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A SPLITTING PROXIMAL POINT METHOD FOR NASH-COURNOT EQUILIBRIUM MODELS INVOLVING NONCONVEX COST FUNCTIONS

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Dedicated to Professor Pham Huu Sach on the occasion of his 70th birthday

Abstract. Unlike convex case, a local equilibrium point of a nonconvex Nash-Cournot oligopolistic equilibrium problem may not be a global one. Finding such a point or even a *stationary point* of this problem is not an easy task. In this paper, we propose a numerical method for finding a stationary point of nonconvex Nash-Cournot equilibrium problems. The convergence of the algorithm is proved and its complexity is estimated under certain assumptions. Numerical examples are implemented to illustrate the convergence properties of the proposed algorithm.

1. Introduction

Nash-Cournot oligopolistic equilibrium models have been widely applied in economics, electricity markets, transportation, networks and environments. Such models can be formulated as a game strategy, where each player has a profit function which can be expressed as the difference of the price and the cost function. In classical models, the price function is affine while the cost function is assumed to be convex. In this case, a local equilibrium point is also a global one. Mathematical programming and variational inequality approaches can be used to treat these problems [3, 4, 8, 9, 14].

In many practical models, the cost per unit usually decreases as the production level increases. To represent this situation, a nonconvex function of the production cost of the model is required. In [15], a global optimization algorithm was developed to find a global equilibrium point of the Nash-Cournot oligopolistic equilibrium market model involving piecewise concave cost functions. However, global algorithms only work well for the problems of moderate size, while it becomes intractable when the size of the problem increases, except for special structures are exploited.

In this paper, we continue the work in [15] by proposing a local solution method for finding a stationary point of Nash-Cournot equilibrium models with nonconvex

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cost functions. We consider a Nash-Cournot model involving an affine price function and nonconvex smooth production cost functions. With this structure, the cost function of the model can be decomposed as the sum of a convex quadratic function and a nonconvex and smooth function. Then, we develop a numerical method for finding a stationary point of this model, which is called a splitting proximal point algorithm. The main idea of this algorithm is to preserve the convexity of the problem, while convexifies the nonconvex part by linearizing it around each iteration point.

Proximal point methods have been well developed in optimization as well as in nonlinear analysis. Myriad of research papers concerned to these methods were published (see, e.g., [11, 12, 21, 22] and the references quoted therein). However, all these papers only deal with a class of monotone problems. Recently, Pennanen in [20] extended the proximal point method to nonmonotone variational inequality problems. Lewis [11] further generalized this algorithm in a unified framework using the prox-regularity concept [22]. A main point of the proximal point methods is to control the proximal parameter sequence. This affects to the performance of the algorithm as well as its global convergence behavior. Güler in [6] investigated the rate of global convergence of the classical proximal point methods for convex programs. He proved that the worst case the complexity bound of this method is $O(1/\varepsilon)$, where ε is a desired accuracy. The author further accelerated the classical proximal point method to get a better complexity bound for the convex programming problems, precisely, $O(1/\sqrt{\varepsilon})$ [7]. Splitting proximal point methods are also developed in many research papers. Such methods were applied to nonlinear optimization by Mine and Fukushima in [13] and, recently, to convex optimization by Nesterov in [17].

This paper contributes a new local method for finding a stationary point of the Nash-Cournot equilibrium models involving the nonconvex cost functions, which is called splitting proximal point method. The convergence of the algorithm is investigated and its global worst-case complexity bound is estimated, which is $O(1/\varepsilon^2)$, where ε is a desired accuracy. To our knowledge, this is the first estimate proposed to the Nash-Cournot equilibrium models.

The rest of the paper is organized as follows. Section 2 presents a formulation of the Nash-Cournot oligopolistic equilibrium model involving nonconvex cost functions. This problem is reformulated as a mixed variational inequality. In Section 3, we define three concepts including local equilibria, global equilibria and stationary points of the Nash-Cournot equilibrium model. Section 4 deals with a gradient mapping and its properties. The splitting algorithm is described in Section 5, where its convergence is proved and the worst-case complexity is estimated. Two numerical examples are implemented in the last section.

2. Mixed variational inequality formulation

We consider a Nash-Cournot oligopolistic equilibrium market models with n-firms producing a common homogeneous commodity in a non-cooperative fashion. The price p of production depends on the total quantity $\sigma := \sum_{i=1}^{n} x_i$ of the commodity. Let $h_i(x_i)$ denote the cost of the firm i when its production level is x_i. Suppose that the profit of firm i is given as

$$f_i(x_1, \cdots, x_n) := x_i p\Big(\sum_{i=1}^{n} x_i\Big) - h_i(x_i), \; i = 1, \ldots, n, \tag{2.1}$$

where h_i is the cost function of the firm i which is assumed to only depend on its production level.

Let $C_i \subset \mathbb{R}$ $(i = 1, \ldots, n)$ denote the strategy set of the firm i, which is assumed to be closed and convex. Each firm seeks on its strategy set to maximize the profit by choosing the corresponding production level under the presumption that the production of the other firms are parametric inputs. In this context, a Nash equilibrium is a production pattern in which no firm can increase its profit by changing its controlled variables. Thus under this equilibrium concept, each firm determines its best response given other firms' actions. Mathematically, a point $x^* = (x_1^*, \ldots, x_n^*)^T \in C := C_1 \times \cdots \times C_n$ is said to be a Nash equilibrium point if

$$f_i(x_1^*, \ldots, x_{i-1}^*, y_i, x_{i+1}^*, \ldots, x_n^*) \leq f_i(x_1^*, \ldots, x_n^*), \; \forall y_i \in C_i \; (i = 1, \ldots, n). \tag{2.2}$$

When h_i is affine, this market problem can be formulated as a special Nash equilibrium problem in the n-person non-cooperative game model, which is in turn a strongly monotone variational inequality (see, e.g., [9]).
Let us define

$$\Psi(x, y) := -\sum_{i=1}^{n} f_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n), \tag{2.3}$$

and

$$\phi(x, y) := \Psi(x, y) - \Psi(x, x). \tag{2.4}$$

Then, as proven in [9], the problem of finding an equilibrium point of this model can be reformulated as the following formulation:

$$\text{Find } \; x^* \in C \text{ such that: } \phi(x^*, y) \geq 0 \text{ for all } y \in C. \tag{EP}$$

This problem is referred to as an equilibrium problem [2, 15].

