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A GENERAL COMPOSITE ITERATION METHOD FOR MONOTONE MAPPINGS AND A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS

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Abstract. We introduce a general composite iterative scheme for an inverse-strongly monotone mapping and a countable family of nonexpansive mappings in Hilbert spaces. It is proved that the sequence generated by the proposed iterative scheme converges strongly to a common point of the set of solutions of variational inequality for an inverse-strongly monotone mapping and the set of fixed points of a countable family of nonexpansive mapping, which is the unique solution of a certain variational inequality being the optimality condition for some minimization problem. Our results substantially improve and develop the corresponding results of Chen et al. [5], Iiduka and Takahashi [9], and Jung [11].

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0, 1)$ such that $\|f(x) - f(y)\| \le k\|x - y\|,\ x,\ y \in C$. We use Π_C to denote the collection of mappings f verifying the above inequality. That is, $\Pi_C = \{f : C \to C \mid f \text{ is a contraction with constant } k\}$. A mapping $S : C \to C$ is called *nonexpansive* if $\|Sx - Sy\| \le \|x - y\|\ \ x,\ y \in C$: see [7,19] for the results of nonexpansive mappings. We denote by $F(S)$ the set of fixed points of S; that is, $F(S) = \{x \in C : x = Sx\}$.

Recall that a linear bounded operator B is strongly positive if there is a constant $\overline{\gamma} > 0$ with property

$$\langle Bx, x\rangle \ge \overline{\gamma}\|x\|^2, \quad \text{for all } x \in H.$$

Recently, iterative methods for nonexpansive mappings have been applied to solve convex minimization problem; see, e.g., [6, 23, 24, 26] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle, \tag{1.1}$$

where C is the fixed point set of a nonexpansive mapping S and b is a given point in H

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Let P_C be the metric projection of H onto C. A mapping A of C into H is called *monotone* if for $x,\ y \in C$, $\langle x-y, Ax-Ay\rangle \geq 0$. The variational inequality problem is to find a $u \in C$ such that

$$\langle v-u, Au\rangle \geq 0$$

for all $v \in C$; see [2, 4, 13, 24]. The set of solutions of the variational inequality is denoted by $VI(C,A)$. A mapping A of C into H is called *inverse-strongly monotone* if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2$$

for all $x,\ y \in C$; see [3, 8, 14]. For such a case, A is called α-inverse-strongly monotone.

In 2005, Iiduka and Takahashi [9] introduced an iterative scheme for finding a common point of the set of fixed points of a nonexapnsive mapping and the set of solutions of the variational inequality for an inverse-strong monotone mapping as follows: for an α-inverse-strongly-monotone mapping A of C into H, a nonexpansive mapping S of C into itself such that $F(S)\cap VI(C,A) \neq \emptyset$, $x_1 = x \in C$, $\{\alpha_n\} \subset [0,1)$, and $\{\lambda_n\} \subset [0,2\alpha]$,

$$x_{n+1} = \alpha_n x + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{1.2}$$

for every $n \geq 1$, where P_C is the metric projection of H onto C. They proved that the sequence generated by (1.2) converges strongly to $P_{F(S)\cap VI(C,A)}x$ under the following conditions on $\{\alpha_n\}$ and $\{\lambda_n\}$: $\lambda_n \in [c,d]$ for some $c,\ d$ with $0<c<d<2\alpha$ and

$$\lim_{n\to\infty}\alpha_n = 0,\ \sum_{n=1}^{\infty}\alpha_n < \infty,\ \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty \text{ and } \sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty. \tag{1.3}$$

On the other hand, the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [16]. In 2004, in order to extend Theorem 2.2 of Moudafi [16] to a Banach space setting, Xu [25] consider the the following explicit iterative process: for $S: C \to C$ a nonexpansive mapping, $f \in \Pi_C$ and $\alpha_n \in (0,1)$,

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Sx_n, \quad n \geq 1. \tag{1.4}$$

Moreover, in [25], he also studied the strong convergence of $\{x_n\}$ generated by (1.4) as $n \to \infty$ in either a Hilbert space or a uniformly smooth Banach space and showed that the strong $\lim_{n\to\infty} x_n$ is the unique solution of certain variational inequality.

In [24], Xu proved that, for a strongly positive bounded linear operator B with constant $\overline{\gamma}$, the sequence $\{x_n\}$ defined by the following iterative method with the initial guess $x_1 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n b + (I-\alpha_n B)Sx_n, \quad n \geq 1,$$

converges strongly to the unique solution of the minimization problem (1.1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. In 2006, Marino and Xu [15] introduced a new iterative scheme by the viscosity approximation method: for a strongly positive bounded linear operator B with constant $\overline{\gamma}$, $f \in \Pi_H$ and $\gamma > 0$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n B)Sx_n, \quad n \geq 1, \tag{1.5}$$

and proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)} \frac{1}{2} \langle Bx, x \rangle - h(x),$$

where h is a potential function for γf (that is, $h'(x) = \gamma f(x)$ for $x \in H$).

In 2007, as the the viscosity iteration method of (1.2), Chen et al. [5] considered the following iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \quad n \geq 1, \tag{1.6}$$

and showed that the sequence $\{x_n\}$ generated by (1.6) strongly converges strongly to a point in $F(S) \cap VI(C, A)$ under the conditions on $\{\alpha_n\}$ and $\{\lambda_n\}$ in (1.3), which is the unique solution of a certain variational inequality.

In 2010, Jung [11] provided a new composite iterative scheme as follows:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S P_C(y_n - \lambda_n A y_n), \quad n \geq 1, \end{cases} \tag{1.7}$$

where $\{\beta_n\} \in [0, 1]$. Also he proved that the sequence $\{x_n\}$ generated by (1.7) strongly converges strongly to a point in $F(S) \cap VI(C, A)$ under the conditions on $\{\alpha_n\}$ and $\{\lambda_n\}$ in (1.3) and suitable conditions on $\{\beta_n\}$, which is the unique solution of a certain variational inequality.

In this paper, motivated by above-mentioned results [5, 9, 11, 15], we introduce a general composite iterative scheme for finding a common point of the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of fixed points of a countable family of nonexapnsive mappings as follows: for an α-inverse-strongly monotone mapping A of C into H, a countable family of nonexpansive mappings S_n of C into itself such that $\cap_{n=1}^{\infty} F(S_n) \cap VI(C, A) \neq \emptyset$, a contraction f of C into itself with constant k, a strongly positive bounded linear operator B on C with constant $\overline{\gamma}$, $0 < \gamma < \frac{\overline{\gamma}}{k}$, $x_1 \in C$, $\{\alpha_n\}$ and $\{\beta_n\} \subset [0, 1)$, and $\{\lambda_n\} \subset [0, 2\alpha]$,

$$\begin{cases} y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S_n P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S_n P_C(y_n - \lambda_n A y_n), \quad n \geq 1. \end{cases} \tag{1.8}$$

Under appropriate conditions on the sequences $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$, we show that the sequence $\{x_n\}$ generated by (1.8) converges strongly to a unique solution of a certain variational inequality, which is the optimality condition for some minimization problem. Using this result, we first obtain a strong convergence result for finding a common fixed point of a strictly pseudo-contractive mapping and a countable family of nonexpansive mappings. Moreover, we investigate the problem of finding a common point of the set of zero of an inverse-strongly monotone mapping and the set of fixed points of a countable family of nonexpansive mappings. The main results improve and complement the corresponding results of Chen et al. [5], Iiduka and Takahashi [9] and Jung [11]. We point out that the iterative scheme

(1.8) is a new approach for finding solutions of variational inequalities for monotone mappings and the fixed points of a countable family of nonexpansive mappings.

