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SOME PROPERTIES OF POLYHEDRAL MULTIFUNCTIONS

N. T. QUI AND N. D. YEN

Dedicated to Professor Pham Huu Sach on the occasion of his 70th birthday

Abstract. This paper establishes several properties of polyhedral multifunctions with respect to: (a) Relationships between the lower semicontinuity and the pseudo-Lipschitz property, (b) Connectedness preservation, (c) Path-connectedness preservation. We make use of a theorem on the existence of Lipschitz selections of Lipschitz continuous multifunctions with nonempty closed convex values in Euclidean spaces.

1. Introduction

The notion of polyhedral multifunctions, which plays one important role in optimization theory and the theory of equilibrium problems, was introduced by S. M. Robinson in 1981.

Definition 1.1 ([10])**.** A set-valued map $\Phi : I\!R^n \rightrightarrows I\!R^m$ is called a *polyhedral multifunction* if $\mathrm{gph}\Phi := \{(x, y) \in I\!R^n \times I\!R^m \mid y \in \Phi(x)\}$ can be represented as the union of finitely many nonempty polyhedral convex sets $Q_1, Q_2, \ldots, Q_k$ of the product space $I\!R^n \times I\!R^m$, i.e.,

$$\mathrm{gph}\Phi = \bigcup_{j=1}^{k} Q_j. \tag{1.1}$$

The normal-cone operator of a polyhedral convex set, the multifunction defining a parametric affine variational inequality (also called a linear generalized equation), as well as the solution map of an affine variational inequality with respect to the linear part of its data are well-known examples of polyhedral multifunctions; see [10], [11], and [7].

By using the Walkup-Wets theorem, Robinson [10] proved that polyhedral multifunctions are locally upper Lipschitz maps. Recently, Dontchev and Rockafellar [4, Corollary 3D.4] have obtained a result on the relationships between the lower semicontinuity and the Lipschitz-like property. Infinite dimensional polyhedral multifunctions are defined and studied by Bonnans and Shapiro in [3].

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By a theorem of Warburton [12, Theorem 3.1] we know that upper semicontinuous multifunctions with connected nonempty values map connected sets to connected sets. Lower semicontinuous (= inner semicontinuous at any point in the graph) multifunctions with connected nonempty values have the same 'connectedness preservation' property (see Hiriart-Urruty [5, Theorem 3.1]).

In some questions related to

- differentiation rules of multifunctions [8],
- stability of nonlinear optimization problems and variational systems (in particular, stability of quadratic programming and affine variational inequality problems), see e.g. [4, 7, 8],

the *Lipschitz-like property* of multifunctions, which was introduced and termed the *pseudo-Lipschitz property* by J.-P. Aubin [1], has shown to be a very useful concept. This property is available if one imposes certain regularity assumptions using derivatives [2] or coderivatives [8] of the set-valued maps. The Lipschitz-like property implies inner semicontinuity property, but the converse is not true in general.

Our aim in this paper is to get some properties of polyhedral multifunctions with respect to:

(a) Relationships between the lower semicontinuity and the Lipschitz-like property,
(b) Connectedness preservation,
(c) Path-connectedness preservation.

Topic (a) is addressed in Section 2. Topics (b) and (c) are studied in Section 3, where we will make use of a theorem on the existence of Lipschitz selections of Lipschitz continuous maps with nonempty closed convex values in Euclidean spaces, which is available in [2, Chapter 8]. Several useful examples will be given.

Let $F : X \rightrightarrows Y$ be a multifunction, where $X \subset I\!\!R^n$ and $Y \subset I\!\!R^m$ are given subsets. We equip X and Y, respectively, with the induced topologies τ_X and τ_Y. (The induced topology is also called the relative topology.) By definition, $U \in \tau_X$ if and only if there is an open set $\Omega \subset I\!\!R^n$ such that $U = \Omega \cap X$. The graph and the effective domain of F are given by $\text{gph}F = \{(x, y) \in I\!\!R^n \times I\!\!R^m \mid y \in F(x)\}$ and $\text{dom}\, F = \{x \in I\!\!R^n \mid F(x) \neq \emptyset\}$.

For any $a \in I\!\!R^n$ and $\varepsilon > 0$, denote the open ball $\{x \in I\!\!R^n \mid \ ||x - a|| < \varepsilon\}$ by $B(a, \varepsilon)$, and the corresponding closed ball by $\bar{B}(a, \varepsilon)$. We will write $\bar{B}_{I\!\!R^n}$ for $\bar{B}(0_{I\!\!R^n}, 1)$. The interior of $\bar{B}_{I\!\!R^n}$ is denoted by $B_{I\!\!R^m}$.

We say that F is *inner semicontinuous* (isc for brevity) at $(x, y) \in \text{gph}F$ if for any $V \in \tau_Y$ containing y there exists $U \in \tau_X$ with $x \in U$ such that $F(u) \cap V \neq \emptyset$ for all $u \in U$. (This isc property is stronger than that one in [8, Def. 1.63] where, in our notation, the condition $F(u) \cap V \neq \emptyset$ is required only for those $u \in U$ taken from $\text{dom}\, F$.) As usual (see e.g. [7, p. 148]), if for any $V \in \tau_Y$ with $F(x) \cap V \neq \emptyset$ there exists $U \in \tau_X$ with $x \in U$ such that $F(u) \cap V \neq \emptyset$ for all $u \in U$, then F is said to be *lower semicontinuous* (lsc) at $x \in X$. In the same manner, one says that F is *upper semicontinuous* (usc) at $x \in X$ if for any $V \in \tau_Y$ containing $F(x)$ there exists $U \in \tau_X$ with $x \in U$ such that $F(u) \subset V$ for all $u \in U$. It is obvious that F is lsc at $x \in X$ if and only if F is isc at any point $(x, y) \in \{x\} \times F(x)$. When F is lsc (resp., usc) at any $x \in X$, we say that F is lsc (resp., usc) on X.

One says that F is *Lipschitz continuous* on X if there exists a constant $\ell > 0$ such that

$$F(x) \subset F(u) + \ell||x-u||\bar{B}_{I\!R^m} \quad \forall x, u \in X.$$

If there is $\ell > 0$ satisfying $F(x) \subset F(u) + \ell||x-u||\bar{B}_{I\!R^m}$ for all x, u from a neighborhood $U \in \tau_X$ of $\overline{x} \in X$, then F is called *locally Lipschitz* at $\overline{x}$. When it holds $F(x) \subset F(\overline{x}) + \ell||x-\overline{x}||\bar{B}_{I\!R^m}$ for some $\ell > 0$ and for all x from a neighborhood $U \in \tau_X$ of $\overline{x} \in X$, F is said to be *locally upper Lipschitz* at $\overline{x}$ with the modulus ℓ. If $(\overline{x}, \overline{y}) \in \operatorname{gph} F$ and there are $\ell > 0$ and neighborhoods $U \in \tau_X$ of $\overline{x}$, $V \in \tau_Y$ of $\overline{y}$ with

$$F(x) \cap V \subset F(u) + \ell||x-u||\bar{B}_{I\!R^m} \quad \forall x, u \in U,$$

then F is termed *Lipschitz-like* at $(\overline{x}, \overline{y})$; see [1, 8].

For a subset $\Omega \subset I\!R^n$, the symbols $\overline{\Omega}$ and $\text{int}\Omega$, respectively, denote the closure and the interior of Ω. The boundary $\partial\Omega$ of Ω is defined by setting $\partial\Omega = \overline{\Omega} \setminus \text{int}\Omega$.

Definition 1.2 ([10])**.** A set-valued map $\Phi : I\!R^n \rightrightarrows I\!R^m$ is a *polyhedral convex multifunction* if $\text{gph}\Phi$ is a nonempty polyhedral convex set in the product space $I\!R^n \times I\!R^m$.

Theorem 1.3 (See for instance [11] and [7, Claim 1, p.126])**.** *If $\Phi : I\!R^n \rightrightarrows I\!R^m$ is a polyhedral convex multifunction, then there exists a constant $\ell > 0$ such that*

$$\Phi(x) \subset \Phi(u) + \ell||x-u||\bar{B}_{I\!R^m} \quad \forall x, u \in \operatorname{dom} \Phi. \tag{1.2}$$

If Φ is nonconvex (i.e., $\text{gph}\Phi$ is nonconvex), then property (1.2) may not hold. However, the following important fact is valid.