In classical Cournot models [4, 9], the price and the cost functions for each firm are assumed to be affine and given as follows:

$$p(\sigma) = \alpha_0 - \beta\sigma, \quad \alpha_0 \geq 0, \quad \beta > 0, \quad \text{with } \sigma = \sum_{i=1}^{n} x_i, \tag{2.5}$$

$$h_i(x_i) = \mu_i x_i + \xi_i, \quad \mu_i > 0, \quad \xi_i \geq 0 \; (i = 1, \ldots, n). \tag{2.6}$$

In this case, by using (2.1), (2.2), (2.3) and (2.4), it is easy to check that

$$\phi(x, y) = (\tilde{B}x + \mu - \alpha)^T(y - x) + \frac{1}{2}y^T By - \frac{1}{2}x^T Bx,$$

where

$$B = \begin{bmatrix} 2\beta & 0 & 0 & \ldots & 0 \\ 0 & 2\beta & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 2\beta \end{bmatrix}; \; \tilde{B} = \begin{bmatrix} 0 & \beta & \beta & \ldots & \beta \\ \beta & 0 & \beta & \ldots & \beta \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta & \beta & \beta & \ldots & 0 \end{bmatrix},$$

$$\alpha = (\alpha_0, \dots, \alpha_0)^T, \text{ and } \mu = (\mu_1, \dots, \mu_n)^T.$$

Then the problem of finding a Nash equilibrium point can be formulated as a mixed variational inequality of the form:

$$\text{Find } x^* \in C \text{ such that:}$$

$$(\tilde{B}x^* + \mu - \alpha)^T(y - x^*) + \frac{1}{2}y^T By - \frac{1}{2}(x^*)^T Bx^* \geq 0, \ \forall y \in C. \tag{2.7}$$

Let $Q := B + \tilde{B}$. Since $\beta > 0$ and matrices $\tilde{B}$ and B are symmetric, it is clear that Q is symmetric and positive definite. This mixed variational inequality can be reformulated equivalently to the following strongly convex quadratic programming problem:

$$\min_{x \in C} \left\{ \frac{1}{2}x^T Qx + (\mu - \alpha)^T x \right\}. \tag{QP}$$

Hence, problem (QP) has a unique optimal solution, which is also the unique equilibrium point of the classical oligopolistic equilibrium market model.

The oligopolistic market equilibrium model, where the profit function f_i of firm i is assumed to be differentiable and convex with respect to its production level x_i, while the other production levels are fixed, is studied in [4, 9]. This convex model is reformulated equivalently to a monotone variational inequality.

In many practical models, the production cost function h_i assumed to be affine is no longer satisfied. Since the cost per a unit of the action does decrease when the quantity of the commodity exceeds a certain amount. Taking into account this fact, in the sequel, we consider market equilibrium models where the cost function h may not be convex, whereas the price function is affine as in (2.5). Typically, the cost function h is given as:

$$h(x) := \sum_{i=1}^{n} h_i(x_i), \tag{2.8}$$

where h_i $(i = 1, \dots, n)$ are differentiable and nonconvex.

Let us denote by

$$F(x) := \tilde{B}x - \tilde{\alpha}$$
$$\varphi(x) := g(x) - h(x),$$

where $\tilde{\alpha} := \alpha - \mu$, $g(x) := \frac{1}{2}x^T Bx$ and $h(x)$ defined as (2.8).

Obviously, matrix $\tilde{B}$ is symmetric, by using the same notation $\tilde{B}$, B and α as in (2.7), we can formulate the nonconvex Nash-Cournot equilibrium model as a mixed variational inequality:

$$\text{Find } x^* \in C \text{ such that:}$$

$$F(x^*)^T(y - x^*) + \varphi(y) - \varphi(x^*) \geq 0 \text{ for all } y \in C. \tag{ncMVIP}$$

Note that mixed variational inequality problems of the form (ncMVIP), where φ is convex, i.e. h is concave, were extensively studied in literature (see, e.g., [1, 4, 5, 9, 10, 16, 18, 23]).

Remark 2.1. If the function φ is convex and differentiable then problem (ncMVIP) can be reformulated equivalently to a classical variational inequality problem. More generally, it can be converted to a generalized variational inequality problem when φ is convex and subdifferentiable (see, e.g. [9]). However, the mixed variational inequality (ncMVIP) can not equivalently transform into a variational inequality if φ is nonconvex.

Note that if we define $\phi(x,y) := F(x)^T(y-x) + \varphi(y) - \varphi(x)$ then problem (ncMVIP) coincides with a nonconvex equilibrium problem of the form (EP).

3. Local equilibria and stationary points

Unlike the convex case, if the cost function φ of the problem (ncMVIP) is nonconvex, it may not have a global equilibria even if C is compact, and F and φ are continuous. Indeed, let us consider $C := [-1,1] \subset \mathbb{R}$, $F(x) := x$ and $\varphi(x) := -\frac{1}{2}x^2$, which is concave, then $F(x)^T(y-x)+\varphi(y)-\varphi(x) = -\frac{1}{2}(y-x)^2$. Therefore, problem (ncMVIP) has no solution.