2. Preliminaries and Lemmas

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \le \|x - y\|$$

for all $y \in C$. P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and P_C satisfies

$$\langle x - y, P_C(x) - P_C(y)\rangle \ge \|P_C(x) - P_C(y)\|^2 \tag{2.1}$$

for every $x,\ y \in H$. Moreover, $P_C(x)$ is characterized by the properties:

$$u = P_C(x) \Leftrightarrow \langle x - u, u - y\rangle \ge 0$$

and

$$\|x - y\|^2 \ge \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \ \text{ for all } x \in H,\ y \in C.$$

In the context of the variational inequality problem for a nonlinear mapping A, this implies that

$$u \in VI(C, A) \Longleftrightarrow u = P_C(u - \lambda A u), \quad \text{for any } \lambda > 0. \tag{2.2}$$

It is also well known that H satisfies the *Opial condition* (cf. [7, 21]), that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

We state some examples for inverse-strongly monotone mappings. If $A = I - T$, where T is a nonexpansive mapping of C into itself and I is the identity mapping of H, then A is $\frac{1}{2}$-inverse-strongly monotone and $VI(C, A) = F(T)$. A mapping A of C into H is called *strongly monotone* if there exists a positive real number η such that

$$\langle x - y, Ax - Ay\rangle \ge \eta\|x - y\|^2$$

for all $x,\ y \in C$. In such a case, we say A is η-strongly monotone. If A is η-strongly monotone and κ-*Lipschitz continuous*, that is, $\|Ax - Ay\| \le \kappa\|x - y\|$ for all $x,\ y \in C$, then A is $\frac{\eta}{\kappa^2}$-inverse-strongly monotone.

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x,\ y \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\le \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2.\end{aligned}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H. The following result for the existence of solutions of the variational inequality problem for inverse strongly-monotone mappings was given in Takahashi and Toyoda [22].

Proposition 2.1. *Let C be a bounded closed convex subset of a real Hilbert space and let A be an α-inverse-strongly monotone mapping of C into H. Then, $VI(C, A)$ is nonempty.*

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x,\ y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the *normal cone* to C at v, that is, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \text{ for all } u \in C\}$, and define

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$: see [19, 20].

We need the following lemmas for the proof of our main results.

Lemma 2.2. *In a real Hilbert space H, there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle,$$

for all $x,\ y \in H$.

Lemma 2.3 (Xu [23])**.** *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \beta_n + \gamma_n, \quad n \geq 1,$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=1}^{\infty}(1 - \lambda_n) = 0$,

(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\lambda_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < \infty$,

(iii) $\gamma_n \geq 0\ (n \geq 1),\ \sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 (Marino and Xu [15])**.** *Assume that B is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.5 (Aoyama et al. [1])**.** *Let C be a nonempty closed convex subset of H and $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself. Suppose that*

$$\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in C\} < \infty.$$

Then, for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Sz - S_n z\| : z \in C\} = 0$.

3. Main results

In this section, we present a new general composite iterative scheme for inverse-strongly monotone mappings and a countable family of nonexpansive mappings.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let A be an α-inverse-strongly monotone mapping of C into H and $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(S_n) \cap VI(C,A) \neq \emptyset$. Let B be a strongly positive bounded linear operator on C with constant $\overline{\gamma} \in (0,1)$ and f be a contraction of C into itself with constant $k \in (0,1)$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{k}$. Let $\{x_n\}$ be a sequence generated by*

$$\text{(IS)} \qquad \begin{cases} x_1 = x \in C, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S_n P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\beta_n) y_n + \beta_n S_n P_C(y_n - \lambda_n A y_n), \qquad n \geq 1, \end{cases}$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1]$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the conditions:

(i) *$\alpha_n \to 0$ $(n \to \infty)$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;*

(ii) *$\beta_n \subset [0,a)$ for all $n \geq 0$ and for some $a \in (0,1)$;*

(iii) *$\lambda_n \in [c,d]$ for some c, d with $0 < c < d < 2\alpha$;*

(iv) *$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.*

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in D\} < \infty$ for any bounded subset D of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $F(S) = \cap_{n=1}^{\infty} F(S_n)$. Then $\{x_n\}$ converges strongly to $q \in \cap_{n=1}^{\infty} F(S_n) \cap VI(C,A)$, where $q = P_{\cap_{n=1}^{\infty} F(S_n) \cap VI(C,A))}(\gamma f + I - B)(q)$, which is the unique solution of a variational inequality

$$\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \quad p \in \cap_{n=1}^{\infty} F(S_n) \cap VI(C,A),$$

which is the optimality condition for the minimization problem

$$\min_{x \in \cap_{n=1}^{\infty} F(S_n) \cap VI(C,A)} \frac{1}{2} \langle Bx, x \rangle - h(x),$$

where h is a potential function for γf.

Proof. Since $\alpha_n \to 0$ by the condition (i), we may assume, with no loss of generality, that $\alpha_n < \|B\|^{-1}$ for all $n \geq 1$. From Lemma 2.4, we know that if $0 < \rho \leq \|B\|^{-1}$, then $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$. We will assume that $\|I - B\| \leq 1 - \overline{\gamma}$. Let $Q = P_{\cap_{n=1}^{\infty} F(S_n) \cap VI(C,A)}$. Then $Q(\gamma f + I - B)$ is a contraction of C into itself. Indeed, for $x, y \in C$, we have

$$\begin{aligned} \|Q(\gamma f + &I - B)(x) - Q(\gamma f + I - B)(y)\| \\ &\leq \|(\gamma f + I - B)(x) - (\gamma f + I - B)(y)\| \\ &\leq \gamma \|f(x) - f(y)\| + \|I - B\| \|x - y\| \\ &\leq \gamma k \|x - y\| + (1 - \overline{\gamma}) \|x - y\| \\ &< \|x - y\|. \end{aligned}$$

Since H is complete, there exists a unique point $q \in C$ such that $q = Q(\gamma f + I - B)(q) = P_{\cap_{n=1}^{\infty} F(S_n) \bigcap VI(C,A)}(\gamma f + I - B)(q)$.