Theorem 1.4 ([10, Proposition 1] (see also [7, Chapter 7]))**.** *If $\Phi : I\!R^n \rightrightarrows I\!R^m$ is a polyhedral multifunction, then there is a constant $\ell > 0$ such that, for any $\overline{x} \in \operatorname{dom} \Phi$, Φ is locally upper Lipschitz at $\overline{x}$ with the modulus ℓ.*

Interestingly, the Lipschitz continuity property of convex polyhedral multifunctions on its effective domain, as stated in Theorem 1.3, and the locally upper Lipschitz continuity of polyhedral multifunctions in Theorem 1.4, hold in an infinite-dimensional setting; see Bonnans and Shapiro [3].

We recall some results from Dontchev and Rockafellar's recent book [4].

Theorem 1.5 ([4, Theorem 3D.3])**.** *Consider a multifunction $F : I\!R^n \rightrightarrows I\!R^m$ and a convex set $D \subset \operatorname{dom} F$ such that $F(x)$ is closed for every $x \in D$. Then F is Lipschitz continuous on D with constant κ if and only if F is both lower semicontinuous on D and upper Lipschitz continuous on D with constant κ.*

Theorem 1.6 ([4, Corollary 3D.4])**.** *Let $\Psi : I\!R^n \rightrightarrows I\!R^m$ be a polyhedral multifunction and $D \subset \operatorname{dom} \Psi$ be convex. Then Ψ is lower semicontinuous on D if and only if Ψ is Lipschitz continuous on D.*

2. Lower semicontinuity

From now on, we fix a polyhedral multifunction $\Phi : I\!R^n \rightrightarrows I\!R^m$ which admits the graph representation in the form (1.1).

For each $j \in J := \{1, \dots, k\}$, consider the multifunction $\Phi_j : I\!R^n \rightrightarrows I\!R^m$ defined by

$$\Phi_j(x) = \{y \in I\!R^m \mid (x, y) \in Q_j\} \quad \forall x \in I\!R^n. \tag{2.1}$$

It is clear that $\mathrm{gph}\Phi_j = Q_j$ and

$$\Phi(x) = \bigcup_{j \in J} \Phi_j(x), \ \ \mathrm{gph}\Phi = \bigcup_{j\in J} \mathrm{gph}\Phi_j. \tag{2.2}$$

The next lemma is useful for our proving subsequent results.

Lemma 2.1. *Let $\overline{x} \in I\!R^n$. If $\overline{x} \notin \bigcup_{j\in J} \partial(\mathrm{dom}\,\Phi_j)$, then Φ is locally Lipschitz at $\overline{x}$.*

Proof. By Theorem 1.3, for each $j \in J$ there exists a constant $\ell_j > 0$ such that $\Phi_j(x) \subset \Phi_j(u) + \ell_j||x - u||\bar{B}_{I\!R^m}$ for all $x, u \in \mathrm{dom}\,\Phi_j$. Setting $\ell = \max\{\ell_j \mid j \in J\}$, we have

$$\Phi_j(x) \subset \Phi_j(u) + \ell||x - u||\bar{B}_{I\!R^m}$$

for any $j \in J$ and $x, u \in \mathrm{dom}\,\Phi_j$. Let $J_0 = \{j \in J \mid \overline{x} \in \mathrm{dom}\,\Phi_j\}$ and $J_1 = J\backslash J_0$. Since $\mathrm{dom}\,\Phi_j = \pi(Q_j)$, where $\pi : I\!R^n \times I\!R^m \to I\!R^n$ is a linear mapping defined by setting $\pi(x, y) = x$, $\mathrm{dom}\,\Phi_j$ is a polyhedral convex set. Hence the set $\bigcup_{j\in J_1} \mathrm{dom}\,\Phi_j$ is closed. Since $\overline{x} \notin \bigcup_{j\in J_1} \mathrm{dom}\,\Phi_j$, we can find $\varepsilon > 0$ such that

$$B(\overline{x}, \varepsilon) \cap \Big(\bigcup_{j\in J_1} \mathrm{dom}\,\Phi_j \Big) = \emptyset.$$

If $J_0 = \emptyset$, then $B(\overline{x}, \varepsilon) \cap \mathrm{dom}\,\Phi = \emptyset$. Thus, for any $\ell > 0$, the inclusion

$$\Phi(x) \subset \Phi(u) + \ell||x - u||\bar{B}_{I\!R^m}$$

holds for all $x, u \in B(\overline{x}, \varepsilon)$.

If $J_0 \neq \emptyset$ then, by the assumption $\overline{x} \notin \bigcup_{j\in J} \partial(\mathrm{dom}\,\Phi_j)$, $\overline{x} \in \mathrm{int}(\mathrm{dom}\,\Phi_j)$ for each $j \in J_0$. Let $\delta > 0$ be such that $B(\overline{x}, \delta) \subset \mathrm{dom}\,\Phi_j$ for all $j \in J_0$. Set $\rho = \min\{\varepsilon, \delta\}$. Let $x, u \in B(\overline{x}, \rho)$ be given arbitrarily. By the choice of δ, we have

$$\{j \in J \mid x \in \mathrm{dom}\,\Phi_j\} = \{j \in J \mid u \in \mathrm{dom}\,\Phi_j\} = J_0.$$

Therefore

$$\begin{aligned} \Phi(u) &= \bigcup_{j\in J} \Phi_j(u) = \bigcup_{j\in J_0} \Phi_j(u) \\ &\subset \bigcup_{j\in J_0} \Big(\Phi_j(x) + \ell||u - x||\bar{B}_{I\!R^m}\Big) = \Phi(x) + \ell||u - x||\bar{B}_{I\!R^m}, \end{aligned}$$

and

$$\begin{aligned} \Phi(x) &= \bigcup_{j\in J} \Phi_j(x) = \bigcup_{j\in J_0} \Phi_j(x) \\ &\subset \bigcup_{j\in J_0} \Big(\Phi_j(u) + \ell||x - u||\bar{B}_{I\!R^m}\Big) = \Phi(u) + \ell||x - u||\bar{B}_{I\!R^m}. \end{aligned}$$

This shows that Φ is Lipschitz on $B(\overline{x}, \rho)$ with the Lipschitz constant ℓ. □

From Lemma 2.1 we get

Corollary 2.2. *If* $(\overline{x},\overline{y}) \in \mathrm{gph}\Phi$ *and if* $\overline{x} \notin \bigcup_{j\in J} \partial(\mathrm{dom}\,\Phi_j)$*, then* Φ *is Lipschitz-like at* $(\overline{x},\overline{y})$.

It is of interest to know when the following properties are equivalent:

(P1) Φ *is isc at* $(\overline{x},\overline{y}) \in \mathrm{gph}\Phi$;

(P2) Φ *is Lipschitz-like at* $(\overline{x},\overline{y}) \in \mathrm{gph}\Phi$.

Note that (P1) yields

$$\overline{x} \in \mathrm{int}(\mathrm{dom}\,\Phi). \tag{2.3}$$

This means that (2.3) is a necessary condition for having (P1). Since (P2) implies (P1), (2.3) is also a necessary condition for the validity of (P2).

We now show that, in general, (P1) $\not\Rightarrow$ (P2).

Example 2.3. Consider the multifunctions $\Psi_i : I\!R \rightrightarrows I\!R,\ i = 1,2$, where $\Psi_1(x) = I\!R$ for $x \in (-\infty, 0]$, $\Psi_1(x) = \emptyset$ for $x \in (0, +\infty)$, $\Psi_2(x) = \{0\}$ for $x \in [0, +\infty)$, $\Psi_2(x) = \emptyset$ for $x \in (-\infty, 0)$. Let $\Psi : I\!R \rightrightarrows I\!R$ be defined by

$$\Psi(x) = \Psi_1(x) \cup \Psi_2(x) \quad \forall x \in I\!R.$$

We have $\mathrm{gph}\Psi_1 = (-\infty, 0] \times I\!R,\ \mathrm{gph}\Psi_2 = [0, +\infty) \times \{0\}$, and

$$\mathrm{gph}\Psi = \mathrm{gph}\Psi_1 \cup \mathrm{gph}\Psi_2 = \Big((-\infty, 0] \times I\!R\Big) \cup \Big([0, +\infty) \times \{0\}\Big).$$

Hence Ψ is a polyhedral multifunction (see Fig. 1).