For a given $x \in C$, let $\mathbb{B}(x,r)$ be an open ball of radius $r > 0$ centered at x in $\mathbb{R}^n$. By borrowing the concepts from classical optimization, we firstly propose a local equilibria and a stationary point of the mixed variational inequality (ncMVIP).

Definition 3.1. A point $x^* \in C$ is called a local solution or a local equilibria to (ncMVIP) if there exists an open ball $\mathbb{B}(x^*,r)$ such that

$$F(x^*)^T(y-x^*) + \varphi(y) - \varphi(x^*) \geq 0 \text{ for all } y \in C \cap \mathbb{B}(x^*,r). \tag{3.1}$$

If $C \subseteq \mathbb{B}(x^*,r)$ then x^* is called a global solution or a global equilibria to (ncMVIP).

Let

$$\psi(x;y) := (\tilde{B}x - \tilde{\alpha})^T(y-x) + \frac{1}{2}y^T By - h(y). \tag{3.2}$$

We consider a function $m : C \times \mathbb{R}_{++} \to \mathbb{R}$ and a mapping $S : C \times \mathbb{R}_{++} \rightrightarrows 2^C$ defined as follows:

$$m(x;r) := \min\left\{\psi(x;y) \mid y \in C \cap \overline{\mathbb{B}}(x,r)\right\}, \tag{3.3}$$

$$S(x;r) := \operatorname{argmin}\left\{\psi(x;y) \mid y \in C \cap \overline{\mathbb{B}}(x,r)\right\}, \tag{3.4}$$

where $\overline{\mathbb{B}}(x,r)$ stands for the closure of the open ball $\mathbb{B}(x,r)$. As usual, we refer to m as a local gap function for problem (ncMVIP). Obviously, if h is continuous and C is closed then the function m and the mapping S are well-defined. If h is concave then S is reduced to a single valued mapping due to the positive definiteness of B.

The following proposition gives a necessary and sufficient condition for a point to be a local or global solution to (ncMVIP).

Proposition 3.2. *The following statements are equivalent:*

a) x^* *is a local solution to* (ncMVIP)*;*
b) *There exists* $\bar{r} > 0$ *such that* $x^* \in C$ *and* $m(x^*;\bar{r}) = 0$*;*
c) *There exists* $\bar{r} > 0$ *such that* $x^* \in C$ *and* $x^* \in S(x^*;\bar{r})$*.*

Proof. Note that since $\psi(x,x) = 0$ for all $x \in C$. The equivalence between b) and c) immediately follows from the definitions (3.3) and (3.4). "We only prove that a) is equivalent to b).

Suppose that $x^* \in C$ is a local equilibria of (ncMVIP). Then, there exists a neighbourhood $\mathbb{B}(x^*, r)$ of x^* such that $\psi(x^*, y) \geq 0$ for all $y \in C \cap \mathbb{B}(x^*, r)$, where $r > 0$. Assume that $0 < \bar{r} < r$. Then, $\overline{\mathbb{B}}(x^*, \bar{r}) \subset \mathbb{B}(x^*, r)$. In particular, $\psi(x^*, y) \geq 0$ for all $y \in C \cap \overline{\mathbb{B}}(x^*, \bar{r})$ which is equivalent to $\psi(x^*, y) \geq \psi(x^*, x^*)$ for all $y \in C \cap \overline{\mathbb{B}}(x^*, \bar{r})$. Hence, $m(x^*, \bar{r}) = 0$ according to (3.3). Conversely, suppose that there exists $\bar{r} > 0$ such that $x^* \in C$ and $m(x^*, \bar{r}) = 0$. From the definition of m, we have $\psi(x^*, y) \geq m(x^*, \bar{r}) = 0$ for all $x \in C \cap \overline{\mathbb{B}}(x^*, \bar{r})$. In particular, $\psi(x^*, y) \geq 0$ for all $y \in \mathbb{B}(x^*, \bar{r})$. Hence, x^* is a local equilibria of (ncMVIP). □

Clearly, if the conclusions of Proposition 3.2 hold for a given $\bar{r} > 0$ and $C \subseteq \mathbb{B}(x^*, \bar{r})$ then x^* is a global solution to (ncMVIP).

Next, let us define

$$\mathcal{F}_C(x) := \{d := t(y - x) \mid y \in C,\ t \geq 0\}, \tag{3.5}$$

the cone of all feasible directions of C starting from $x \in C$. The dual cone of $\mathcal{F}_C(x)$ is the normal cone of C at x which is defined as

$$N_C(x) := \begin{cases} \left\{w \in \mathbb{R}^n \mid w^T(y - x) \geq 0,\ y \in C\right\}, & \text{if } x \in C, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{3.6}$$

By Proposition 3.2, a point $x \in C$ is a local solution to (ncMVIP) if and only if

$$x \in \operatorname{argmin} \left\{ (\tilde{B}x - \alpha)^T(y - x) + \frac{1}{2}y^T By - h(y) \mid y \in C \cap \overline{\mathbb{B}}(x, \bar{r}) \right\}, \tag{3.7}$$

for some $\bar{r} > 0$. Since h is not necessarily concave, finding such a point x satisfying (3.7), in general, is not an easy task. In this paper, we concentrate in finding a stationary point rather than a local equilibria. We develop a numerical method to compute such a point for (ncMVIP). By borrowing the concept of *stationary points* in optimization, we define a stationary point (or a *critical point*) of the mixed variational inequality (ncMVIP) as follows.

Definition 3.3. A point $x \in C$ is called a stationary point or a critical point to the problem (ncMVIP) if

$$0 \in Qx - \tilde{\alpha} - \nabla h(x) + N_C(x), \tag{3.8}$$

where N_C is defined by (3.6) and $Q := \tilde{B} + B$.