Let $z_n = P_C(x_n - \lambda_n A x_n)$ and $w_n = P_C(y_n - \lambda_n A y_n)$ for every $n \geq 1$. Let $u \in \cap_{n=1}^{\infty} F(S_n) \bigcap VI(C, A)$. Since $I - \lambda_n A$ is nonexpansive and $u = P_C(u - \lambda_n A u)$ from (2.2), we have

$$\begin{aligned}\|z_n - u\| &= \|P_C(x_n - \lambda_n A x_n) - P_C(u - \lambda_n A u)\| \\ &\leq \|(x_n - \lambda_n A x_n) - (u - \lambda_n A u)\| \\ &\leq \|x_n - u\|.\end{aligned}$$

Similarly we have $\|w_n - u\| \leq \|y_n - u\|$.

Now we divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. In fact, put $M = \max\{\|x_1 - u\|, \frac{\|\gamma f(u) - Bu\|}{\overline{\gamma} - \gamma k}\}$. It is obvious that $\|x_1 - u\| \leq M$. Suppose that $\|x_n - u\| \leq M$. Then, we have

$$\begin{aligned}\|y_n - u\| &= \|\alpha_n(\gamma f(x_n) - Bu) + (I - \alpha_n B)(S_n z_n - u)\| \\ &\leq \alpha_n \|\gamma f(x_n) - Bu\| + \|I - \alpha_n B\| \|z_n - u\| \\ &\leq \alpha_n[\gamma \|f(x_n) - f(u)\| + \|\gamma f(u) - Bu\|] + (1 - \alpha_n \overline{\gamma})\|x_n - u\| \\ &\leq \alpha_n \gamma k \|x_n - u\| + (1 - \alpha_n \overline{\gamma})\|x_n - u\| + \alpha_n \|\gamma f(u) - Bu\| \\ &= (1 - (\overline{\gamma} - \gamma k)\alpha_n)\|x_n - u\| + \alpha_n(\overline{\gamma} - \gamma k)\frac{1}{\overline{\gamma} - \gamma k}\|\gamma f(u) - Bu\| \\ &\leq (1 - (\overline{\gamma} - \gamma k)\alpha_n)M + (\overline{\gamma} - \gamma k)\alpha_n M = M,\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - u\| &= \|(1 - \beta_n)(y_n - u) + \beta_n(S_n w_n - u)\| \\ &\leq (1 - \beta_n)\|y_n - u\| + \beta_n \|w_n - u\| \\ &\leq (1 - \beta_n)\|y_n - u\| + \beta_n \|y_n - u\| \\ &= \|y_n - u\| \leq M.\end{aligned}$$

So, we have that $\|x_n - u\| \leq M$ for $n \geq 0$ and hence $\{x_n\}$ is bounded and so $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{BS_n z_n\}$, $\{Ax_n\}$ and $\{Ay_n\}$ are bounded. Moreover, since $\|S_n z_n - u\| \leq \|x_n - u\|$ and $\|S_n w_n - u\| \leq \|y_n - u\|$, $\{S_n z_n\}$ and $\{S_n w_n\}$ are also bounded. By condition (i), we also obtain

$$\|y_n - S_n z_n\| = \alpha_n \|\gamma f(x_n) - BS_n z_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.1}$$

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. From (IS), we have

$$\begin{cases} y_{n+1} = \alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}B)S_{n+1}z_{n+1} \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B)S_n z_n. \end{cases}$$

Simple calculations show that

$$\begin{aligned}y_{n+1} - y_n = {} & (I - \alpha_{n+1}B)(S_{n+1}z_{n+1} - S_n z_n) - (\alpha_{n+1} - \alpha_n)BS_n z_n \\ & + \gamma[\alpha_n(f(x_{n+1}) - f(x_n)) + (\alpha_{n+1} - \alpha_n)f(x_n)].\end{aligned}$$

Since

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_nAx_n)\| \\ &\le \|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_{n+1}Ax_n)\|+|\lambda_n-\lambda_{n+1}|\|Ax_n\| \\ &\le \|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|\|Ax_n\|\end{aligned}$$

for every $n \ge 1$, we have

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|(I-\alpha_nB)(S_{n+1}z_{n+1}-S_nz_n)-(\alpha_{n+1}-\alpha_n)BS_nz_n \\ &\quad +\gamma[\alpha_{n+1}(f(x_{n+1})-f(x_n))+(\alpha_{n+1}-\alpha_n)f(x_n)]\| \\ &\le (1-\alpha_{n+1}\overline{\gamma})\|S_{n+1}z_{n+1}-S_nz_n\|+|\alpha_{n+1}-\alpha_n|\|BS_nz_n\| \\ &\quad +\gamma[\alpha_{n+1}\|f(x_{n+1})-f(x_n)\|+|\alpha_{n+1}-\alpha_n|\|f(x_n)\|] \\ &\le (1-\alpha_{n+1}\overline{\gamma})[\|S_{n+1}z_{n+1}-S_{n+1}z_n\|+\|S_{n+1}z_n-S_nz_n\|] \\ &\quad +|\alpha_{n+1}-\alpha_n|\|BS_nz_n\|+\gamma[\alpha_{n+1}k\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|f(x_n)\|] \\ &\le (1-\alpha_{n+1}\overline{\gamma})\|z_{n+1}-z_n\|+\|S_{n+1}z_n-S_nz_n\|+|\alpha_{n+1}-\alpha_n|\|BS_nz_n\| \\ &\quad +\gamma[\alpha_{n+1}k\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|f(x_n)\|] \\ &\le (1-\alpha_{n+1}\overline{\gamma})[\|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|\|Ax_n\|] \\ &\quad +\gamma\alpha_{n+1}k\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|L_1+\|S_{n+1}z_n-S_nz_n\| \\ &\le (1-(\overline{\gamma}-\gamma k)\alpha_{n+1})\|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|L_1 \\ &\quad +|\alpha_{n+1}-\alpha_n|M_1+\|S_{n+1}z_n-S_nz_n\|\end{aligned} \tag{3.2}$$

for every $n \ge 1$, where $M_1 = \sup\{\gamma\|f(x_n)\|+\|BS_nz_n\| : n \ge 1\}$ and $L_1 = \sup\{\|Ax_n\| : n \ge 1\}$.

On the other hand, from (IS) we have

$$\begin{cases} x_{n+1} = (1-\beta_n)y_n+\beta_nS_nw_n \\ x_n = (1-\beta_{n-1})y_{n-1}+\beta_{n-1}S_{n-1}w_{n-1}. \end{cases}$$

Also, simple calculations show that

$$\begin{aligned}x_{n+1}-x_n &= (1-\beta_n)(y_n-y_{n-1})+\beta_n(S_nw_n-S_{n-1}w_{n-1}) \\ &\quad +(\beta_n-\beta_{n-1})(S_{n-1}w_{n-1}-y_{n-1}).\end{aligned}$$