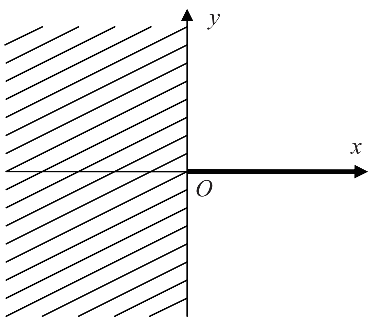

FIGURE 1. The graph of Ψ

It is easy to verify that Ψ is isc at $(\overline{x},\overline{y}) := (0,0) \in \mathrm{gph}\Psi$, but Ψ is not Lipschitz-like at $(\overline{x},\overline{y})$.

Remark 2.4. In the above example, $\overline{x} = 0 \in \partial(\mathrm{dom}\,\Psi_1) \cup \partial(\mathrm{dom}\,\Psi_2)$.

It is worthy to observe that (P2) does not guarantee the lsc property of Φ at $\overline{x}$.

Example 2.5. Let multifunctions $\widetilde{\Psi}_i : I\!R \rightrightarrows I\!R,\ i = 1,2$, be given by setting $\widetilde{\Psi}_1(x) = I\!R$ for $x \in (-\infty, 0]$, $\widetilde{\Psi}_1(x) = \emptyset$ for $x \in (0, +\infty)$, $\widetilde{\Psi}_2(x) = [-1, 1]$ for $x \in [0, +\infty)$, $\widetilde{\Psi}_2(x) = \emptyset$ for $x \in (-\infty, 0)$. Define $\widetilde{\Psi} : I\!R \rightrightarrows I\!R$ by the formula $\widetilde{\Psi}(x) = \widetilde{\Psi}_1(x) \cup \widetilde{\Psi}_2(x)$. We have

$$\mathrm{gph}\widetilde{\Psi} = \mathrm{gph}\widetilde{\Psi}_1 \cup \mathrm{gph}\widetilde{\Psi}_2 = \Big((-\infty, 0] \times I\!R\Big) \cup \Big([0, +\infty) \times [-1, 1]\Big).$$

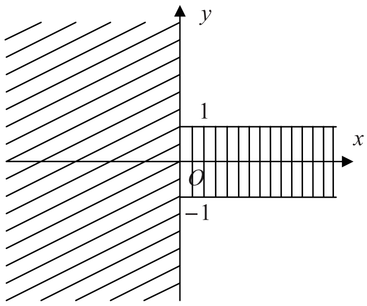

FIGURE 2. The graph of $\widetilde{\Psi}$

Hence $\widetilde{\Psi}$ is a polyhedral multifunction (see Fig. 2).

Note that $\widetilde{\Psi}$ is Lipschitz-like at $(\overline{x}, \overline{y}) := (0, 0) \in \text{gph}\widetilde{\Psi}$, but it is not lsc at $\overline{x}$.

We now study (P1) and (P2) in the case of one-variable polyhedral multifunctions, i.e., the case where $n = 1$.

Theorem 2.6. *Let $n = 1$ and let $\overline{x}$ be such that (2.3) holds. Then Φ is isc at any point $(\overline{x}, \overline{y}) \in \{\overline{x}\} \times \Phi(\overline{x})$ (that is, Φ is lsc at $\overline{x}$) if and only if Φ is Lipschitz-like at any point $(\overline{x}, \overline{y}) \in \{\overline{x}\} \times \Phi(\overline{x})$. Moreover, if Φ is lsc at $\overline{x}$ then it is locally Lipschitz at $\overline{x}$.*

Proof. First, if Φ is Lipschitz-like at $(\overline{x}, \overline{y})$ for every $\overline{y} \in \Phi(\overline{x})$ then Φ is obviously lsc at $\overline{x}$.

Suppose now that Φ is lsc at $\overline{x}$. If $\overline{x} \notin \bigcup_{j \in J} \partial(\text{dom}\, \Phi_j)$, then by Corollary 2.2 we can infer that Φ is Lipschitz-like at $(\overline{x}, \overline{y})$ for every $\overline{y} \in \Phi(\overline{x})$. Next, let us consider the case where $\overline{x} \in \bigcup_{j \in J} \partial(\text{dom}\, \Phi_j)$. For every $j \in J$, since $\text{dom}\, \Phi_j \subset I\!R$ is polyhedral convex set, the boundary $\partial(\text{dom}\, \Phi_j)$ of $\text{dom}\, \Phi_j$ can have at most two elements. Hence $\bigcup_{j \in J} \partial(\text{dom}\, \Phi_j)$ can have at most $2|J|$ elements, where $|J|$ denotes the number of elements of J. Thus $\bigcup_{j \in J} \partial(\text{dom}\, \Phi_j)$ is a finite set. This fact and the inclusion $\overline{x} \in \text{int}(\text{dom}\, \Phi)$ imply that there exists $\delta > 0$ satisfying $B(\overline{x}, \delta) \subset \text{dom}\, \Phi$ and

$$B(\overline{x}, \delta) \cap \Big(\bigcup_{j \in J} \partial(\text{dom}\, \Phi_j) \Big) = \{\overline{x}\}.$$

Therefore, by Corollary 2.2, for all $x \in B(\overline{x}, \delta) \backslash \{\overline{x}\}$, Φ is Lipschitz-like at (x, y) for every $y \in \Phi(x)$. Consequently, Φ is lsc at every $x \in B(\overline{x}, \delta) \backslash \{\overline{x}\}$. Besides, Φ is lsc at $\overline{x}$ by our assumption. Hence Φ is lsc on the convex set $B(\overline{x}, \delta) \subset \text{dom}\, \Phi$. Then, according to Theorem 1.6, Φ is Lipschitz continuous on $B(\overline{x}, \delta)$. In particular, Φ is Lipschitz-like at any point $(\overline{x}, \overline{y}) \in \{\overline{x}\} \times \Phi(\overline{x})$. □

We are going to establish a sufficient condition, which is based on the lower semicontinuity, for Φ to be Lipschitz-like at every point $(x, y) \in \text{gph}\Phi$ where $x \in \text{int}(\text{dom}\, \Phi)$.

Theorem 2.7. *The polyhedral multifunction* Φ *is Lipschitz-like at every point* $(x, y) \in \mathrm{gph}\Phi$ *where* $x \in \mathrm{int}(\mathrm{dom}\,\Phi)$ *if and only if it is lsc at every point in the set*

$$M := \Big(\bigcup_{j\in J} \partial(\mathrm{dom}\,\Phi_j)\Big) \cap \mathrm{int}(\mathrm{dom}\,\Phi). \tag{2.4}$$

Proof. Clearly, if Φ is Lipschitz-like at every $(x, y) \in \mathrm{gph}\Phi$ where $x \in \mathrm{int}(\mathrm{dom}\,\Phi)$ then Φ is lsc at every $x \in M$.

Assume now that Φ is lsc at every $x \in M$. Fix any $(\overline{x}, \overline{y}) \in \mathrm{gph}\Phi$ where $\overline{x} \in \mathrm{int}(\mathrm{dom}\,\Phi)$. If $\overline{x} \notin \bigcup_{j\in J} \partial(\mathrm{dom}\,\Phi_j)$ then, by Corollary 2.2, Φ is Lipschitz-like at $(\overline{x}, \overline{y})$. Let us consider the case $\overline{x} \in \bigcup_{j\in J} \partial(\mathrm{dom}\,\Phi_j)$. Choose $\delta > 0$ such that $B(\overline{x}, \delta) \subset \mathrm{dom}\,\Phi$. For every $x \in B(\overline{x}, \delta)\backslash\big(\bigcup_{j\in J} \partial(\mathrm{dom}\,\Phi_j)\big)$, Corollary 2.2 ensures that Φ is Lipschitz-like at (x, y) for all $y \in \Phi(x)$. Hence Φ is lsc on $B(\overline{x}, \delta)\backslash\big(\bigcup_{j\in J} \partial(\mathrm{dom}\,\Phi_j)\big)$. Combining this with the assumption that Φ is lsc on M, we deduce that Φ is lsc on $B(\overline{x}, \delta)$. Since $B(\overline{x}, \delta) \subset \mathrm{dom}\,\Phi$ is convex, Φ is Lipschitz continuous on $B(\overline{x}, \delta)$ by Theorem 1.6. In particular, Φ is Lipschitz-like at $(\overline{x}, \overline{y})$. □

The isc property of Φ *at any point* $(\overline{x}, \overline{y})$ *in the fiber* $\{\overline{x}\} \times \Phi(\overline{x}) \subset \mathrm{gph}\Phi$ *does not imply that* Φ *is Lipschitz-like at every point belonging to the fiber.* The next example is designed for proving the claim. Moreover, via this example, we will see that the lower semicontinuity of Φ on the set M given by (2.4) is essential for the Lipschitz-like property of Φ at every point $(x, y) \in \mathrm{gph}\Phi$ where $x \in \mathrm{int}(\mathrm{dom}\,\Phi)$.