Since N_C is a cone, for any $c > 0$, the inclusion (3.8) is equivalent to

$$0 \in c[(\tilde{B} + B)x - \tilde{\alpha} - \nabla h(x)] + N_C(x). \tag{3.9}$$

Let

$$D\phi(x; d) := [(\tilde{B} + B)x - \tilde{\alpha} - \nabla h(x)]^T d, \tag{3.10}$$

for any $x \in C$ and $d \in \mathcal{F}_C(x)$. Then the condition (3.8) is equivalent to

$$D\phi(x^*; d) \geq 0,\ \forall d \in \mathcal{F}_C(x^*). \tag{3.11}$$

Let us denote by S^* the set of stationary points of (ncMVIP). The following lemma shows that every local equilibria of problem (ncMVIP) is its stationary point.

Lemma 3.4. *Suppose that h is continuous differentiable on its domain. Then, every local equilibria is a stationary point of problem* (ncMVIP).

Proof. Suppose that x^* is a local equilibria of (ncMVIP). Then there exists a ball $\mathbb{B}(x^*, r)$ such that $F(x^*)^T(y - x^*) + \varphi(y) - \varphi(x^*) \geq 0$ for all $y \in \mathbb{B}(x^*, r) \cap C$. However, this inequality can be written equivalently to $\psi(x^*, y) \geq \psi(x^*, x^*)$ for all $y \in C \cap \mathbb{B}(x^*, r)$. In other words, $x^* \in \operatorname{argmin} \{\psi(x^*, y) \mid x \in C \cap \mathbb{B}(x^*, r)\}$. Using the optimality condition for this convex optimization, we have:

$$0 \in \partial\psi(x^*, x^*) + N_{C\cap\mathbb{B}(x^*,r)}(x^*). \tag{3.12}$$

Since $x^* \in C\cap\mathbb{B}(x^*, r)$ and $\mathbb{B}(x^*, r)$ is open, it implies that $N_{C\cap\mathbb{B}(x^*,r)}(x^*) = N_C(x^*)$. Moreover, $\partial\psi(x, x) = (\tilde{B} + B)x - \tilde{\alpha} - \nabla h(x)$. Therefore, the condition (3.12) is equivalent to (3.8). □

Let $\partial\delta_C(x)$ denote the subdifferential of the indicator function δ_C of C at x, one has $\partial\delta_C(x) = N_C(x)$. Since matrix B is symmetric and positive definite, if we define $g_1(x) := \frac{1}{2}x^T Bx + \delta_C(x)$ then $\partial g_1(x) = Bx + \partial\delta_C(x)$ and this mapping is maximal monotone. Consequently, $T_c^{-1} := (I + c\partial g_1)^{-1}$ is well-defined and single valued, where I is the identity mapping (see [21, 22]).

The following proposition provides a necessary and sufficient condition for a stationary point of (ncMVIP).

Proposition 3.5. *A necessary and sufficient condition for a point $x \in C$ to be a stationary point to problem* (ncMVIP) *is:*

$$x = (I + c\partial g_1)^{-1}\left(x - c(\tilde{B}x - \tilde{\alpha}) + c\nabla h(x)\right), \tag{3.13}$$

where $c > 0$ and I stands for the identity mapping.

Proof. Since g_1 is proper closed convex, the inverse $\big(I + \partial g_1\big)^{-1}$ is single valued and defined everywhere [21]. Thus x satisfies (3.13) if and only if $x - c(\tilde{B}x - \tilde{\alpha}) + c\nabla h(x) \in (I + c\partial g_1)(x)$. Moreover, since $N_C(x)$ is a cone and $\partial g_1(x) = Bx + \partial\delta_C(x) = Bx + N_C(x)$, the latter inclusion is equivalent to $0 \in \tilde{B}x - \tilde{\alpha} + Bx - \nabla h(x) + N_C(x)$, which shows that x is a stationary point of (ncMVIP). □

Now, if we define $y_c(x) := x - c(\tilde{B}x - \alpha) + c\nabla h(x)$ and

$$S_c(x) := \big(I + c\partial g_1\big)^{-1}\big(x - c(\tilde{B}x - \alpha) + c\nabla h(x)\big), \tag{3.14}$$

then, it follows from Proposition 3.5 that $x = S_c(x)$. Therefore, every stationary point x of (ncMVIP) is a fixed-point of $S_c(\cdot)$. To compute $S_c(x)$ it requires to solve the following strongly convex quadratic problem over a convex set:

$$\min\left\{\frac{1}{2}y^T By + \frac{1}{2c}\|y - y_c(x)\|^2 \mid y \in C\right\}, \tag{3.15}$$

This problem has a unique solution for any $c > 0$.

Finally, we introduce the following concept, which will be used in the sequel. For a given tolerance $\varepsilon \geq 0$, a point $x^* \in C$ is said to be an ε-stationary point to (ncMVIP) if

$$D\phi(x^*; d) \geq -\varepsilon, \ \forall d \in \mathcal{F}_C(x^*), \ \|d\| = 1. \tag{3.16}$$

4. Gradient mapping and its properties

By substituting $y_c(x)$ into (3.15), after a simple rearrangement, we can write problem (3.15) as

$$\min \left\{ \frac{1}{2} y^T B y + [\tilde{B}x - \tilde{\alpha} - \nabla h(x)]^T (y - x) + \frac{1}{2c}\|y - x\|^2 \mid y \in C \right\}. \tag{4.1}$$

Now, we consider the following mappings:

$$m_c(x; y) := \frac{1}{2} y^T B y + [\tilde{B}x - \tilde{\alpha} - \nabla h(x)]^T (y - x) - h(x) + \frac{1}{2c}\|y - x\|^2, \tag{4.2}$$

$$\text{and } s_c(x) := \operatorname{argmin} \{m_c(x; y) \mid y \in C\}. \tag{4.3}$$

