_i,Q_i,R_i,S_i,T_i,Z_i,\mathcal U_i)
$$

$$+\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j\Big(\Xi_i(P_j,Q_j,R_j,S_j,T_j,Z_j,\mathcal{U}_j)+\Xi_j(P_i,Q_i,R_i,S_i,T_i,Z_i,\mathcal{U}_i)\Big)\Big]\eta(t)$$
$$+\sum_{i=1}^{p}\xi_i\eta^{\mathsf{T}}(t)\Psi_i\eta(t),$$

where,

$$\Psi_i=\mathrm{diag}\Big\{2\alpha P_i,\ (1-e^{-2\alpha\tau})R_i,\ 0,\ (1-e^{-2\alpha h_m})Q_i,$$
$$h_M^2(e^{2\alpha h_M}-1)T_i+(h_M-h_m)^2(e^{2\alpha h_M}-1)Z_i,\ (1-e^{-2\alpha\tau})S_i,\ 0\Big\}.$$

Therefore,

$$\dot V(t,x_t)+2\alpha V(t,x_t)\le\eta^{\mathsf{T}}(t)\left(-\sum_{i=1}^{p}\xi_i^2+\frac{2}{p-1}\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j\right)\mathbb{M}\eta(t)$$
$$-\delta\Big(\sum_{i=1}^{p}\xi_i^2\Big)\|\eta(t)\|^2+\rho(\alpha)\|\eta(t)\|^2. \quad (3.12)$$

Observe that,

$$(p-1)\sum_{i=1}^{p}\xi_i^2-2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j=\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}(\xi_i-\xi_j)^2\ge 0,$$
$$p\Big(\sum_{i=1}^{p}\xi_i^2\Big)\ge\Big(\sum_{i=1}^{p}\xi_i\Big)^2=1,$$

then from (3.12) we have

$$\dot V(t,x_t)+2\alpha V(t,x_t)\le\Big[\rho(\alpha)-\frac{\delta}{p}\Big]\|\eta(t)\|^2,\quad t\ge 0. \quad (3.13)$$

For any $\alpha\in(0,\alpha_*]$ we have

$$\dot V(t,x_t)+2\alpha V(t,x_t)\le\eta^{\mathsf{T}}(t)\Big[\rho(\alpha)-\frac{\delta}{p}\Big]\|\eta(t)\|^2\le 0$$

which implies

$$V(t,x_t)\le V(0,x_0)e^{-2\alpha t},t\ge 0.$$

Taking (3.4) into account, we obtain

$$\|x(t,\phi)\|\le\sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|e^{-\alpha t},\quad t\ge 0$$

which concludes the proof. □

Remark 3.2. In this paper, by using an improved Lyapunov-Krasovskii functional we obtain directly the exponential estimate for solutions of polytopic neutral system (2.1) without using assumptions on the stability of neutral operator [8].

Remark 3.3. It is worth noting that the condition (3.1) means the asymptotic stability of each i^{th}-subsystem, while the condition (3.2) implies the asymptotic stability of the ij^{th}-subsystems and if $p = 1$ this condition is automatically removed. Then theorem 3.1 is reduced to the exponential stability criterion for neutral systems with nonlinear perturbations and time-varying delays.

Remark 3.4. Theorem 3.1 gives conditions for the exponential stability of neutral systems with nonlinear, polytopic type uncertainties and interval-time varying state delay. These conditions are derived in terms of linear matrix inequalities which can be solved effectively by various computation tools [1].

4. An example

Consider the nonlinear neutral system (2.1), where

$$A_{01} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad A_{02} = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}, \quad A_{03} = \begin{bmatrix} -3 & -1 \\ 0 & -1 \end{bmatrix};$$
$$A_{11} = \begin{bmatrix} -0.5 & 0.1 \\ -0.1 & 0.2 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -0.5 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -0.2 & 0.4 \\ -0.5 & 0.4 \end{bmatrix};$$
$$D_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad D_3 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.1 \end{bmatrix};$$
$$a_{0i} = a_{1i} = 0.1, \quad i = 1, 2, 3.$$

Delay function $h(t) = 0.5 + d(t)$, where

$$\begin{cases} d(t) = 0.5 \sin^2 t \quad if \quad i \in \mathcal{I} = \cup_{k \geq 0} [2k\pi, (2k+1)\pi] \\ d(t) = 0 \quad if \quad t \in R^+ \setminus \mathcal{I}. \end{cases}$$

