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COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS SATISFYING THE EXPANSIVE CONDITION

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Abstract. We obtain several fixed point theorems for a class of operators called occasionally weakly compatible maps defined on a symmetric space satisfying a generalized expansive condition. These results are some of the most general fixed point theorems for four maps which satisfy expansive condition.

Prior to 1968 all work involving fixed points used the Banach contraction principle. In 1968 Kannan [9] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next two decades. (See, e.g., [12] for a listing and comparison of many of these definitions.). Sessa [14] coined the notion of weakly commuting. Then Jungck generalized this idea, first to compatible mappings [6] and then to weakly compatible mappings [7]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. We shall list here only the definition of weakly compatible. Also during this time a number of authors established fixed point theorems for pairs of maps (See for example, [4], [11] and references therein). Thagafi and Shahzad [3] gave a definition which is proper generalization of nontrivial weakly compatible maps which have coincidence points. Zhang [16] obtained common fixed point theorems for some new generalized contractive type mappings. Recently, Abbas and Rhoades [2] obtained common fixed point theorems for occasionally weakly compatible mappings satisfying a generalized contractive condition. A survey of the literature regarding common fixed points of two and four mappings shows that most of the results do not involve expansion conditions. Rhoades [13] proved a common fixed point theorem for a pair of expansive maps (see also [15]).

The aim of this paper is to obtain some fixed points theorem involving occasionally weakly compatible maps in the setting of a symmetric space satisfying a generalized expansion condition.

Two maps S and T are said to be weakly compatible if they commute at coincidence points (see [10]).

Definition 1. Let X be a set and f, g be selfmaps of X. A point x in X is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.

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The following concept [3] is a proper generalization of nontrivial weakly compatible maps which have a coincidence point.

Definition 2. Two selfmaps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

We shall also need the following lemma from [8] (see also [1]).

Lemma 3. *Let X be a set, f, g owc selfmaps of X. If f and g have a unique point of coincidence, $w := fx = gx$, then w is the unique common fixed point of f and g.*

Our theorems are proved in symmetric spaces which are more general than metric spaces.

Definition 4. Let X be a set. A mapping $d : X \times X \to [0, \infty)$ is said to be symmetric on X if it satisfies the conditions;

$$d(x, y) = 0 \quad iff \quad x = y, \quad and \quad d(x, y) = d(y, x) \quad for \quad x, y \in X.$$

Let $A \in (0, \infty]$, $R_A^+ = [0, A)$. Define,

$\digamma[0, A) = \{F : R_A^+ \longrightarrow R : F$ is nondecreasing, $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, A)\}$ and

$\Psi[0, A) = \{\psi : R_A^+ \longrightarrow R : \psi$ is nondecreasing $\psi(t) > t$, for each $t \in (0, A)\}$.

Theorem 5. *Let X be a set with a symmetric d. Let $D = \sup\{d(x, y) : x, y \in X\}$. Suppose that f, g, S, T are selfmaps of X and that the pairs $\{f, S\}$ and $\{g, T\}$ are each owc. If for each $x, y \in X$ satisfying $fx \neq gy$ we have*

$$F(d(fx, gy)) \geq \psi(F(M(x, y))), \tag{1}$$

$F \in \digamma[0, A)$ and $\psi \in \Psi[0, F(A-0))$, where $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$, and

$$M(x, y) := \max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), \\ d(Sx, gy), d(Ty, fx)\},$$

then there is a unique point $w \in X$ such that $fw = gw = w$ and a unique point $z \in X$ such that $gz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of f, g, S, and T.

Proof. Since the pairs $\{f, S\}$ and $\{g, T\}$ are owc, there exist points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. We claim that $fx = gy$. If not, then we consider,

$$\begin{aligned} M(x, y) :&= \max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), d(Sx, gy), d(Ty, fx)\} \\ &= d(fx, gy). \end{aligned}$$

Then (1) implies

$$\begin{aligned} F(d(fx, gy)) &\geq \psi(F(M(x, y))) \\ &= \psi(F(d(fx, gy))) > F(d(fx, gy)), \end{aligned}$$

which is a contradiction. Therefore, $fx = gy$; i.e., $fx = Sx = gy = Ty$. Moreover, if there is another point z such that $fz = Sz$, then, using (1) it follows that $fz = Sz = gy = Ty$, or $fx = fz$, and $w = fx = Sx$ is the unique point of coincidence of f and S. By Lemma 3, w is the only common fixed point of f and S. By symmetry there is a unique point $z \in X$ such that $z = gz = Tz$.

Suppose that $w \neq z$. Using (1),

$$\begin{aligned} F(d(w,z)) &= F(d(fw,gz)) \geq \psi(F(M(w,z))) \\ &> F(d(w,z)), \end{aligned}$$

which is a contradiction. Therefore $w = z$ and w is a common fixed point. By the preceding argument it is clear that w is unique. □

Corollary 6. *Let X be a set with a symmetric d. Let $D = \sup\{d(x,y) : x, y \in X\}$. Suppose that f, g, S, T are selfmaps of X such that $\{f, S\}$ and $\{g, T\}$ are owc. If*

$$F(d(fx,gy)) \geq \psi(F(m(x,y))), \tag{2}$$

for each $x, y \in X$, $F \in \digamma[0, A)$ and $\psi \in \Psi[0, F(A-0))$, where $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$, and

$$m(x,y) = h \max\{d(Sx,Ty), d(Sx,fx), d(Ty,gy), d(Sx,gy), d(Ty,fx)\}, h > 1,$$

then f, g, S, T have a unique common fixed point.