Then, since problem (4.3) is strongly convex, $s_c(x)$ is well-defined and single-valued. Let us define

$$G_c(x) := \frac{1}{c}[x - s_c(x)]. \tag{4.4}$$

The mapping $G_c(\cdot)$ is referred to as a *gradient-type mapping* of (3.3) [17]. By using the optimality condition for (4.3) we have

$$\left[B s_c(x) + \tilde{B}x - \tilde{\alpha} - \nabla h(x) - G_c(x) \right]^T (y - s_c(x)) \geq 0, \ \forall y \in C. \tag{4.5}$$

From now on, we further suppose that the cost function h is Lipschitz continuous differentiable on C with a Lipschitz constant $L_h > 0$, i.e.

$$\|\nabla h(x) - \nabla h(y)\| \leq L_h \|x - y\|, \ \forall x, y \in C. \tag{4.6}$$

By using the mean-valued theorem, it is easy to show that the condition (4.6) implies

$$\left| h(y) - h(x) - \nabla h(x)^T (y - x) \right| \leq \frac{1}{2} L_h \|y - x\|^2, \ \forall x, y \in C. \tag{4.7}$$

The following lemma provides some properties of $D\phi(\cdot; \cdot)$.

Lemma 4.1. *For any $x \in C$, we have*

$$D\phi(s_c(x); x - s_c(x)) \geq \frac{1 - c(L_h + \|\tilde{B}\|)}{c^2} \|G_c(x)\|^2, \tag{4.8}$$

$$D\phi(s_c(x); y - s_c(x)) \geq -[1 + c(L_h + \|\tilde{B}\|)] \|G_c(x)\| \|y - s_c(x)\|, \ \forall y \in C. \tag{4.9}$$

As a consequence, for any $d \in \mathcal{F}_C(s_c(x))$ with $\|d\| = 1$, we have

$$D\phi(s_c(x); d) \geq -[1 + c(L_h + \|\tilde{B}\|)] \|G_c(x)\|. \tag{4.10}$$

Proof. From the definition of $D\phi$ in (3.10), we have

$$D\phi(s_c(x); x - s_c(x)) = \left[\tilde{B} s_c(x) - \tilde{\alpha} + B s_c(x) - \nabla h(s_c(x)) \right]^T (x - s_c(x))$$

$$= \left[\tilde{B}x - \tilde{\alpha} - \nabla h(x) + B s_c(x) \right]^T (x - s_c(x))$$

$$
\begin{aligned}
(4.11)\quad & -\Big[\tilde{B}s_c(x)-\tilde{\alpha}-\nabla h(s_c(x))-\tilde{B}x-\tilde{\alpha}+\nabla h(x)\Big]^T(s_c(x)-x)\\
& \geq \Big[\tilde{B}x-\tilde{\alpha}-\nabla h(x)+Bs_c(x)\Big]^T(x-s_c(x))-(L_h+\|\tilde{B}\|)\|x-s_c(x)\|^2.
\end{aligned}
$$

Substituting (4.5) into (4.11) we obtain

$$
\begin{aligned}
D\phi(s_c(x);x-s_c(x)) &\geq (\frac{1}{c}-[L_h+\|\tilde{B}\|)]\|x-s_c(x)\|^2\\
&= \frac{1-c(L_h+\|\tilde{B}\|)}{c^2}\|G_c(x)\|^2,
\end{aligned}
$$

which proves (4.8).

Using again (4.5) and (4.6) we have

$$
\begin{aligned}
& D\phi(s_c(x);y-s_c(x)) = \Big[\tilde{B}s_c(x)-\tilde{\alpha}+Bs_c(x)-\nabla h(s_c(x)\Big]^T(y-s_c(x))\\
&= \Big[\tilde{B}s_c(x)-\tilde{\alpha}-\nabla h(s_c(x))\Big]^T(y-s_c(x))+(Bs_c(x))^T(y-x)\\
&\geq \Big[\tilde{B}s_c(x)-\tilde{\alpha}-\nabla h(s_c(x))\Big]^T(y-s_c(x))\\
&+\Big[\tilde{B}x-\nabla h(x)+\frac{1}{c}(s_c(x)-x)\Big]^T(s_c(x)-y)\\
&= \Big[\tilde{B}s_c(x)-\nabla h(s_c(x))-(\tilde{B}x-\tilde{\alpha})+\nabla h(x)\Big]^T(y-s_c(x))+G_c(x)^T(s_c(x)-y)\\
&\geq -(L_h+\|\tilde{B}\|)\|x-s_c(x)\|\|y-s_c(x)\|-\|G_c(x)\|\|y-s_c(x)\|\\
&\geq -\Big[1+c(L_h+\|\tilde{B}\|)\Big]\|G_c(x)\|\|y-s_c(x)\|,
\end{aligned}
$$

which proves (4.9).

By the convexity of C, there exists $t \geq 0$ such that $s_c(x)+td \in C$, where $\|d\| = 1$. If we substitute $y := s_c(x)+td \in C$ into (4.9) then we get

$$
D\phi(x;td) \geq -t(1+cL_fh+c\|\tilde{B}\|)\|G_c(x)\|. \tag{4.12}
$$

If $t = 0$ then (4.9) automatically holds. If $t > 0$ then by the linearity of D with respect to the second argument, we divide both sides of (4.12) by $t > 0$ to get (4.9). □

Remark 4.2. For a fixed $x \in C$, if we define $e_c(x) := \|G_c(x)\|$ and $r_c(x) := \|x-s_c(x)\|$ then $e_c(x)$ decreases in c and $r_c(x)$ increases in c.