It is worth noting that, the delay function $h(t)$ is non-differentiable, interval time varying in R^+. Therefore, the stability criteria proposed in [9, 13, 15, 16] are not applicable to this system. We have $h_m = 0.5, h_M = 1, \tau = 1$. By LMI toolbox of Matlab, we find that LMI conditions (3.1), (3.2) are satisfied with $M = I, \epsilon = 10^2$ and

$$P_1 = \begin{bmatrix} 85.5661 & 6.9397 \\ 6.9397 & 52.6648 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 74.6947 & 9.2670 \\ 9.2670 & 147.8895 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 137.3857 & 13.1879 \\ 13.1879 & 41.1895 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 28.8406 & 11.1200 \\ 11.1200 & 21.8652 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 32.6350 & 16.3100 \\ 16.3100 & 82.0828 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 127.5799 & 2.2804 \\ 2.2804 & 18.3659 \end{bmatrix},$$
$$R_1 = \begin{bmatrix} 27.1430 & 12.2736 \\ 12.2736 & 15.5309 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 29.9222 & 27.8139 \\ 27.8139 & 78.6832 \end{bmatrix},$$
$$R_3 = \begin{bmatrix} 120.2473 & 18.2163 \\ 18.2163 & 8.4003 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 12.4857 & 1.8011 \\ 1.8011 & 19.8902 \end{bmatrix},$$
$$S_2 = \begin{bmatrix} 17.2561 & 0.5387 \\ 0.5387 & 13.0135 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 13.0859 & 1.1669 \\ 1.1669 & 11.1598 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 33.9760 & -6.1901 \\ -6.1901 & 15.8206 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 13.7778 & 19.4778 \\ 19.4778 & 77.8855 \end{bmatrix},$$
$$T_3 = \begin{bmatrix} 12.2321 & -4.8051 \\ -4.8051 & 14.9685 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} 26.7516 & -1.4499 \\ -1.4499 & 56.5555 \end{bmatrix},$$
$$Z_2 = \begin{bmatrix} 47.3321 & -9.1867 \\ -9.1867 & 50.2910 \end{bmatrix}, \quad Z_3 = \begin{bmatrix} 65.5498 & -16.5890 \\ -16.5890 & 40.6353 \end{bmatrix},$$
$$U_{11} = \begin{bmatrix} 11.4849 & -10.1346 \\ -10.3104 & 5.9741 \end{bmatrix}, \; U_{12} = \begin{bmatrix} 11.4849 & -10.1346 \\ -10.3104 & 5.9741 \end{bmatrix},$$
$$U_{13} = \begin{bmatrix} 11.4849 & -10.1346 \\ -10.3104 & 5.9741 \end{bmatrix}, \; U_{31} = \begin{bmatrix} 11.4849 & -10.1346 \\ -10.3104 & 5.9741 \end{bmatrix},$$
$$U_{32} = \begin{bmatrix} 2.3684 & 0.2624 \\ 4.3405 & -2.9379 \end{bmatrix}, \; U_{33} = \begin{bmatrix} -1.1523 & 3.9144 \\ -6.6555 & 0.8054 \end{bmatrix},$$
$$U_{51} = \begin{bmatrix} 55.9962 & -9.3215 \\ -8.3719 & 41.0290 \end{bmatrix}, \quad U_{71} = \begin{bmatrix} 14.9702 & 5.0667 \\ -2.6189 & 17.8526 \end{bmatrix},$$
$$U_{72} = \begin{bmatrix} 19.3078 & 0.3527 \\ -0.2927 & 22.2819 \end{bmatrix}, \; U_{73} = \begin{bmatrix} 18.6734 & -3.1088 \\ -4.6628 & 17.0935 \end{bmatrix},$$
$$U_{2i} = U_{4i} = U_{6i} = 0, i = 1, 2, 3, U_{52} = U_{53} = 0.$$

Moreover, from (3.1) we find that

$$\Xi_i(P_i, Q_i, R_i, S_i, T_i, Z_i, \mathcal{U}_i) \leq -\mathbb{M} - \delta I, \quad i = 1, 2, 3,$$

where $\delta = 1.6079$. Taking some computation by theorem 3.1, we have

$$\rho(\alpha) = 298.098\alpha + 127.6275\Big(1 - e^{-\alpha}\Big) + 143.4444\Big(1 - e^{-2\alpha}\Big) + 101.7989\Big(e^{2\alpha} - 1\Big)$$

and the positive solution of the equation $\rho(\alpha) = \delta/3$ is $\alpha_* = 0.5850722618 \times 10^{-3}$. Applying Theorem 3.1, the system is globally exponentially stable with any convergence rate $\alpha \in (0, \alpha_*]$. For $\alpha = 0.00058$, every solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \leq 3.2327\|\phi\|e^{-0.00058t}, \quad t \geq 0.$$

5. Conclusions

In this paper, the problem of the robust exponential stability for nonlinear neutral differential equations with interval non-differentiable time-varying delays and polytopic uncertainties has been studied. By constructing a set of new parameter-dependent Lyapunov functionals, novel delay-dependent conditions for the robust exponential stability are derived in terms of linear matrix inequalities, which allow simultaneous computation of two bounds that characterize the exponential stability rate of the solution and can be easily determined by utilizing MATLABs LMI Control Toolbox.

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