Example 2.8. Given an $\alpha \in (0, +\infty)$, we consider the sets

$$\Omega_1 = \big\{(x_1, x_2) \in I\!R^2 \mid x_1 + x_2 \geq 0,\ x_1 + 2x_2 \leq \alpha,\ x_2 \geq 0\big\},$$
$$\Omega_2 = \big\{(x_1, x_2) \in I\!R^2 \mid x_1 - x_2 \geq 0,\ x_1 - 2x_2 \leq \alpha,\ x_2 \leq 0\big\},$$
$$\Omega_3 = \big\{(x_1, x_2) \in I\!R^2 \mid x_1 + x_2 \leq 0,\ x_1 - x_2 \leq 0,\ x_1 \geq -\alpha\big\}.$$

Let the polyhedral convex multifunctions $\Phi_j : I\!R^2 \rightrightarrows I\!R$ $(j = 1, \ldots, 5)$ be determined by setting

$$\Phi_1(x) = \begin{cases} \{0\} & \text{if } x = (x_1, x_2) \in \Omega_1 \\ \emptyset & \text{if } x \notin \Omega_1, \end{cases}$$

$$\Phi_2(x) = \begin{cases} \{0\} & \text{if } x = (x_1, x_2) \in \Omega_2 \\ \emptyset & \text{if } x \notin \Omega_2, \end{cases}$$

$$\Phi_3(x) = \begin{cases} \{-\sqrt{2}x_1\} & \text{if } x = (x_1, x_2) \in \Omega_3 \\ \emptyset & \text{if } x \notin \Omega_3, \end{cases}$$

$$\Phi_4(x) = \begin{cases} \big[0, -\sqrt{2}x_1\big] & \text{if } x = (x_1, x_2) \in \Omega_1 \cap \Omega_3 \\ \emptyset & \text{if } x \notin \Omega_1 \cap \Omega_3, \end{cases}$$

$$\Phi_5(x) = \begin{cases} \big[0, -\sqrt{2}x_1\big] & \text{if } x = (x_1, x_2) \in \Omega_2 \cap \Omega_3 \\ \emptyset & \text{if } x \notin \Omega_2 \cap \Omega_3. \end{cases}$$

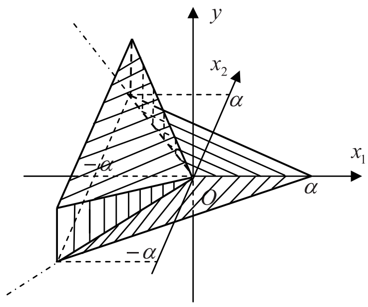

FIGURE 3. The graph of Φ

Putting

$$\Phi(x) = \bigcup_{j=1}^{5} \Phi_j(x) \quad \forall x \in I\!R^2,$$

we get a polyhedral multifunction $\Phi : I\!R^2 \rightrightarrows I\!R$, whose graph is the striped domain in Fig. 3. Obviously, Φ is isc at $(0,0,0)$, which is the unique point in the fiber $\{(0,0)\} \times \Phi(0,0)$; i.e., Φ is lsc at $\overline{x} := (0,0)$. However, Φ is not Lipschitz-like at $(0,0,0)$. Indeed, suppose on the contrary that there exist a neighborhood U of $0 \in I\!R^2$ and constants $\ell > 0$, $\mu > 0$ such that

$$\Phi(v) \cap (-\mu, \mu) \subset \Phi(u) + \ell ||v - u|| \bar{B}_{I\!R} \tag{2.5}$$

for all $u, v \in U$.

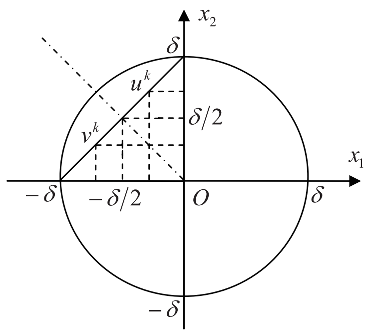

FIGURE 4. Sequences $\{u^k\}_{k\in I\!N}$ and $\{v^k\}_{k\in I\!N}$

Choose $\delta \in (0, \mu)$ as small as $\delta \bar{B}_{I\!R^2} \subset U$. Consider the sequences $\{u^k\}_{k\in I\!N}$, $\{v^k\}_{k\in I\!N}$ where $u^k = (u_1^k, u_2^k) = \left(-\frac{\delta}{2}+\frac{1}{k}, \frac{\delta}{2}+\frac{1}{k}\right)$ and $v^k = (v_1^k, v_2^k) = \left(-\frac{\delta}{2}-\frac{1}{k}, \frac{\delta}{2}-\frac{1}{k}\right)$. We see that there exists $k_0 > 0$ such that $u^k \in \delta \bar{B}_{I\!R^2}$, $v^k \in \delta \bar{B}_{I\!R^2}$, and $\sqrt{2}(\frac{\delta}{2} + \frac{1}{k}) \in (-\mu, \mu)$ for all $k \geq k_0$. Clearly,

$$\lim_{k\to\infty} u^k = \lim_{k\to\infty} v^k = \left(-\frac{\delta}{2}, \frac{\delta}{2}\right),$$

hence $\lim_{k\to\infty} ||u^k - v^k|| = 0$. Note that $u^k \in \text{int}\Omega_1$ and $v^k \in \text{int}\Omega_3$ for all $k \geq k_0$, so one has $\Phi(u^k) = \{0\}$ and $\Phi(v^k) = \{-\sqrt{2}v_1^k\} = \big\{\sqrt{2}(\frac{\delta}{2} + \frac{1}{k})\big\}$ for all $k \geq k_0$. By the choice of δ, k_0 and by (2.5), we get

$$\Phi(v^k) \cap (-\mu, \mu) \subset \Phi(u^k) + \ell||v^k - u^k||\bar{B}_{I\!R},$$

or,

$$\left\{\sqrt{2}\Big(\frac{\delta}{2} + \frac{1}{k}\Big)\right\} \subset \{0\} + \ell\frac{2\sqrt{2}}{k}\bar{B}_{I\!R}$$

for all $k \geq k_0$; a contradiction.

3. Upper semicontinuity

The following well-known fact is a direct consequence of Theorem 1.4 and the definition of the usc property of multifunctions.

Fact. *If $\Phi(\overline{x})$ is bounded, then the polyhedral multifunction Φ is usc at $\overline{x}$.*

If $\Phi(\overline{x})$ is unbounded, then Φ may not be usc at $\overline{x}$.

Example 3.1. Consider the polyhedral multifunction $\Phi : I\!R \rightrightarrows I\!R^2$ with $\Phi(x) = \{x\} \times [0, +\infty)$ for all $x \in I\!R$ (see Fig. 5). Since

$$\begin{aligned} \text{gph}\Phi &= \big\{(x, x, y) \in I\!R^3 \mid x \in I\!R,\ y \geq 0\big\} \\ &= \big\{\alpha(1, 1, 0) + \beta(-1, -1, 0) + \gamma(0, 0, 1) \mid \alpha \geq 0, \beta \geq 0, \gamma \geq 0\big\} \\ &= \text{pos}\big\{(1, 1, 0), (-1, -1, 0), (0, 0, 1)\big\}, \end{aligned}$$

where $\text{pos}\,\Omega$ denotes the convex cone hull of Ω, is a polyhedral convex set, Φ is a polyhedral convex multifunction. Note that the open set $V := \{y = (y_1, y_2) \in I\!R^2 \mid\ y_2|y_1| < 1\}$ contains $\Phi(0)$, but for every $x \neq 0$ it holds $\Phi(x) \not\subset V$. Hence Φ is not usc at $\overline{x} = 0$.