Since

$$\begin{aligned}\|w_n-w_{n-1}\| &\le \|(y_n-\lambda_nAy_n)-(y_{n-1}-\lambda_{n-1}Ay_{n-1})\| \\ &\le \|(y_n-\lambda_nAy_n)-(y_{n-1}-\lambda_nAy_{n-1})\|+|\lambda_{n-1}-\lambda_n|\|Ay_{n-1}\| \\ &\le \|y_n-y_{n-1}\|+|\lambda_{n-1}-\lambda_n|\|Ay_{n-1}\|\end{aligned}$$

for every $n \ge 2$, it follows that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (1-\beta_n)\|y_n-y_{n-1}\| \\ &\quad +\beta_n[\|S_nw_n-S_{n-1}w_n\|+\|S_{n-1}w_n-S_{n-1}w_{n-1}\|] \\ &\quad +|\beta_n-\beta_{n-1}|\|S_{n-1}w_{n-1}-y_{n-1}\| \\ &\le (1-\beta_n)\|y_n-y_{n-1}\|+\beta_n\|w_n-w_{n-1}\|+\beta_n\|S_nw_n-S_{n-1}w_n\| \\ &\quad +|\beta_n-\beta_{n-1}|\|S_{n-1}w_{n-1}-y_{n-1}\| \\ &\le (1-\beta_n)\|y_n-y_{n-1}\|+\beta_n(\|y_n-y_{n-1}\|+|\lambda_{n-1}-\lambda_n|\|Ay_{n-1}\|) \\ &\quad +\beta_n\|S_nw_n-S_{n-1}w_n\|+|\beta_n-\beta_{n-1}|\|S_{n-1}w_{n-1}-y_{n-1}\|\end{aligned} \tag{3.3}$$

$$\leq \|y_n - y_{n-1}\| + |\lambda_{n-1} - \lambda_n| \|Ay_{n-1}\| + |\beta_n - \beta_{n-1}| \|S_{n-1}w_{n-1} - y_{n-1}\| + \|S_n w_n - S_{n-1}w_n\|.$$

Substituting (3.2) into (3.3), we derive

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - (\overline{\gamma} - \gamma k)\alpha_n)\|x_n - x_{n-1}\| + |\lambda_{n-1} - \lambda_n| L_1 + |\alpha_n - \alpha_{n-1}| M_1 \\ &\quad + |\lambda_{n-1} - \lambda_n| \|Ay_{n-1}\| + |\beta_n - \beta_{n-1}| \|S_{n-1}w_{n-1} - y_{n-1}\| \\ &\quad + 2\sup\{\|S_n z - S_{n-1}z\| : z \in D\} \\ &\leq (1 - (\overline{\gamma} - \gamma k)\alpha_n)\|x_n - x_{n-1}\| + L_2|\lambda_{n-1} - \lambda_n| + M_1|\alpha_n - \alpha_{n-1}| \\ &\quad + M_2|\beta_n - \beta_{n-1}| + 2\sup\{\|S_n z - S_{n-1}z\| : z \in D\}, \end{aligned} \tag{3.4}$$

where D is a bounded subset of C containing $\{w_n\}$, $L_2 = \sup\{L_1 + \|Ay_n\| : n \geq 1\}$, and $M_2 = \sup\{\|S_n w_n - y_n\| : n \geq 1\}$. From the conditions (i) and (iv), it is easy to see that

$$\lim_{n\to\infty} (\overline{\gamma} - \gamma k)\alpha_n = 0, \quad \sum_{n=1}^{\infty} (\overline{\gamma} - \gamma k)\alpha_n = \infty$$

and

$$\sum_{n=1}^{\infty} (M_1|\alpha_{n+1} - \alpha_n| + M_2|\beta_{n+1} - \beta_n| + L_2|\lambda_{n+1} - \lambda_n| + 2\sup\{\|S_{n+1}z - S_n z\| : z \in D\}) < \infty.$$

Applying Lemma 2.3 to (3.4), we have

$$\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty.$$

By (3.2), we also have that $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.

Step 3. We show that $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and $\lim_{n\to\infty} \|x_n - S_n z_n\| = 0$. Indeed,

$$\begin{aligned} \|x_{n+1} - y_n\| &= \beta_n \|S_n w_n - y_n\| \\ &\leq \beta_n (\|S_n w_n - S_n z_n\| + \|S_n z_n - y_n\|) \\ &\leq a(\|w_n - z_n\| + \|S_n z_n - y_n\|) \\ &\leq a(\|y_n - x_n\| + \|S_n z_n - y_n\|) \\ &\leq a(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|S_n z_n - y_n\|) \end{aligned}$$

which implies that

$$\|x_{n+1} - y_n\| \leq \frac{a}{1-a}(\|x_{n+1} - x_n\| + \|S_n z_n - y_n\|).$$

Obviously, by (3.1) and Step 2, we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. This implies that that

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.5}$$

By (3.1) and (3.5), we also have

$$\|x_n - S_n z_n\| \leq \|x_n - y_n\| + \|y_n - S_n z_n\| \to 0 \text{ as } n \to \infty. \tag{3.6}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and $\lim_{n\to\infty}\|y_n - z_n\| = 0$. To this end, let $u \in \cap_{n=1}^{\infty} F(S_n) \cap VI(C, A)$. Then, we have

$$\begin{aligned}\|y_n - u\|^2 &= \|\alpha_n(\gamma f(x_n) - Bu) + (I - \alpha_n B)(S_n z_n - u)\|^2\\ &\le (\alpha_n\|\gamma f(x_n) - Bu\| + \|I - \alpha_n B\|\|S_n z_n - u\|)^2\\ &\le \alpha_n\|\gamma f(x_n) - Bu\|^2 + (1 - \alpha_n\overline{\gamma})\|z_n - u\|^2\\ &\quad + 2\alpha_n(1 - \alpha_n\overline{\gamma})\|\gamma f(x_n) - Bu\|\|z_n - u\|\\ &\le \alpha_n\|\gamma f(x_n) - Bu\|^2 + (1 - \alpha_n\overline{\gamma})[\|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Au\|^2]\\ &\quad + 2\alpha_n(1 - \alpha_n\overline{\gamma})\|\gamma f(x_n) - Bu\|\|z_n - u\|\\ &\le \alpha_n\|\gamma f(x_n) - Bu\|^2 + \|x_n - u\|^2 + (1 - \alpha_n\overline{\gamma})c(d - 2\alpha)\|Ax_n - Au\|^2\\ &\quad + 2\alpha_n\|\gamma f(x_n) - Bu\|\|z_n - u\|.\end{aligned}$$

So we obtain

$$\begin{aligned}&- (1 - \alpha_n\overline{\gamma})c(d - 2\alpha)\|Ax_n - Au\|^2\\ &\le \alpha_n\|\gamma f(x_n) - Bu\|^2 + (\|x_n - u\| + \|y_n - u\|)(\|x_n - u\| - \|y_n - u\|)\\ &\quad + 2\alpha_n\|\gamma f(x_n) - Bu\|\|z_n - u\|\\ &\le \alpha_n\|\gamma f(x_n) - Bu\|^2 + (\|x_n - u\| + \|y_n - u\|)\|x_n - y_n\|\\ &\quad + 2\alpha_n\|\gamma f(x_n) - Bu\|\|z_n - u\|.\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_n - y_n\| \to 0$ by the condition (i) and Step 3, respectively, we have $\|Ax_n - Au\| \to 0$ $(n \to \infty)$. Moreover, from (2.1) we obtain

$$\begin{aligned}\|z_n - u\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(u - \lambda_n Au)\|^2\\ &\le \langle x_n - \lambda_n Ax_n - (u - \lambda_n Au), z_n - u\rangle\\ &= \frac{1}{2}\{\|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au)\|^2 + \|z_n - u\|^2\\ &\quad - \|(x_n - \lambda_n Ax_n) - (u - \lambda_n Au) - (z_n - u)\|^2\}\\ &\le \frac{1}{2}\{\|x_n - u\|^2 + \|z_n - u\|^2 - \|x_n - z_n\|^2\\ &\quad + 2\lambda_n\langle x_n - z_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2\}.\end{aligned}$$

and so

$$\begin{aligned}\|z_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, Ax_n - Au\rangle\\ - \lambda_n^2\|Ax_n - Au\|^2.\end{aligned}$$