Indeed, let $q(y,c) := (\tilde{B}x-\tilde{\alpha})^T(y-x)+\frac{1}{2}y^TBy-h(x)-\nabla h(x)^T(y-x)+\frac{1}{2c}\|y-x\|^2$. Then q is convex jointly in two arguments y and c. Thus $\omega(c) := \min_{y\in C} q(y,c)$ is convex. It is easy to see that $\omega'(c) = -\frac{1}{2}\|G_c(x)\|^2$ increases in c. Hence, $e_c(x)$ decreases in c. If we replace c by $1/c$ in $q(y,c)$, this function becomes concave in c, then by the same argument as $\omega(\cdot)$, we conclude that $r_c(x)$ increases in c.

Since B and $\tilde{B}$ are symmetric, we consider a potential function defined as follows:

$$
\gamma(x) := \frac{1}{2}x^TBx+\frac{1}{2}x^T\tilde{B}x-\tilde{\alpha}^Tx-h(x), \tag{4.13}
$$

Then, γ is nonconvex but Lipschitz continuous differentiable. We have the following statement.

Lemma 4.3. *For $x, y \in C$, we have*

$$m_c(x; s_c(x)) + x^T\tilde{B}x - \alpha^T x \le \gamma(x) - \frac{c}{2}\|G_c(x)\|^2. \tag{4.14}$$

Moreover, if $c(L_h + \|\tilde{B}\|) \le 1$ then

$$m_c(x; s_c(x)) + \frac{1}{2}x^T\tilde{B}x - \tilde{\alpha}^T x \ge \gamma(s_c(x)). \tag{4.15}$$

Proof. It is obvious from the definition of $m_c(x;x)$ that $\phi(x) = m_c(x;x) + \frac{1}{2}x^T\tilde{B}x - \tilde{\alpha}^T x$. Since $m_c(x;\cdot)$ is strongly convex quadratic with modulus $\frac{1}{2c}$, using (4.13) we have

$$\begin{aligned}\gamma(x) - m_c(x; s_c(x)) - \frac{1}{2}x^T\tilde{B}x + \tilde{\alpha}^T x &= m_c(x;x) - m_c(x; s_c(x)) \\ &\ge \frac{1}{2c}\|x - s_c(x)\|^2 = \frac{c}{2}\|G_c(x)\|^2,\end{aligned}$$

which proves (4.14).

To prove (4.15), from (4.7) and the definition of γ we have

$$\begin{aligned}&\gamma(s_c(x)) - m_c(x; s_c(x)) - \frac{1}{2}x^T\tilde{B}x + \tilde{\alpha}^T x \\ &= \frac{1}{2}\left[s_c(x)\tilde{B}s_c(x) - x^T\tilde{B}x - 2(\tilde{B}x)^T(s_c(x) - x)\right] \\ &\quad -h(s_c(x)) + h(x) + \nabla h(x)^T(s_c(x) - x) - \frac{1}{2c}\|s_c(x) - x\|^2 \\ &\le \frac{1}{2}(s_c(x) - x)^T\tilde{B}(s_c(x) - x) + \frac{(cL_h - 1)}{2c}\|s_c(x) - x\|^2 \\ &\le -\frac{1 - c(L_h + \|\tilde{B}\|)}{2c}\|s_c(x) - x\|^2.\end{aligned}$$

By assumption $c(L_h + \|\tilde{B}\|) \le 1$, we obtain (4.15). □

If we combine the inequalities (4.15) and (4.14) in Lemma 4.3 then:

$$\gamma(s_c(x)) \le \gamma(x) - \frac{c}{2}\|G_c(x)\|^2. \tag{4.16}$$

This inequality plays an important role in proving the convergence of the splitting proximal point algorithm in the next section.

For a given starting point $x^0 \in C$, let us define the level set of γ with respect to C as

$$\mathcal{L}_\gamma(\gamma(x^0)) := \left\{x \in C \mid \gamma(x) \le \gamma(x^0)\right\}. \tag{4.17}$$

From (4.16), it is obvious that if $x^0 \in \mathcal{L}_\gamma(\gamma(x^0))$ then $s_c(x^0) \in \mathcal{L}_\gamma(\gamma(x^0))$ provided that $c(L_h + \|\tilde{B}\|) \le 1$.

5. A splitting proximal point algorithm and its convergence

Proposition 3.5 suggests that a proximal point method can be applied to find a stationary point of (ncMVIP). For the implementation purpose, the proximal mapping defined by (3.14) is extracted to the expression (3.15). The *splitting proximal point algorithm* constructs an iterative sequence as follows:

Algorithm 1. (The splitting proximal-point algorithm)

Initialization: Choose a positive number $c_0 > 0$. Find an initial point $x^0 \in C$ and set $k := 0$.
Iteration k**:** For a given x^k, execute the three steps below.

Step 1: Evaluate $\nabla h(x^k)$ and set $y_k := x^k - c_k(\tilde{B}x^k - \alpha) + c_k\nabla h(x^k)$.
Step 2: Compute x^{k+1} by solving the following convex quadratic program over a convex set:

$$\min\{\frac{1}{2}y^TBy + \frac{1}{2c_k}\|y - y^k\|^2 \mid y \in C\} \tag{5.1}$$

Step 3: If $\|x^{k+1} - x^k\| \leq \varepsilon$ for a given tolerance $\varepsilon > 0$ then terminate, x^k is an ε-stationary point of (ncMVIP). Otherwise, update c_k, increase k by 1 and go back to Step 1.

In Algorithm 1 we left unspecified the way to update c_k. If the Lipschitz constant L_h is provided then we can choose $c_k = \frac{1}{L_\gamma}$ for all k, where $L_\gamma := L_h + \|\tilde{B}\|$. Otherwise, a line-search procedure can be used to update c_k. The latter procedure is briefly described as follows. First, we choose two constants $\underline{c}$ and $\bar{c}$ such that $\underline{c} > 0$ and $\frac{1}{L_\gamma} \leq \bar{c} < +\infty$. Then we perform the following steps.