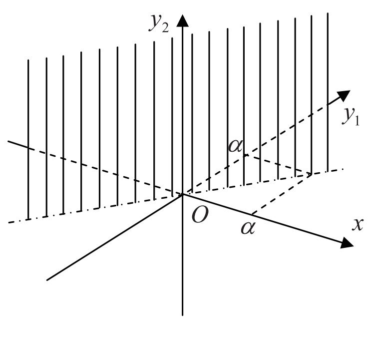

Figure 5. The graph of Φ

It is known [5, 12] that if $F : I\!R^n \rightrightarrows I\!R^m$ is such a multifunction that $\text{dom}\,F$ is connected, $F(x)$ is connected for all $x \in \text{dom}\,F$, and F is usc on $\text{dom}\,F$, then

$$\text{rge}\,F := \bigcup_{x\in\text{dom}F} F(x) \quad \text{(the range of } F)$$

is connected. In other words, upper semicontinuous multifunctions with nonempty connected values preserve connectedness.

The next example shows that H-upper semicontinuous multifunctions with connected values may not preserve connectedness. By definition, a multifunction $F : I\!R^n \rightrightarrows I\!R^m$ is Hausdorff-upper semicontinuous (H-usc) at $\overline{x}$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$F(x) \subset F(\overline{x}) + \varepsilon B_{I\!R^m} \quad \forall x \in B(\overline{x}, \delta).$$

From Theorem 1.4 it follows that polyhedral multifunctions are H-usc on their effective domains.

Example 3.2. [9] Consider the multifunction $F : I\!R \rightrightarrows I\!R^2$, $F(x) = \{(x, \frac{1}{x})\}$ for $x \neq 0$ and $F(0) = \{0\} \times I\!R$. It is easy to verify that F is H-usc on $\operatorname{dom} F = I\!R$, but $\operatorname{rge} F$ has three connected components.

Since polyhedral multifunctions with nonempty connected values are not usc in general (see Example 3.1), we may think that they do not preserve connectedness.

In connection with Example 3.2 and the preceding remarks, the three forthcoming 'connectedness preservation' and 'path-connectedness preservation' theorems seem to be interesting. Recall that a topological space (X, τ) is said to be path-connected if for any two points $x, y \in X$ there exists a continuous function $f : [0,1] \to X$ satisfying $f(0) = x$ and $f(1) = y$.

Theorem 3.3. *If* $\operatorname{dom}\Phi$ *is connected and* $\Phi(x)$ *is connected for all* $x \in \operatorname{dom}\Phi$*, then* $\operatorname{rge}\Phi$ *is connected.*

Theorem 3.4. *Let* $D \subset \operatorname{dom}\Phi$ *be such a connected set that*

(i) $\Phi(x)$ *is connected for all* $x \in D$*, and*
(ii) *For every* $j \in J$*, the number of connected components of* $D \cap \operatorname{dom}\Phi_j$ *is finite.*

Then the image set $\Phi(D)$ *is connected.*

Theorem 3.5. *Suppose that, for every* $x \in \operatorname{dom}\Phi$*,* $\Phi(x)$ *is connected. If* D *is a connected component of* $\operatorname{dom}\Phi$*, then the image set* $\Phi(D)$ *is path-connected.*

Taking $D = \operatorname{dom}\Phi$ one can easily deduce Theorem 3.3 from Theorem 3.4. However, we prefer to formulate separately the two theorems and give each of them an independent proof. The reason for doing so is that the first proof is elementary in some sense, while the second one relies on an advanced tool: a theorem on the existence of Lipschitz continuous selections of Lipschitz multifunctions [2, Chapter 8].

Proof of Theorem 3.3 Since $\operatorname{dom}\Phi_j$ and $\operatorname{rge}\Phi_j$ are polyhedral convex sets, they are connected for all $j \in J = \{1, 2, \ldots, k\}$.

We divide the proof into several steps.

Step 1. If $J = \{1\}$ then $\operatorname{rge}\Phi = \operatorname{rge}\Phi_1$, a connected set. Consider the case where J has two elements, or more. Since $\operatorname{dom}\Phi$ is connected and all the sets $\operatorname{dom}\Phi_j$ ($j \in J$) are closed, there must exist $j_1 \in J\backslash\{1\}$ satisfying $\operatorname{dom}\Phi_1 \cap \operatorname{dom}\Phi_{j_1} \neq \emptyset$. Pick any $x_1 \in \operatorname{dom}\Phi_1 \cap \operatorname{dom}\Phi_{j_1}$ and let $K_1 = \{j \in J \mid x_1 \in \operatorname{dom}\Phi_j\}$. For each $j \in K_1$,

$\Phi(x_1) = \bigcup_{\ell\in K_1} \Phi_\ell(x_1)$ and $\Phi(x_1) \cap \operatorname{rge}\Phi_j \neq \emptyset$. As $\Phi(x_1)$ and $\operatorname{rge}\Phi_j$ are connected, $\Phi(x_1) \cup \operatorname{rge}\Phi_j$ is connected by [6, Theorem 21, p. 54]. From $\Phi(x_1) \subset \bigcup_{j\in K_1} \operatorname{rge}\Phi_j$ it follows that

$$\bigcup_{j\in K_1} \operatorname{rge}\Phi_j = \bigcup_{j\in K_1} \big(\Phi(x_1) \cup \operatorname{rge}\Phi_j\big). \tag{3.1}$$

Since $\bigcap_{j\in K_1} \big(\Phi(x_1) \cup \operatorname{rge}\Phi_j\big) \supset \Phi(x_1) \neq \emptyset$, by (3.1) and [6, Theorem 21, p. 54] we infer that $\bigcup_{j\in K_1} \operatorname{rge}\Phi_j$ is a connected set.

Step 2. If $K_1 = J$, the proof is complete because $\operatorname{rge}\Phi = \bigcup_{j\in K_1} \operatorname{rge}\Phi_j$ and the set on the right-hand side has been shown to be connected. Suppose now that $K_1 \neq J$. As $\operatorname{dom}\Phi$ is connected and the sets $\operatorname{dom}\Phi_j$ $(j \in J)$ are closed, there exists $j_2 \in J\backslash K_1$ such that

$$\Big(\bigcup_{j\in K_1} \operatorname{dom}\Phi_j\Big) \cap \operatorname{dom}\Phi_{j_2} \neq \emptyset.$$

Fix a point $x_2 \in \big(\bigcup_{j\in K_1} \operatorname{dom}\Phi_j\big) \cap \operatorname{dom}\Phi_{j_2}$ and put $K_2 = \{j \in J \mid x_2 \in \operatorname{dom}\Phi_j\}$. Then $K_1 \subsetneqq K_1 \cup K_2$, $\Phi(x_2) = \bigcup_{j\in K_2} \Phi_j(x_2) \subset \bigcup_{j\in K_2} \operatorname{rge}\Phi_j$ and $\Phi(x_2) \cap \operatorname{rge}\Phi_j \neq \emptyset$ for all $j \in K_2$. By the argument described in Step 1, we deduce that $\bigcup_{j\in K_2} \operatorname{rge}\Phi_j$ is connected. Moreover, since $x_2 \in \bigcup_{j\in K_1} \operatorname{dom}\Phi_j$, there is $j_* \in K_1$ such that $x_2 \in \operatorname{dom}\Phi_{j_*}$. This means that $j_* \in K_2$, hence

$$\Big(\bigcup_{j\in K_1} \operatorname{rge}\Phi_j\Big) \cap \Big(\bigcup_{j\in K_2} \operatorname{rge}\Phi_j\Big) \supset \Phi_{j_*}(x_2) \neq \emptyset.$$

Thus $\bigcup_{j\in K_1\cup K_2} \operatorname{rge}\Phi_j$ is connected.

Suppose that, by the above construction, we have got a maximal finite sequence $K_1, K_2, \dots, K_r$ of subsets of J such that $\bigcup_{i=1}^{r-1} K_i \subsetneqq \bigcup_{i=1}^{r} K_i$ and the set

$$\bigcup \big\{\operatorname{rge}\Phi_j \mid j \in K_1 \cup K_2 \cup \cdots \cup K_r\big\}$$

is connected.