Thus it follows that

$$\begin{aligned}\|y_n - u\|^2 &\le (\alpha_n\|\gamma f(x_n) - Bu\| + (1 - \alpha_n\overline{\gamma})\|z_n - u\|)^2\\ &\quad + 2\alpha_n(1 - \alpha_n\overline{\gamma})\|\gamma f(x_n) - Bu\|\|z_n - u\|\\ &\le \alpha_n\|\gamma f(x_n) - Bu\|^2 + \|x_n - u\|^2 - (1 - \alpha_n\overline{\gamma})\|x_n - z_n\|^2\\ &\quad + 2(1 - \alpha_n\overline{\gamma})\lambda_n\langle x_n - z_n, Ax_n - Au\rangle - (1 - \alpha_n\overline{\gamma})\lambda_n^2\|Ax_n - Au\|^2\\ &\quad + 2\alpha_n\|\gamma f(x_n) - Bu\|\|z_n - u\|.\end{aligned}$$

Then, we have

$$
\begin{aligned}
&(1-\alpha_n\overline{\gamma})\|x_n - z_n\|^2 \\
&\le \alpha_n\|\gamma f(x_n) - Bu\|^2 + (\|x_n - u\| + \|y_n - u\|)(\|x_n - u\| - \|y_n - u\|) \\
&\quad + 2(1-\alpha_n\overline{\gamma})\lambda_n\langle x_n - z_n, Ax_n - Au\rangle - (1-\alpha_n\overline{\gamma})\lambda_n^2\|Ax_n - Au\|^2 \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Bu\|\|z_n - u\| \\
&\le \alpha_n\|\gamma f(x_n) - Bu\|^2 + (\|x_n - u\| + \|y_n - u\|)\|x_n - y_n\| \\
&\quad + 2(1-\alpha_n\overline{\gamma})\lambda_n\langle x_n - z_n, Ax_n - Au\rangle - (1-\alpha_n\overline{\gamma})\lambda_n^2\|Ax_n - Au\|^2 \\
&\quad + 2\alpha_n\|\gamma f(x_n) - Bu\|\|z_n - u\|.
\end{aligned}
$$

Since $\alpha_n \to 0$, $\|x_n - y_n\| \to 0$ and $\|Ax_n - Au\| \to 0$, we get $\|x_n - z_n\| \to 0$. Also by (3.5)

$$
\|y_n - z_n\| \le \|y_n - x_n\| + \|x_n - z_n\| \to 0 \ (n \to \infty). \tag{3.7}
$$

Step 5. We show that $\lim_{n\to\infty}\|Sz_n - z_n\| = 0$. In fact, since

$$
\begin{aligned}
\|S_n z_n - z_n\| &\le \|S_n z_n - y_n\| + \|y_n - z_n\| \\
&\le \alpha_n\|\gamma f(x_n) - BS_n z_n\| + \|y_n - z_n\|,
\end{aligned}
$$

from (3.7), we have $\lim_{n\to\infty}\|S_n z_n - z_n\| = 0$. Observe that

$$
\begin{aligned}
\|Sz_n - z_n\| &\le \|Sz_n - S_n z_n\| + \|S_n z_n - z_n\| \\
&\le \sup\{\|Sz - S_n z\| : z \in D\} + \|S_n z_n - z_n\|.
\end{aligned}
$$

By Lemma 2.5, we have $\lim_{n\to\infty}\|Sz_n - z_n\| = 0$.

Step 6. We show that $\limsup_{n\to\infty}\langle \gamma f(q) - Bq, y_n - q\rangle \le 0$ for $q \in \cap_{n=1}^{\infty}F(S_n) \cap VI(C,A)$, where $q = P_{\cap_{n=1}^{\infty}F(S_n)\cap VI(C,A)}(\gamma f + I - B)(q)$. To this end, choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$
\limsup_{n\to\infty}\langle \gamma f(q) - Bq, z_n - q\rangle = \lim_{i\to\infty}\langle \gamma f(q) - Bq, z_{n_i} - q\rangle.
$$

Since $\{z_{n_i}\}$ is bounded, there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which converges weakly to z. We may assume without loss of generality that $z_{n_i} \rightharpoonup z$. Since $\|Sz_{n_i} - z_{n_i}\| \to 0$ by Step 5, we have $Sz_{n_i} \rightharpoonup z$. Then we can obtain $z \in \cap_{n=1}^{\infty}F(S_n) \cap VI(C,A)$. Indeed, let us first show that $z \in VI(C,A)$. Let

$$
Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset & v \notin C. \end{cases}
$$

Then T is maximal monotone. Let $(v,w) \in G(T)$. Since $w - Av \in N_C v$ and $z_n \in C$, we have

$$
\langle v - z_n, w - Av\rangle \ge 0.
$$

On the other hand, from $z_n = P_C(x_n - \lambda_n Ax_n)$, we have $\langle v - z_n, z_n - (x_n - \lambda_n Az_n)\rangle \ge 0$ and hence

$$
\langle v - z_n, \frac{z_n - x_n}{\lambda_n} + Ax_n\rangle \ge 0.
$$

Therefore we have

$$\begin{aligned}\langle v - z_{n_i}, w\rangle &\geq \langle v - z_{n_i}, Av\rangle \\ &\geq \langle v - z_{n_i}, Av\rangle - \langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i}\rangle \\ &= \langle v - z_{n_i}, Av - Ax_{n_i} - \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \\ &= \langle v - z_{n_i}, Av - Az_{n_i}\rangle + \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle \\ &\quad - \langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle - \langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle.\end{aligned}$$

Since $\|z_n - x_n\| \to 0$ in Step 4 and A is α-inverse-strongly monotone, we have $\langle v - z, w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$. Next we show that $z \in F(S)$. Assume $z \notin F(S)$. Since $z_{n_i} \rightharpoonup z$ and $z \neq Sz$, from Opial condition and Step 5, we have

$$\begin{aligned}\liminf_{i\to\infty} \|z_{n_i} - z\| &< \liminf_{i\to\infty} \|z_{n_i} - Sz\| \\ &\leq \liminf_{i\to\infty} (\|z_{n_i} - Sz_{n_i}\| + \|Sz_{n_i} - Sz\|) \\ &\leq \liminf_{i\to\infty} \|z_{n_i} - z\|.\end{aligned}$$