- Given a constant $\tau_c \in (0, 1)$. Choose an initial value of c in $[\underline{c}, \bar{c}]$.
- Compute $s_c(x^k)$. While the decreasing condition

$$\gamma(s_c(x^k)) \leq m_c(x^k; s_c(x^k)) + \frac{1}{2}(x^k)^T\tilde{B}x^k - \tilde{\alpha}^Tx^k. \tag{5.2}$$

  does not satisfy and $c > \underline{c}$, decrease c by $c := \tau c$ and recompute $s_c(x^k)$.
- Set $c_{k+1} := c$.

Now, we define $\Delta x^k := x^{k+1} - x^k$ and

$$\delta_k := \min_{0\leq i\leq k} \frac{\|\Delta_i\|^2}{2c_i}. \tag{5.3}$$

The convergence of the *splitting proximal point algorithm* is stated as follows.

Theorem 5.1. *Suppose that the function h is Lipschitz continuous differentiable on C with a Lipschitz constant $L_h \geq 0$. Suppose further that for a given $x^0 \in C$ the level set $\mathcal{L}_\gamma(\gamma(x^0))$ is bounded (particularly, C is bounded). Then the sequence $\{x^k\}_{k\geq 0}$ generated by Algorithm 1 starting from x^0 satisfies:*

$$\delta_k \leq \frac{(\gamma(x^0) - \underline{\gamma})}{k + 1}, \ \forall k \geq 0, \tag{5.4}$$

where $\underline{\gamma} := \inf_{x\in\mathcal{L}_\gamma(\gamma(x^0))} \gamma(x)$*. Moreover, for any* $d \in \mathcal{F}_C(x^{i_k})$ *with* $\|d\| = 1$*, we have*

$$D\phi(x^{i_k}; d) \geq -(1 + \underline{c}L_h + \underline{c}\|\tilde{B}\|)\sqrt{\frac{2(\gamma(x^0) - \underline{\gamma})}{k+1}}, \tag{5.5}$$

where i_k *is the index such that* $c_{i_k}\|G_{c_{i_k}}(x^{i_k})\|^2 = \Delta_k$*.*

As a consequence, if the sequence $\{x^k\}$ *generated by* (5.1) *is bounded, then every limit point of this sequence is a stationary point of* (ncMVIP)*. The set of limit points is connected and if it is finite then the whole sequence* $\{x^k\}$ *converges to a stationary point of* (ncMVIP)*.*

Proof. Since $\mathcal{L}_\gamma(\gamma(x^0))$ is bounded by assumption, we have $\underline{\gamma} := \inf_{x\in\mathcal{L}_\gamma(\gamma(x^0))} \gamma(x)$ is well-defined due to the continuity of γ and the closedness and nonemptiness of $\mathcal{L}_\gamma(\gamma(x^0))$ (since $x^0 \in \mathcal{L}_\gamma(\gamma(x^0))$. From Step 3 of Algorithm 1, if either the constant parameter $c_k = \frac{1}{L_\gamma}$ or the line search procedure is used then it implies

$$\gamma(x^{k+1}) + \frac{1}{2c_k}\|x^{k+1} - x^k\|^2 \leq m_{c_k}(x^{k+1}) + (\tilde{B}x^k - \tilde{\alpha})^T x^k \leq \gamma(x^k), \ \forall k \geq 0. \tag{5.6}$$

Note that the whole sequence $\{x^k\}$ is contained in $\mathcal{L}_\gamma(\gamma(x^0))$. Rearrange and sum up these inequalities for $k = 0$ to $k = K$ we get

$$\sum_{k=0}^{K} \frac{1}{2c_k}\|x^{k+1} - x^k\|^2 \leq \gamma(x^0) - \gamma(x^{K+1}) \leq \gamma(x^0) - \underline{\gamma}. \tag{5.7}$$

Then the inequality (5.4) directly follows from the definition of δ in (5.3). Combining (5.4) and (4.10) in Lemma 4.1 we obtain (5.5).

To prove the remainder, by taking into account Remark 4.2 and then passing to the limit as k tends to ∞ the resulting inequality of (5.7), we get

$$\sum_{k=0}^{\infty} \frac{1}{2\bar{c}}\|x^{k+1} - x^k\|^2 < +\infty.$$

Since $\bar{c} < +\infty$, this inequality implies that $\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0$. Therefore, the set of limit points is connected. Combine this relation and the assumption of boundedness of $\{x^k\}$ it is easy to show that every limit point of $\{x^k\}$ is a stationary point of (ncMVIP). When the set of the limit points is finite, the last statement of the theorem is proved similarly by using the same technique as in [19][Chapt. 28]. □

Remark 5.2. For a given tolerance $\varepsilon > 0$, according to Theorem 5.1, the number of iterations k to get an ε-stationary point is $O(\varepsilon^2)$. Consequently, the worst-case complexity of Algorithm 1 is $O(1/\varepsilon^2)$.

6. Numerical test

In this section, we consider to numerical examples involving concave cost functions. The aim of these examples is to estimate the number of iterations of Algorithm 1 in a certain case compared to the worst-case complexity given in Theorem

5.1. In addition, we also test the time profile of the algorithm when the size of problem increases.

The algorithm is implemented in Matlab 7.8.0 (R2009a) running on a Pentium IV PC desktop with 2.6GHz and 512Mb RAM. We assume that the feasible set C of (ncMVIP) is a box in $\mathbb{R}^n$. Therefore, the convex problem (3.15) reduces to quadratic programming. We solve this problem by using the `quadprog` solver (a built-in Matlab solver).