Step 3. If $\bigcup_{i=1}^{r} K_i = J$, then the proof is complete because

$$\operatorname{rge}\Phi = \bigcup \big\{\operatorname{rge}\Phi_j \mid j \in K_1 \cup K_2 \cup \cdots \cup K_r\big\}.$$

Suppose that $\bigcup_{i=1}^{r} K_i \neq J$. Then there is $j_{r+1} \in J\backslash\big(\bigcup_{i=1}^{r} K_i\big)$ such that

$$\Big(\bigcup \big\{\operatorname{dom}\Phi_j \mid j \in K_1 \cup K_2 \cup \cdots \cup K_r\big\}\Big) \cap \operatorname{dom}\Phi_{j_{r+1}} \neq \emptyset.$$

Take any $x_{r+1} \in \big(\bigcup \big\{\operatorname{dom}\Phi_j \mid j \in K_1 \cup K_2 \cup \cdots \cup K_r\big\}\big) \cap \operatorname{dom}\Phi_{j_{r+1}}$ and put

$$K_{r+1} = \{j \in J \mid x_{r+1} \in \operatorname{dom}\Phi_j\}.$$

Applying the argument of Step 1 once more, we can show that $\bigcup_{j\in K_{r+1}} \operatorname{rge}\Phi_j$ is connected. Since

$$x_{r+1} \in \bigcup \big\{\operatorname{dom}\Phi_j \mid j \in K_1 \cup K_2 \cup \cdots \cup K_r\big\},$$

there is $j_* \in \bigcup_{i=1}^{r} K_i$ such that $x_{r+1} \in \operatorname{dom}\Phi_{j_*}$. Hence we have $j_* \in K_{r+1}$ and

$$\Big(\bigcup \big\{\operatorname{rge}\Phi_j \mid j \in K_1 \cup K_2 \cup \cdots \cup K_r\big\}\Big) \cap \Big(\bigcup_{j\in K_{r+1}} \operatorname{rge}\Phi_j\Big) \supset \Phi_{j_*}(x_{r+1}) \neq \emptyset.$$

It follows that the set $\bigcup\big\{\mathrm{rge}\,\Phi_j \mid j \in K_1 \cup K_2 \cup \cdots \cup K_r \cup K_{r+1}\big\}$ is connected. We have seen that the sequence $K_1, K_2, \ldots, K_r, K_{r+1}$ of subsets of J have the following properties:

(i) $\bigcup_{i=1}^{r} K_i \subsetneqq \bigcup_{i=1}^{r+1} K_i$,

(ii) $\bigcup\{\mathrm{rge}\,\Phi_j \mid j \in K_1 \cup K_2 \cup \cdots \cup K_{r+1}\}$ is a connected set.

This contradicts the choice of the maximal sequence $K_1, K_2, \ldots, K_r$ in Step 2, and completes the proof. □

In order to prove Theorem 3.4, we first establish an auxiliary result.

Lemma 3.6. *Let $D \subset \mathrm{dom}\,\Phi$ be a connected set such that $\Phi(x)$ is connected for all $x \in D$. For every $j \in J$, if M is a connected component of $D \cap \mathrm{dom}\,\Phi_j$, then $\Phi_j(M)$ is connected.*

Proof. Fix any $j \in J$. By Theorem 1.3 we know that $\Phi_j : \mathrm{dom}\,\Phi_j \rightrightarrows I\!R^m$ is a closed, convex-valued Lipschitz multifunction. According to [2, Theorem 9.4.3], the multifunction $\Phi_j|_M : M \rightrightarrows I\!R^m$, which is the restriction of Φ_j on M, has a Lipschitz selection $f : M \to I\!R^m$. Since f is continuous and M is connected, the set $f(M)$ is connected. It is easy to see that

$$\Phi_j(M) = \bigcup_{x\in M} \Big(\Phi_j(x) \cup f(M)\Big). \tag{3.2}$$

For every $x \in M$, from the connectedness of the sets $\Phi_j(x)$ and $f(M)$, and from the property $\Phi_j(x) \cap f(M) \ni f(x)$, it follows that $\Phi_j(x) \cup f(M)$ is connected. Since the family $\{\Phi_j(x) \cup f(M)\}_{x\in M}$ of connected sets has nonempty intersection, by (3.2) we infer that $\Phi_j(M)$ is connected. □

Proof of Theorem 3.4 By [2, Theorem 9.4.3], for each index $j \in J$ we can find a Lipschitz function $f_j : \mathrm{dom}\,\Phi_j \to I\!R^m$ such that $f_j(x) \in \Phi_j(x)$ for every $x \in \mathrm{dom}\,\Phi_j$.

Let $J_0 = \{j \in J \mid D \cap \mathrm{dom}\,\Phi_j \neq \emptyset\}$ and $D_j = D \cap \mathrm{dom}\,\Phi_j$ for all $j \in J_0$. Denote by τ_D and τ_{D_j} the induced topologies of D and D_j. For each j, we express the set of all the connected components of D_j in the form $\{M_\alpha \mid \alpha \in \Gamma_j\}$, where Γ_j is an index set. As $D \subset \mathrm{dom}\,\Phi$, it holds $D = \bigcup_{\alpha\in\Gamma} M_\alpha$, where $\Gamma := \bigcup_{j\in J_0} \Gamma_j$.

By the assumption (ii), Γ_j is a finite set for every $j \in J_0$. Hence Γ is also finite. For any $j \in J_0$, since $\mathrm{dom}\,\Phi_j$ is closed in $I\!R^n$, $D_j = D \cap \mathrm{dom}\,\Phi_j$ is τ_D-closed in D. Next, for every $\alpha \in \Gamma_j$, as M_α is a connected component of D_j, M_α is τ_{D_j}-closed in D_j. Hence M_α is τ_D-closed in D. Therefore, $\{M_\alpha \mid \alpha \in \Gamma\}$ is a family of finitely many τ_D-closed subsets of D.

For any $\alpha \in \Gamma$, M_α is a connected component of D_j, where $j \in J_0$ is such an index that $\alpha \in \Gamma_j$. By Lemma 3.6 and the assumption (i), the image set $\Phi_j(M_\alpha)$ is connected. In other words, we have shown that $\Phi_j(M_\alpha)$ is connected for any $j \in J_0$ and for any $\alpha \in \Gamma_j$.

Now, in order to show that $\Phi(D)$ is connected, we can proceed similarly as in the proof of Theorem 3.3.

Step 1. Pick any $x_1 \in D = \bigcup_{\alpha\in\Gamma} M_\alpha$ and select an index $\alpha_1 \in \Gamma$ such that $x_1 \in M_{\alpha_1}$. Set $K_1 = \{\alpha \in \Gamma \mid x_1 \in M_\alpha\}$. Since M_α is connected for all $\alpha \in K_1$ and $x_1 \in \bigcap_{\alpha\in K_1} M_\alpha$, $\bigcup_{\alpha\in K_1} M_\alpha$ is connected. Moreover, $\bigcup_{\alpha\in K_1} \Phi(M_\alpha)$ is also connected. Indeed, for each $\alpha \in K_1$, there exists $j_\alpha \in J_0$ such that $\alpha \in \Gamma_{j_\alpha}$. Then M_α is a connected component of D_{j_α} and $x_1 \in M_\alpha$. Note that

$$\Phi(M_\alpha) = \bigcup_{x \in M_\alpha} \Phi(x) = \bigcup_{x \in M_\alpha} \Big(\Phi(x) \cup \Phi_{j_\alpha}(M_\alpha)\Big). \tag{3.3}$$

For every $x \in M_\alpha \subset D_{j_\alpha}$, the set $\Phi(x) \cup \Phi_{j_\alpha}(M_\alpha)$ is connected because $\Phi(x)$ and $\Phi_{j_\alpha}(M_\alpha)$ are connected, and $f_{j_\alpha}(x) \in \Phi(x) \cap \Phi_{j_\alpha}(M_\alpha)$, where f_{j_α} is the chosen Lipschitz selection of Φ_{j_α}. In addition,

$$\bigcap_{x \in M_\alpha} \Big(\Phi(x) \cup \Phi_{j_\alpha}(M_\alpha)\Big) \supset \Phi_{j_\alpha}(M_\alpha) \neq \emptyset.$$

Thus (3.3) implies that $\Phi(M_\alpha)$ is connected. Combining the last fact with the obvious property

$$\bigcap_{\alpha \in K_1} \Phi(M_\alpha) \supset \Phi(x_1) \neq \emptyset,$$

we obtain the desired connectedness of $\bigcup_{\alpha \in K_1} \Phi(M_\alpha)$.