Proof. Since F and ψ are nondecreasing, from $m(x,y) \geq M(x,y)$, we have

$$\psi(F(m(x,y))) \geq \psi(F(M(x,y))).$$

Therefore if (2) holds, then (1) holds and so the result follows immediately from Theorem 5. □

Theorem 7. *Let X be a symmetric space with symmetric d. Let $D = \sup\{d(x,y) : x, y \in X\}$. Suppose that f, S are selfmaps of X such that f and S are owc, and*

$$F(d(fx,fy)) \geq \psi(F(M(x,y))), \tag{3}$$

for each $x, y \in X$, $fx \neq fy$, $F \in \digamma[0, A)$ and $\psi \in \Psi[0, F(A-0))$, where $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$, and

$$\begin{aligned} M(x,y) = ad(Sx,Sy) &+ b\max\{d(fx,Sx), d(fy,Sy)\} \\ &+ c\max\{d(Sx,Sy), d(Sx,fx), d(Sy,fy)\} \end{aligned} \tag{4}$$

for all $x, y \in X$, where $a, b, c > 0, a + c > 1$. Then f and S have a unique common fixed point.

Proof. Since the pair $\{f, S\}$ is owc, so there exists a point $x \in X$ such that $fx = Sx = w$ (say). Suppose that there exist another point y in X such that $fy = Sy = v$ (say). Now we claim that $w = v$, that is, $Sx = Sy$. If not, then we consider

$$\begin{aligned} M(x,y) &= ad(Sx,Sy) + b\max\{d(fx,Sx), d(fy,Sy)\} \\ &\quad + c\max\{d(Sx,Sy), d(Sx,fx), d(Sy,fy)\} \\ &= (a+c)\max\{d(Sx,Sy), d(fx,Sx), d(fy,Sy)\} \\ &= (a+c)d(Sx,Sy). \end{aligned}$$

Thus

$$\begin{aligned} F(d(Sx,Sy)) &\geq \psi(F((a+c)d(Sx,Sy))) \\ &> F((a+c)d(Sx,Sy)), \end{aligned}$$

which is a contradiction. Thus $w = v$. Therefore the result now follows from Lemma 3. □

Theorem 8. *Let X be a symmetric space with symmetric d. Let $D = \sup\{d(x, y) : x, y \in X\}$. Suppose that f, g, S, and T are selfmaps of X and*

$$F(d(fx, gy))^p) \geq \psi(F(M_p(x, y))), \tag{5}$$

for each $x, y \in X$ for which $fx \neq gy$, $F \in \digamma[0, A)$ and $\psi \in \Psi[0, F(A - 0))$, where $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$, and

$$\begin{aligned} M_p(x, y) = a(d(fx, Ty))^p \\ + b \max\{(d(fx, Sx))^p, (d(gy, Ty))^p, \\ (d(fx, Sx))^{p/2}(d(fx, Ty))^{p/2}, \\ (d(Ty, fx))^{p/2}(d(Sx, gy))^{p/2}\}, \end{aligned} \tag{6}$$

for all $x, y \in X$, where $a + b > 1$, and $p > 0$. If $\{f, S\}$ and $\{g, T\}$ are owc, then f, g, S, and T have a unique common fixed point.

Proof. By hypothesis, there exist points x and y such that $fx = Sx$ and $gy = Ty$. Suppose that $fx \neq gy$. Then from (6),

$$\begin{aligned} M_p(x, y) &= a(d(fx, gy))^p \\ &\quad + b \max\{0, 0, 0, (d(fx, gy))^p\} \\ &= (a + b)(d(fx, gy))^p \end{aligned}$$

and

$$\begin{aligned} F(d(fx, gy))^p) &\geq \psi(F(M_p(x, y))) \\ &= \psi(F((a + b)(d(fx, gy))^p)) \\ &\geq \psi(F((d(fx, gy))^p)) > F((d(fx, gy))^p), \end{aligned}$$

which is a contradiction. Therefore $d(fx, gy) = 0$, which implies that $fx = gy$. Suppose that there exists another point z such that $fz = Sz$. Then, using (5) one obtains $fz = Sz = gy = Ty = fx = Sx$ and hence $w = fx = fz$ is the unique point of coincidence of f and S. By symmetry there exists a unique point $v \in X$ such that $v = gz = Tv$. It then follows that $w = v$, w is a common fixed point of f, g, S, and T, and w is unique. □

Define $\dot{g} : \mathbb{R}^5 \to \mathbb{R}^5$such that if $u \in \mathbb{R}^+$ is such that $u \geq \dot{g}(u, 0, 0, u, u)$ or $u \geq \dot{g}(0, u, 0, u, u)$ or $u \geq \dot{g}(0, 0, u, u, u)$, then $u = 0$.