This is a contradiction. So, we obtain $z \in F(S) = \cap_{n=1}^{\infty} F(S_n)$. Therefore, $z \in \cap_{n=1}^{\infty} F(S_n) \cap VI(C, A)$. Since $q = P_{\cap_{n=1}^{\infty} F(S_n)\cap VI(C,A)}(\gamma f + I - B)(q)$, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle \gamma f(q) - Bq, z_n - q\rangle &= \lim_{i\to\infty} \langle \gamma f(q) - Bq, z_{n_i} - q\rangle \\ &= \langle \gamma f(q) - Bq, z - q\rangle \leq 0.\end{aligned}$$

Thus, from (3.7) we obtain

$$\begin{aligned}&\limsup_{n\to\infty} \langle \gamma f(q) - Bq, y_n - q\rangle \\ \leq\ &\limsup_{n\to\infty} \langle \gamma f(q) - Bq, y_n - z_n\rangle + \limsup_{n\to\infty} \langle \gamma f(q) - Bq, z_n - q\rangle \\ \leq\ &\limsup_{n\to\infty} \|\gamma f(q) - Bq\| \|y_n - z_n\| + \limsup_{n\to\infty} \langle \gamma f(q) - Bq, z_n - q\rangle \\ \leq\ &0.\end{aligned}$$

Step 7. We show that $\lim_{n\to\infty} \|x_n - q\| = 0$ for $q \in \cap_{n=1}^{\infty} F(S_n) \cap VI(C, A)$, where $q = P_{\cap_{n=1}^{\infty} F(S_n)\cap VI(C,A)}(\gamma f + I - B)(q)$. Indeed, since $\|x_{n+1} - q\| \leq \|y_n - q\|$, $\|z_n - q\| \leq \|x_n - q\|$ and $y_n - q = \alpha_n(\gamma f(x_n) - Bq) + (I - \alpha_n B)(Sz_n - q)$, by Lemma

2.2 we have

$$\begin{aligned}
&\|x_{n+1}-q\|^2\\
&\le \|y_n-q\|^2=\|\alpha_n(\gamma f(x_n)-Bq)+(I-\alpha_nB)(S_nz_n-q)\|^2\\
&\le \|(I-\alpha_nB)(S_nz_n-q)\|^2+2\alpha_n\langle\gamma f(x_n)-Bq,y_n-q\rangle\\
&\le (1-\alpha_n\overline{\gamma})^2\|z_n-q\|^2+2\alpha_n\gamma\langle f(x_n)-f(q),y_n-q\rangle\\
&\le (1-\alpha_n\overline{\gamma})^2\|x_n-q\|^2+2\alpha_n\gamma\langle f(x_n)-f(q),y_n-q\rangle\\
&\qquad +2\alpha_n\langle\gamma f(q)-Bq,y_n-q\rangle)\\
&\le (1-\alpha_n\overline{\gamma})^2\|x_n-q\|^2+2\alpha_n\gamma k\|x_n-q\|\|y_n-q\|\\
&\quad +2\alpha_n\langle\gamma f(q)-Bq,y_n-q\rangle\\
&\le (1-\alpha_n\overline{\gamma})^2\|x_n-q\|^2+2\alpha_n\gamma k\|x_n-q\|(\|y_n-x_n\|+\|x_n-q\|)\\
&\quad +2\alpha_n\langle\gamma f(q)-Bq,y_n-q\rangle\\
&\le\ (1-2(\overline{\gamma}-\gamma k)\alpha_n)\|x_n-q\|^2+\alpha_n^2\overline{\gamma}^2\|x_n-q\|^2+2\alpha_n\gamma k\|y_n-x_n\|\|x_n-q\|\\
&\qquad +2\alpha_n\langle\gamma f(q)-Bq,y_n-q\rangle\\
&\le (1-\overline{\alpha_n})\|x_n-q\|^2+\overline{\beta_n},
\end{aligned}$$

where

$$\begin{aligned}
\overline{\alpha_n}&=2(\overline{\gamma}-\gamma k)\alpha_n,\\
\overline{\beta_n}&=\alpha_n^2\overline{\gamma}^2M_1^2+2\alpha_n\gamma k\|y_n-x_n\|M_1+2\alpha_n\langle\gamma f(q)-Bq,y_n-q\rangle,
\end{aligned}$$

and $M_1=\sup\{\|x_n-q\|:n\ge 1\}$. From (i), Step 4 and Step 6, it is easily seen that $\overline{\alpha_n}\to 0$, $\sum_{n=1}^{\infty}\overline{\alpha_n}=\infty$, and $\limsup_{n\to\infty}\frac{\overline{\beta_n}}{\overline{\alpha_n}}\le 0$. Thus, by Lemma 2.3, we obtain $x_n\to q$. This completes the proof. □

Remark 3.2. We can obtain that if q solves the minimization problem

$$\min_{x\in\cap_{n=1}^{\infty}F(S_n)\cap VI(C,A)}\frac{1}{2}\langle Bx,x\rangle-h(x),$$

where h is a potential function for γf, then

$$\langle\gamma f(q)-Bq,p-q\rangle\le 0,\quad p\in\cap_{n=1}^{\infty}F(S_n)\cap VI(C,A).$$

For this fact, we also refer [10, 17].

As direct consequences of Theorem 3.1, we have the following results.

Corollary 3.3. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty}F(S_n)\cap VI(C,A)\ne\emptyset$. Let f be a contraction of C into itself with constant $k\in(0,1)$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases}x_1=x\in C,\\ y_n=\alpha_nf(x_n)+(1-\alpha_n)S_nP_C(x_n-\lambda_nAx_n),\\ x_{n+1}=(1-\beta_n)y_n+\beta_nS_nP_C(y_n-\lambda_nAy_n),\qquad n\ge 1,\end{cases}$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, $\{\alpha_n\} \subset [0, 1)$ *and* $\{\beta_n\} \subset [0, 1]$. *Let* $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\beta_n\}$ *satisfy the conditions:*

(i) $\alpha_n \to 0$ $(n \to \infty)$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\beta_n \subset [0, a)$ *for all* $n \geq 0$ *and for some* $a \in (0, 1)$;

(iii) $\lambda_n \in [c, d]$ *for some* c, d *with* $0 < c < d < 2\alpha$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in D\} < \infty$ *for any bounded subset* D *of* C. *Let* S *be a mapping of* C *into itself defined by* $Sz = \lim_{n\to\infty} S_n z$ *for all* $z \in C$ *and suppose that* $F(S) = \cap_{n=1}^{\infty} F(S_n)$. *Then* $\{x_n\}$ *converges strongly to* $q \in \cap_{n=1}^{\infty} F(S_n) \cap VI(C, A)$, *where* $q = P_{\cap_{n=1}^{\infty} F(S_n) \cap VI(C,A))} f(q)$, *which solves a variational inequality*

$$\langle f(q) - q, p - q\rangle \leq 0, \quad p \in F(S) \cap VI(C, A).$$

Proof. Taking $B = I$ and $\gamma = 1$ in Theorem 3.1, we can obtain the desired result. □