Step 2. If $K_1 = \Gamma$ then the conclusion of our theorem follows from the formulae

$$\Phi(D) = \bigcup_{\alpha \in \Gamma} \Phi(M_\alpha) = \bigcup_{\alpha \in K_1} \Phi(M_\alpha)$$

and the result of Step 1. Suppose now that $K_1 \neq \Gamma$. As $D = \bigcup_{\alpha \in \Gamma} M_\alpha$ is assumed connected and M_α is τ_D-closed in D for every $\alpha \in \Gamma$, there must exist $\alpha_2 \in \Gamma \backslash K_1$ satisfying $M_{\alpha_2} \cap \big(\bigcup_{\alpha \in K_1} M_\alpha\big) \neq \emptyset$. (The fact that Γ is finite plays a crucial role here!) Pick any $x_2 \in M_{\alpha_2} \cap \big(\bigcup_{\alpha \in K_1} M_\alpha\big)$ and put $K_2 = \{\alpha \in \Gamma \mid x_2 \in M_\alpha\}$. The argument described in Step 1 shows that the sets $\bigcup_{\alpha \in K_2} M_\alpha$ and $\bigcup_{\alpha \in K_2} \Phi(M_\alpha)$ are connected. Since

$$\Big(\bigcup_{\alpha \in K_1} M_\alpha\Big) \cap \Big(\bigcup_{\alpha \in K_2} M_\alpha\Big) \supset \{x_2\} \neq \emptyset$$

and

$$\Big(\bigcup_{\alpha \in K_1} \Phi(M_\alpha)\Big) \cap \Big(\bigcup_{\alpha \in K_2} \Phi(M_\alpha)\Big) \supset \Phi(x_2) \neq \emptyset,$$

we can assert that $\bigcup_{\alpha \in K_1 \cup K_2} M_\alpha$ and $\bigcup_{\alpha \in K_1 \cup K_2} \Phi(M_\alpha)$ are connected sets. Note that $K_1 \subsetneqq K_1 \cup K_2 \subseteq \Gamma$.

We continue the above construction until obtaining a maximal finite sequence $K_1, K_2, \ldots, K_r$ of subsets of Γ such that $\bigcup_{i=1}^{r-1} K_i \subsetneqq \bigcup_{i=1}^{r} K_i$, and the sets

$$\bigcup \big\{M_\alpha \mid \alpha \in K_1 \cup \cdots \cup K_r\big\}, \quad \bigcup \big\{\Phi(M_\alpha) \mid \alpha \in K_1 \cup \cdots \cup K_r\big\}$$

are connected.

Step 3. If $\bigcup_{i=1}^{r} K_i = \Gamma$ then the proof is complete because

$$\Phi(D) = \bigcup \big\{\Phi(M_\alpha) \mid \alpha \in K_1 \cup \cdots \cup K_r\big\}$$

is connected. We now suppose that $\bigcup_{i=1}^{r} K_i \neq \Gamma$. Since $D = \bigcup_{\alpha \in \Gamma} M_\alpha$ is connected and M_α is τ_D-closed in D for all $\alpha \in \Gamma$, there must be $\alpha_{r+1} \in \Gamma \backslash \bigcup_{i=1}^{r} K_i$ such that

$$M_{\alpha_{r+1}} \cap \Big(\bigcup \big\{M_\alpha \mid \alpha \in K_1 \cup \cdots \cup K_r\big\}\Big) \neq \emptyset.$$

Select any $x_{r+1} \in M_{\alpha_{r+1}} \cap \big(\bigcup\{M_\alpha \mid \alpha \in K_1 \cup \cdots \cup K_r\}\big)$ and put

$$K_{r+1} = \{\alpha \in \Gamma \mid x_{r+1} \in M_\alpha\}.$$

By the argument of Step 1, we infer that $\bigcup_{\alpha\in K_{r+1}} M_\alpha$ and $\bigcup_{\alpha\in K_{r+1}} \Phi(M_\alpha)$ are connected. Since

$$\Big(\bigcup\{M_\alpha \mid \alpha \in K_1 \cup \cdots \cup K_r\}\Big) \cap \Big(\bigcup_{\alpha\in K_{r+1}} M_\alpha\Big) \supset \{x_{r+1}\} \neq \emptyset$$

and

$$\Big(\bigcup\{\Phi(M_\alpha) \mid \alpha \in K_1 \cup \cdots \cup K_r\}\Big) \cap \Big(\bigcup_{\alpha\in K_{r+1}} \Phi(M_\alpha)\Big) \supset \Phi(x_{r+1}) \neq \emptyset,$$

the sets $\bigcup\{M_\alpha \mid \alpha \in K_1 \cup\cdots\cup K_r \cup K_{r+1}\}$, $\bigcup\{\Phi(M_\alpha) \mid \alpha \in K_1 \cup\cdots\cup K_r \cup K_{r+1}\}$ are connected. Hence the sequence $K_1, K_2, \ldots, K_r, K_{r+1}$ of subsets of Γ satisfies the properties:

(i) $\bigcup_{i=1}^{r} K_i \subsetneqq \bigcup_{i=1}^{r+1} K_i$,
(ii) $\bigcup\{M_\alpha \mid \alpha \in K_1 \cup \cdots \cup K_r \cup K_{r+1}\}$ is a connected set,
(iii) $\bigcup\{\Phi(M_\alpha) \mid \alpha \in K_1 \cup \cdots \cup K_r \cup K_{r+1}\}$ is a connected set.

Due to the choice of $K_1, K_2, \ldots, K_r$, this is impossible. The proof is complete. □

The forthcoming lemma will be needed in the proof of Theorem 3.5.

Lemma 3.7. *Let $\{P_1, P_2, \ldots, P_s\}$ be a family of closed, convex subsets of $\mathbb{R}^n$ and let $P = \bigcup_{i=1}^{s} P_i$. If P is connected, then it is path-connected. More specifically, any two points in P can be joined by line segments in P.*

Proof. Given any $x_0, x_1 \in P$, we find $i_0, i_1 \in \{1, 2, \ldots, s\}$ such that $x_0 \in P_{i_0}$ and $x_1 \in P_{i_1}$. Let

$$\begin{aligned} X_{0,1} &= \bigcup\big\{P_i \mid P_i \cap P_{i_0} \neq \emptyset\big\},\\ X_{0,2} &= \bigcup\big\{P_i \mid P_i \cap X_{0,1} \neq \emptyset\big\},\\ &\cdots\cdots\\ X_{0,r} &= \bigcup\big\{P_i \mid P_i \cap X_{0,r-1} \neq \emptyset\big\},\\ &\cdots\cdots \end{aligned}$$

Suppose that the process terminates at a step $r = r_0$. Similarly, let the process

$$\begin{aligned} X_{1,1} &= \bigcup\big\{P_i \mid P_i \cap P_{i_1} \neq \emptyset\big\},\\ X_{1,2} &= \bigcup\big\{P_i \mid P_i \cap X_{1,1} \neq \emptyset\big\},\\ &\cdots\cdots\\ X_{1,r} &= \bigcup\big\{P_i \mid P_i \cap X_{1,r-1} \neq \emptyset\big\},\\ &\cdots\cdots \end{aligned}$$

be terminated at a step $r = r_1$. Then $X_{0,r_0} \cap X_{1,r_1} \neq \emptyset$. Indeed, suppose on the contrary that $X_{0,r_0} \cap X_{1,r_1} = \emptyset$. Put

$$A_0 = X_{0,r_0} \quad \text{and} \quad A_1 = X_{1,r_1} \cup \Big(\bigcup\big\{P_i \mid P_i \cap X_{0,r_0} = \emptyset\big\}\Big).$$

Since P_i is closed for all $i = 1, 2, \ldots, s$, we see that both A_0 and A_1 are closed in the induced topology τ_P of $P = \bigcup_{i=1}^{s} P_i$. As $A_0 \cap A_1 = \emptyset$ and $A_0 \cup A_1 = P$, the connectedness of P is violated. We have thus proved that $X_{0,r_0} \cap X_{1,r_1} \neq \emptyset$.