Theorem 9. *Let X be a set, d a symmetric on X. Let $D = \sup\{d(x, y) : x, y \in X\}$. Suppose that f, g, S, T are selfmaps of X satisfying*

$$\begin{aligned} F(d(fx, gy)) \geq \dot{g}(F(d(Sx, Ty)), F(d(fx, Sx)), F(d(gy, Ty)), \\ F(d(fx, Ty)), F(d(gy, Sx))), \end{aligned} \tag{7}$$

for all $x, y \in X$, $F \in \digamma[0, A)$, where $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$. If $\{f, S\}$ and $\{g, T\}$ are owc, then f, g, S, T have a unique common fixed point.

Proof. By hypothesis there exist points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. Now we show that $fx = gy$. From (7),

$$F(d(fx, gy)) \geq \dot{g}(F(d(fx, gy)), 0, 0, F(d(fx, gy)), F(d(gy, fx))),$$

which implies that $F(d(fx, gy)) = 0$ and hence $d(fx, gy) = 0$. Hence $fx = gy$. As in the previous theorems it can then be shown that fx is unique and that $u = fx$ is a common fixed point of the four mappings. Condition (7) implies uniqueness. □

A control function Φ is defined by $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies $\Phi(t) = 0$ iff $t = 0$.

Theorem 10. *Let $\{f, S\}$ and $\{g, T\}$ be owc pairs of selfmaps of a space X with symmetric d. Let $D = \sup\{d(x, y) : x, y \in X\}$, and satisfying*

$$F(\Phi(d(fx, gy))) \geq \psi(F(M_\Phi(x, y))), \tag{8}$$

for each $x, y \in X$ for which $fx \neq gy$, $F \in \mathcal{F}[0, A)$ and $\psi \in \Psi[0, F(A - 0))$, where $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$, and

$$\begin{aligned} M_\Phi(x, y) := \max\{&\Phi(d(Sx, Ty)), \Phi(d(Sx, fx)), \Phi((d(gy, Ty)), \\ &[\Phi(d(fx, Ty)), +\Phi(d(Sx, gy))]/2\}. \end{aligned} \tag{9}$$

Then f, g, S, and T have a unique common fixed point.

Proof. By hypothesis, there exist points $x, y \in X$ for which $fx = Sx$ and $gy = Ty$. Suppose that $fx \neq gy$. Then, from (9),

$$\begin{aligned} M_\Phi(x, y) := \max\{&\Phi(d(fx, gy)), \Phi(0), \Phi(0), \\ &\Phi(d(fx, gy))\}. \end{aligned}$$

Thus

$$\begin{aligned} F(\Phi(d(fx, gy))) &\geq \psi(F(M_\Phi(x, y))) \\ &= \psi(F(\Phi(d(fx, gy))) \\ &> F(\Phi(d(fx, gy))), \end{aligned}$$

which is a contradiction. Therefore

$$\Phi(d(fx, gy)) = 0,$$

which implies that $d(fx, gy) = 0$, which implies that $fx = gy$. It then follows that f, g, S, and T have a common fixed point. Condition (8) gives uniqueness. □

References

[1] M. Abbas and G. Jungck, *Common fixed point results for non commuting mappings without continuity in cone metric space,* J. Math. Anal. Appl. **341** (2008), 416–420.

[2] M. Abbas and B. E. Rhoades, *Common fixed point theorems for occasionally weakly compatible mappings satisfying a generalized contractive condition*, Mathematical Communications **13** (2008), 295–301.

[3] M. A. Al-Thagafi and N. Shahzad, *Generalized I-nonexpansive selfmaps and invariant approximations,* Acta Math. Sinica **24** (2008), 867–876.

[4] I. Beg and M. Abbas, *Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition*, Fixed Point Theory and Applications, 2006 Article ID 74503, pages 1–7.

[5] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **73** (1976), 261–263.

[6] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. & Math. Sci. **9** (1986), 771–779.
[7] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci. **4** (1996), 19–215.
[8] G. Jungck and B. E. Rhoades, *Fixed point theorems for occasionally weakly compatible mappings*, Fixed Point Theory **7** (2006), 287–296.
[9] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **60** (1968), 71–76.
[10] J. K. Kim, S. Sedghi and N. Shobe, *Common fixed point theorems for a class of weakly compatible mappings in D-metric spaces*, East Asian Mathematical Journal **25** (2009), 107–117
[11] R. P. Pant, *Common fixed point of contractive maps*, J. Math. Anal. Appl. **226** (1998), 251–258.
[12] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **26** (1977), 257–290.
[13] B. E. Rhoades, *Some fixed point theorems for pair of mappings*, Jnanabha **15** (1985), 151-156.
[14] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. **32** (1982), 149–153.
[15] S. Z. Wang, B. Yi, Z. M. Gao and K. Iseki, *Some fixed point theorems on expansion mappings*, Math. Japon. **29** (1984), 631–635.
[16] X. Zhang, *Common fixed point theorems for some new generalized contractive type mappings*, J. Math. Anal. Appl. **333** (2007), 780–786.

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