Corollary 3.4. *Let* C *be a closed convex subset of a real Hilbert space* H *such that* $C \pm C \subset C$. *Let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *and* S *be a nonexpansive mapping of* C *into itself such that* $F(S) \cap VI(C, A) \neq \emptyset$. *Let* B *be a strongly positive bounded linear operator on* C *with constant* $\overline{\gamma} \in (0, 1)$ *and* f *be a contraction of* C *into itself with constant* $k \in (0, 1)$. *Assume that* $0 < \gamma < \frac{\overline{\gamma}}{k}$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S P_C(y_n - \lambda_n A y_n), \quad n \geq 1, \end{cases}$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, $\{\alpha_n\} \subset [0, 1)$ *and* $\{\beta_n\} \subset [0, 1]$. *If* $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\beta_n\}$ *satisfy the conditions:*

(i) $\alpha_n \to 0$ $(n \to \infty)$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\beta_n \subset [0, a)$ *for all* $n \geq 0$ *and for some* $a \in (0, 1)$;

(iii) $\lambda_n \in [c, d]$ *for some* c, d *with* $0 < c < d < 2\alpha$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

then $\{x_n\}$ *converges strongly to* $q \in F(S) \cap VI(C, A)$, *where* $q = P_{F(S)\cap VI(C,A))}(\gamma f + I - B)(q)$, *which is the unique solution of a variational inequality*

$$\langle \gamma f(q) - Bq, p - q\rangle \leq 0, \quad p \in F(S) \cap VI(C, A).$$

Corollary 3.5. *Let* C *be a closed convex subset of a real Hilbert space* H *such that* $C \pm C \subset C$. *Let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *such that* $VI(C, A) \neq \emptyset$. *Let* B *be a strongly positive bounded linear operator on* C *with constant* $\overline{\gamma} \in (0, 1)$ *and* f *be a contraction of* C *into itself with constant* $k \in (0, 1)$. *Assume that* $0 < \gamma < \frac{\overline{\gamma}}{k}$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B) P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n P_C(y_n - \lambda_n A y_n), \quad n \geq 1, \end{cases}$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, $\{\alpha_n\} \subset [0, 1)$ *and* $\{\beta_n\} \subset [0, 1]$. *If* $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\beta_n\}$ *satisfy the conditions:*

(i) $\alpha_n \to 0$ $(n \to \infty)$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\beta_n \subset [0, a)$ *for all* $n \geq 0$ *and for some* $a \in (0, 1)$;

(iii) $\lambda_n \in [c, d]$ *for some* c, d *with* $0 < c < d < 2\alpha$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

then $\{x_n\}$ *converges strongly to* $q \in VI(C, A)$, *which is the unique solution in* $VI(C, A)$ *to the following variational inequality*

$$\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \quad p \in VI(C, A).$$

Remark 3.6. (1) Theorem 3.1 (and Corollary 3.4) improves the corresponding results in Chen et al. [5], Iiduka and Takahashi [9], and Jung [11].

(2) Theorem 3.1 of Jung [11] is a special case of Corollary 3.3 with $S_n = S$ for $n \geq 1$. Also, if $S_n = S$, $\beta_n = 0$ and $f(x_n) = x$ is constant in Corollary 3.3, then Corollary 3.3 reduces to Theorem 3.1 of Iiduka and Takahashi [9].

(3) As in Remark 3.1 of Peng and Yao [18], we can obtain a sequence $\{W_n\}$ of nonexpansive mappings satisfying the condition $\sum_{n=1}^{\infty} \sup\{\|W_{n+1}z - W_n\| : z \in D\} < \infty$ for any bounded subset D of H. So, by replacing $\{S_n\}$ by $\{W_n\}$ in the iterative scheme (IS) in Theorem 3.1, we can obtain the corresponding results of the so-called W-mapping.

(4) Other example of a sequence of nonexpansive mappings satisfying the condition in Theorem 3.1 can be also found in [1, Section 4].

(5) We obtain a new composite iterative scheme for nonexpansive mapping if $A = 0$ in Theorem 3.1 as follows:

$$\begin{cases} x_1 = x \in C, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S_n x_n, \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S_n y_n. \end{cases}$$

This composite iterative scheme reduces to an iterative scheme (1.5) of Marino and Xu [15] if $\beta_n = 0$ and $S_n = S$ for $n \geq 1$.

4. Applications

In this section, as in [5, 9, 11], we prove two theorems in a Hilbert space by using Theorem 3.1.

A mapping $T : C \to C$ is called *strictly pseudo-contractive* if there exists α with $0 \leq \alpha < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \alpha \|(I - T)x - (I - T)y\|^2$$

for every x, $y \in C$. If $\alpha = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudo-contractive mapping with α. Then A is $\frac{1-\alpha}{2}$-inverse-strongly monotone; see [3]. Actually, we have, for all x, $y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + \alpha \|Ax - Ay\|^2.$$

On the other hand, since H is a real Hilbert space, we have

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay \rangle.$$

Hence we obtain

$$\langle x-y, Ax-Ay\rangle \geq \frac{1-\alpha}{2}\|Ax-Ay\|^2.$$

Using Theorem 3.1, we first establish a strong convergence theorem for finding a common fixed point of a countable family of nonexpansive mapping and a strictly pseudo-contractive mapping.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let T be an α-strictly pseudo-contractive mapping of C into itself and $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(S_n) \cap F(T) \neq \emptyset$. Let B be a strongly positive bounded linear operator on C with constant $\overline{\gamma} \in (0,1)$ and f be a contraction of C into itself with constant $k \in (0,1)$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{k}$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B)S_n((1-\lambda_n)x_n + \lambda_n T x_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S_n((1-\lambda_n)y_n + \lambda_n T y_n), \qquad n \geq 1, \end{cases}$$

where $\{\lambda_n\} \subset [0, 1-\alpha)$, $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1]$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the conditions:

(i) $\alpha_n \to 0$ $(n \to \infty)$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\beta_n \subset [0,a)$ *for all* $n \geq 0$ *and for some* $a \in (0,1)$;

(iii) $\lambda_n \in [c,d]$ *for some* $c,\ d$ *with* $0 < c < d < 1-\alpha$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in D\} < \infty$ for any bounded subset D of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $F(S) = \cap_{n=1}^{\infty} F(S_n)$. Then $\{x_n\}$ converges strongly to $q \in \cap_{n=1}^{\infty} F(S_n) \cap F(T)$, which is the unique solution in $\cap_{n=1}^{\infty} F(S_n) \cap F(T)$ to the following variational inequality

$$\langle \gamma f(q) - Bq, p - q\rangle \leq 0, \quad p \in \cap_{n=1}^{\infty} F(S_n) \cap F(T).$$

Proof. Put $A = I - T$. Then A is $\frac{1-\alpha}{2}$-inverse-strongly monotone. We have $F(T) = VI(C,A)$ and $P_C(x_n - \lambda_n A x_n) = (1-\lambda_n)x_n + \lambda_n T x_n$. Thus, the desired result follows from Theorem 3.1. □

Using Theorem 3.1, we also obtain the following result.