Example 1. Suppose that the cost function $h_i(x_i)$ of the firm i is given as $h_i(x_i) = c_i^0 + c_i \ln(1 + r_i x_i)$, where $c_i^0 \geq 0$ is the ceiling cost, $c_i > 0$ and $r_i > 0$ are given. The function h becomes

$$h(x) = c^0 + \sum_{i=1}^{n} c_i \ln(1 + r_i x_i) = c^0 + \ln \prod_{i=1}^{n} (1 + r_i x_i)^{c_i}, \tag{6.1}$$

where $c^0 = \sum_{i=1}^n c_i^0$. It is obvious that h_i is well-defined if $x_i \geq 0$ and $h_i'(x_i) = \frac{c_i r_i}{1+r_i x_i}$, which implies that h is differentiable on $\mathbb{R}^n_+$ and

$$\nabla h(x) = (\frac{c_1 r_1}{1 + r_1 x_1}, \ldots, \frac{c_n r_n}{1 + r_n x_n})^T. \tag{6.2}$$

Since $h_i''(x_i) = -c_i r_i^2/(1+r_i x_i)^2$, we have $|h_i''(x_i)| \leq c_i r_i^2$ and h is concave. Moreover, ∇h is Lipschitz continuous with a Lipschitz constant $L_h := \max\{c_i r_i^2 \mid i = 1, \ldots, n\}$.

In this example, we choose $\beta = 0.1 > 0$, $\alpha = 10$, $c_i^0 = 2$, $c_i = 1.5$ for all $i = 1, \ldots, n$, and $r_i = 1 + \omega_i$, where ω_i is randomly generated in $(0, 1)$ $(i = 1, \ldots, n)$. The strategy set of the firm i is defined by $C_i := [0, 10]$ for all $i = 1, \ldots, n$.

We test Algorithm 1 for problem (ncMVIP) with the size increasing from 10 to 1000. The tolerance ε is 10^{-3}. The number of iterations as well as the CPU time with respect to the size of the problems is visualized in Fig1. and Fig2., respectively.

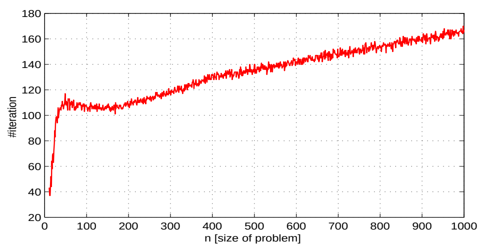

Fig1. Number of iterations depending on n [Ex. 1]

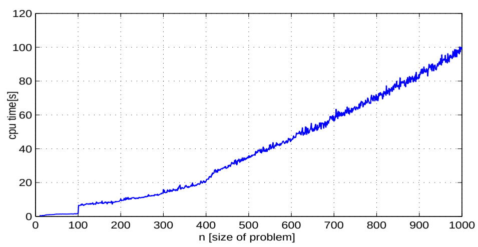

Fig2. CPU time depending on n [Ex. 1]

From (5.5) of Theorem 5.1, it implies that the number of iterations k to reach an ε-stationary point depends on the structure of the function h and the value $\gamma(x^0) - \underline{\gamma}$, L_h and $\|\tilde{B}\|$. Since h is a logarithmic function , the value of h slowly increases in n, while the Lipschitz constant $L_h \leq 1.5 \times 2^2 = 6$ for all n and the norm $\|\tilde{B}\| = (n-1)\beta$. Consequently, the worst-case complexity bound increases almost linearly in n. As can be seen from the first figure, the number of iterations increases with a small slope when the size of problem grows up. The curvature of this graph stays below a linear line generated by the worst-case complexity bound. The CPU time also increases almost linearly in the size of problem.

Example 2. In this example, we choose the cost function h_i as $h_i(x_i) = c_i^0 - c_i e^{-r_i x_i}$, where $c_i^0 \geq c_i > 0$ and $r_i > 0$ given. It is easy to see that $h_i''(x_i) = -c_i \alpha_i^2 e^{-r_i x_i} < 0$, then h_i is concave. Since h_i is differentiable on $\mathbb{R}$, it means that h is differentiable on $\mathbb{R}^n$ and ∇h is expressed by

$$\nabla h(x) = (c_1 r_1 e^{-r_1 x_1}, \ldots, c_n r_n e^{-r_n x_n})^T. \tag{6.3}$$

We have $|h_i''(x_i)| \leq c_i r_i^2$ for all $i = 1, \ldots, n$, thus ∇h is Lipschitz continuous on $\mathbb{R}^n$ with a Lipschitz constant $L_h := \max\{c_i r_i^2 \mid 1 \leq i \leq n\}$.

To compare with the previous example, we choose the value of the parameters α and β as in Example 1. The parameters c_i^0 and c_i are given by $c_i^0 = 4$ and $c_i = 2$ for all $i = 1, \ldots, n$. The parameter $r_i := 0.1 + 0.1\text{rand}_i$, where rand_i is generated randomly in $(0, 1)$.

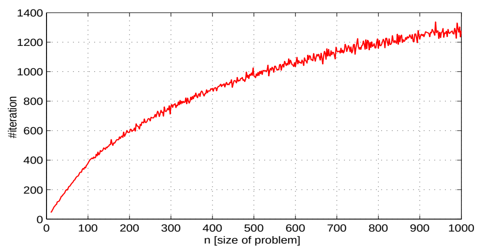

Fig3. Iterations depending on n [Ex. 2]

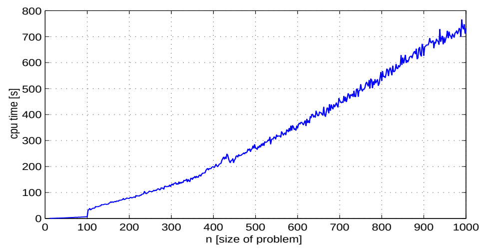

Fig4. CPU time depending on n [Ex. 2]

We also test Algorithm 1 for the problem size from 10 to 1000. The number of iterations and the CPU time are plotted in Fig3. and Fig4., respectively. Since the function γ rapidly increases in n compared to the previous case, the number of iteration also increases. Consequently, the CPU time respectively increases as can be seen in Fig4.

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