Pick any $z \in X_{0,r_0} \cap X_{1,r_1}$. By the construction of X_{0,r_0} and the convexity of $P_i, i = 1, \ldots, s$, x_0 can be joined with z by some line segments belonging to X_{1,r_0}. Similarly, x_1 can be joined with z by some line segments belonging to X_{0,r_1}. Hence x_0 can be joined with x_1 by some line segments belonging to $X_{0,r_0} \cup X_{1,r_1}$. □

Proof of Theorem 3.5 As in the proof of Theorem 3.4, for each $j \in J$ we find a Lipschitz function $f_j : \operatorname{dom} \Phi_j \to I\!R^m$ such that $f_j(x) \in \Phi_j(x)$ for every $x \in \operatorname{dom} \Phi_j$.

Let $y^0, y^1 \in \Phi(D)$ be given arbitrarily. Choose $x^0, x^1 \in D$ such that $y^0 \in \Phi(x^0)$ and $y^1 \in \Phi(x^1)$. Since D is a connected component of $\operatorname{dom} \Phi = \bigcup_{j \in J} \operatorname{dom} \Phi_j$, there exists $J_0 \subset J$ such that $D = \bigcup_{j \in J_0} \operatorname{dom} \Phi_j$. By Lemma 3.7, x^0 can be joined with x^1 by some line segments $L_1, \ldots, L_s$ belonging to D. Suppose that $L_\ell \cap L_{\ell+1} = \{z^\ell\}$ for $\ell = 1, 2, \ldots, s-1$. Set $z^0 = x^0$ and $z^s = x^1$. Without loss of generality, we may assume that for every $\ell \in \{1, 2, \ldots, s\}$ there is $j_\ell \in J_0$ with the property that $L_\ell \subset \operatorname{dom} \Phi_{j_\ell}$. Clearly, $f_{j_\ell}(L_\ell)$ is a continuous curve in $\operatorname{rge} \Phi_{j_\ell} \subset \Phi(D)$ joining $f_{j_\ell}(z^{\ell-1})$ with $f_{j_\ell}(z^\ell)$.

We have $y^0, f_{j_1}(x^0) \in \Phi(x^0) = \bigcup_{j \in J_0} \Phi_j(x^0)$ and $\Phi_j(x^0)$ is closed, convex for all $j \in J_0$. Invoking the assumed connectedness of $\Phi(x^0)$, we can apply Lemma 3.7 to assert that y^0 can be joined with $f_{j_1}(x^0)$ by some line segments belonging to $\Phi(x^0) \subset \Phi(D)$. In the same manner, we can join y^1 with $f_{j_s}(x^1)$ by some line segments belonging to $\Phi(x^1) \subset \Phi(D)$. And, for each $\ell \in \{1, 2, \ldots, s-1\}$, we can join $f_{j_\ell}(z^\ell)$ with $f_{j_{\ell+1}}(z^\ell)$ by some line segments belonging to $\Phi(z^\ell) \subset \Phi(D)$. Therefore, y^0 can be joined with y^1 by a continuous curve belonging to $\Phi(D)$. □

Remark 3.8. Theorem 3.5 can be proved also by applying Theorem 3.3 and Lemma 3.7. Indeed, since D is a connected component of $\operatorname{dom} \Phi$, there exists $J_0 \subset J$ such that $D = \bigcup_{j \in J_0} \operatorname{dom} \Phi_j$. Note that $\Psi : I\!R^n \rightrightarrows I\!R^m$ defined by

$$\Psi(x) = \bigcup_{j \in J_0} \Phi_j(x) \quad \forall x \in I\!R^n,$$

is a polyhedral multifunction. According to Theorem 3.3, $\operatorname{rge} \Psi = \bigcup_{j \in J_0} \operatorname{rge} \Phi_j$ is connected. As $\operatorname{rge} \Phi_j$ are polyhedral convex sets for all $j \in J_0$, Lemma 3.7 tells us that $\operatorname{rge} \Psi$ is path-connected. Hence the desired conclusion follows from the obvious equality $\Phi(D) = \operatorname{rge} \Psi$.

Remark 3.9. By Theorem 3.4, a polyhedral multifunction with nonempty connected values preserves connectedness of a set, provided that the mild additional condition (ii) is satisfied. A natural question arises here: *Does such a polyhedral multifunction preserves path-connectedness of a set contained in its effective domain?* At first glance, we thought that the answer should be affirmative. However, our further investigation shows that *such a polyhedral multifunction may not preserve the path-connectedness of a path-connected subset of its effective domain.*

We conclude this paper by a tailor-made (counter)example.

Example 3.10. Let convex polyhedral multifunctions $\Phi_1, \Phi_2, \Phi_3 : I\!R^2 \rightrightarrows I\!R^3$ be defined by setting

$$\Phi_1(x) = \begin{cases} \{(x_1, x_2, 1)\} & \text{if } x = (x_1, x_2) \in I\!R_+ \times I\!R_+ \\ \emptyset & \text{if } x \notin I\!R_+ \times I\!R_+, \end{cases}$$

$$\Phi_2(x) = \begin{cases} \{(x_1, x_2, 0)\} & \text{if } x = (x_1, x_2) \in I\!R_+ \times I\!R_- \\ \emptyset & \text{if } x \notin I\!R_+ \times I\!R_-, \end{cases}$$

$$\Phi_3(x) = \begin{cases} \{(x_1, 0)\} \times [0, 1] & \text{if } x = (x_1, x_2) \in I\!R_+ \times \{0\} \\ \emptyset & \text{if } x \notin I\!R_+ \times \{0\}. \end{cases}$$

Then the formula

$$\Phi(x) = \bigcup_{j=1}^{3} \Phi_j(x) \quad \forall x \in I\!R^2$$

defines a polyhedral multifunction $\Phi : I\!R^2 \rightrightarrows I\!R^3$. The set

$$D := \left\{ \left(t, t \sin \frac{1}{t}\right) \mid 0 < t \le 1 \right\} \cup \{(0, 0)\}$$

is a continuous curve joining $x^0 := (0, 0)$ and $x^1 := (1, \sin 1)$. Meanwhile, $\Phi(D)$ is not path-connected because there is no continuous curve joining $y^0 := (0, 0, 0) \in \Phi(x^0)$ and $y^1 := (1, \sin 1, 1) \in \Phi(x^1)$ which lies entirely in $\Phi(D)$. Indeed, suppose to the contrary that there exists a continuous map $\beta : [0, 1] \to \Phi(D)$ such that $\beta(0) = y^0$ and $\beta(1) = y^1$. By the continuity of β we can find $\delta \in (0, 1)$ with the property that

$$\beta(t) \in \Omega := \left\{ z = (z_1, z_2, z_3) \in \Phi(D) \mid z_3 < \frac{1}{2} \right\} \quad \forall t \in [0, \delta).$$

From the constructions of β and Ω it follows that the connected component of Ω containing y^0, denoted by $\Omega(y^0)$, is a subset of the segment $Q := \{(0, 0, \alpha) \mid \alpha \in [0, 1]\}$. Since $\{\beta(t) \mid t \in [0, \delta)\}$ is a connected subset of Ω and $\beta(0) = y^0$, we can assert that $\{\beta(t) \mid t \in [0, \delta)\} \subset \Omega(y^0)$. Hence $[0, \delta) \subset T := \{t \in [0, 1] \mid \beta(t) \in Q\}$. Using some standard arguments, we can easily show that T is both open and closed in the induced topology of the interval $[0, 1] \subset I\!R$. Hence $T = [0, 1]$. This implies that $\beta(1) = y^1$ is an element of Q, an absurd.

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N. T. Qui
College of Information and Communication Technology, Can Tho University, 1 Ly Tu Trong, Can Tho, Vietnam
E-mail address: `ntqui@cit.ctu.edu.vn`

N. D. Yen
Institute of Mathematics, Vietnamese Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi 10307, Vietnam
E-mail address: `ndyen@math.ac.vn`