Theorem 4.2. *Let H be a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of H into itself and $\{S_n\}$ be a sequence of nonexpansive mappings of H into itself such that $\cap_{n=1}^{\infty} F(S_n) \cap A^{-1}0 \neq \emptyset$. Let B be a strongly positive bounded linear operator on H with constant $\overline{\gamma} \in (0,1)$ and f be a contraction of H into itself with constant $k \in (0,1)$. Assume that $0 < \gamma < \frac{\overline{\gamma}}{k}$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in H, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B)S_n(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S_n(y_n - \lambda_n A y_n), \qquad n \geq 1, \end{cases}$$

where $\{\lambda_n\} \subset [0, 2\alpha)$, $\{\alpha_n\} \subset [0, 1)$ *and* $\{\beta_n\} \subset [0, 1]$. *Let* $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\beta_n\}$ *satisfy the conditions:*

(i) $\alpha_n \to 0$ $(n \to \infty)$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\beta_n \subset [0, a)$ *for all* $n \geq 0$ *and for some* $a \in (0, 1)$;

(iii) $\lambda_n \in [c, d]$ *for some* c, d *with* $0 < c < d < 2\alpha$;

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in D\} < \infty$ *for any bounded subset* D *of* H. *Let* S *be a mapping of* H *into itself defined by* $Sz = \lim_{n\to\infty} S_n z$ *for all* $z \in H$ *and suppose that* $F(S) = \cap_{n=1}^{\infty} F(S_n)$. *Then* $\{x_n\}$ *converges strongly to* $q \in \cap_{n=1}^{\infty} F(S_n) \cap A^{-1}0$, *which is the unique solution in* $\cap_{n=1}^{\infty} F(S_n) \cap A^{-1}0$ *to the following variational inequality*

$$\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \quad p \in \cap_{n=1}^{\infty} F(S_n) \cap A^{-1}0.$$

Proof. We have $A^{-1}0 = VI(H, A)$. So, putting $P_H = I$, by Theorem 3.1, we obtain the desired result. □

Remark 4.3. (1) Theorems 4.1 and 4.2 improve and extend Theorems 4.1 and 4.2 in Chen et al. [5] and Jung [11] from one nonexpansive mapping to a countable family of nonexpansive mapping. In particular, if $B = I$, $\gamma = 1$, and $S_n = S$ for $n \geq 1$ in Theorems 4.1 and 4.2, we obtain Theorems 4.1 and 4.2 in Jung [11].

(2) If $B = I$, $\gamma = 1$, $\beta_n = 0$ and $S_n = S$ for $n \geq 1$ in Theorems 4.1 and 4.2, then we also get Theorems 4.1 and 4.2 in Chen et al [5].

(3) Theorems 4.1 and 4.2 also extend Theorem 4.1 and 4.2 in Iiduka and Takahashi [9] to the viscosity methods in general composite iterative schemes with a countable family of nonexpansive mappings.

(4) In all our results, we can replace the condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ on the control parameter $\{\alpha_n\}$ by the condition $\alpha_n \in (0, 1]$ for $n \geq 1$, $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$ ([23, 24]) or by the perturbed control condition $|\alpha_{n+1} - \alpha_n| < o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=1}^{\infty} \sigma_n < \infty$ ([12]).

References

[1] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space,* Nonlinear Anal. **67** (2007), 2350–2360.

[2] F. E. Browder, *Nonlinear monotone operators and convex sets in Banach spaces,* Bull. Amer. Math. Soc. **71** (1965), 780–785.

[3] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces,* J. Math. Anal. Appl. **20** (1967), 197–228.

[4] R. E. Bruck, *On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space,* J. Math. Anal. Appl. **61** (1977), 159–164.

[5] J. Chen, L. Zhang and T. Fan, *Viscosity approximation methods for nonexpansive mappings and monotone mappings,* J. Math. Anal. Appl. **334** (2007), 1450–1461.

[6] F. Deutsch and I. Yamada, *Minimizing certain convex functions over the intersection of the fixed point set of nonexpansive mappings,* Numer. Funct. Anal. Optim. **19** (1998), 33–56.

[7] K. Goebel and W. A. Kirk, Topics in Metric Fixed Point Theory, in Cambridge Studies in Advanced Mathematics Vol. 28, Cambridge Univ. Press, Cambridge, UK, 1990.

[8] H. Iiduka, W. Takahashi and M. Toyoda, *Approximation of solutions of variational inequalities for monotone mappings,* Panamer. Math. J. **14** (2004), 49–61.

[9] H. Iiduka and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-stronly monotone mappings,* Nonlinear Anal. **61** (2005), 341–350.

[10] J. S. Jung, *Iterative algorithms with some control conditions for quadratic optimizations,* Panamer. Math. J. **16** (2006), 13–25.

[11] J. S. Jung, *A new iteration method for nonexpansive mappings and monotone mappings in Hilbert spaces,* J. Inequal. Appl. **2010** (2010) Article ID 251761, 16 pages.

[12] J. S. Jung, Y. J. Cho and R. P. Agarwal, *Iterative schemes with some control conditions for a family of finite nonexpansive mappings in Banach spaces,* Fixed Point Theory Appl. **2005** (2005), 125–135.

[13] P. L. Lions and G. Stampacchia, *Variational inequalities,* Comm. Pure Appl. Math. **20** (1967), 493–517.

[14] F. Liu and M. Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates,* Set-Valued Anal. **6** (1998), 313–344.

[15] G. Marino and H. K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces,* J. Math. Anal. Appl. **318** (2006), 43–52.

[16] A. Moudafi, *Viscosity approximation methods for fixed-points problems,* J. Math. Anal. Appl. **241** (2000), 46–55.

[17] J. T. Oden, *Qualitative methods on nonlinear mechanics,* Prentice-Hall, Englewood Cliffs, New Jersey, 1986.

[18] J. W. Peng and J. C. Yao, *A viscosity approximation scheme for system of equilibrium problems, nonexpansive mappings and monotone mappings,* Nonlinear Anal. **71** (2009), 6001–6010.

[19] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators,* Trans. Amer. Math. Soc. **149** (1970), 75–88.

[20] R. T. Rockafellar, *Monotone operators and the proximal point theorems,* SIAM J. Control Optim. **14** (1976), 877–898.

[21] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, Japan, 2000.

[22] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings,* J. Optim. Theory Appl. **118** (2003), 417–428.

[23] H. K. Xu, *An iterative algorithm for nonlinear operator,* J. London Math. Soc. **66** (2002), 240–256.

[24] H. K. Xu, *An iterative approach to quadratic optimization,* J. Optim. Theory Appl. **116** (2003), 659–678.

[25] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings,* J. Math. Anal. Appl. **298** (2004), 279–291.

[26] I. Yamada, *The hybrid steepest descent method for the variational inequality of the intersection of fixed point sets of nonexpansive mappings,* in D. Butnariu, Y. Censor, S. Reich (Eds), Inherently Parallel Algorithm for Feasibility and Optimization, and Their Applications, Kluwer Academic Publishers, Dordrecht, Holland, 2001, pp. 473–504.

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