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ON ϵ-OPTIMALITY THEOREMS FOR CONVEX VECTOR OPTIMIZATION PROBLEMS

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Dedicated to Professor Pham Huu Sach on the occasion of his 70th birthday

Abstract. A convex vector optimization problem, which consists of more than two convex objective functions and finitely many convex constraint functions, is considered. In this paper, we discuss ϵ-efficient solutions and weakly ϵ-efficient solutions for the convex vector optimization problem and obtain ϵ-optimality theorems for such solutions of which hold without any constraint conditions and are expressed by sequences. Moreover, we obtain ϵ-optimality theorems for the convex vector optimization problem which hold under certain constraint qualifications.

1. Introduction

Many authors have studied existence of ϵ-approximate solutions, ϵ-optimality conditions and ϵ-duality results for several kinds of optimization problems([2, 3, 4, 10, 11, 12, 13, 16, 17, 18, 20]).

It is well known that constraint qualifications (for example, the Slater condition) should be imposed on convex optimization problems to obtain ϵ-optimality conditions for its ϵ-approximate solutions.

To get an optimality condition for an efficient solution of a vector optimization problem, we often formulate an corresponding scalar problem. However, it is so difficult that such scalar program satisfies a constraint qualification which we need to derive an optimality condition. Hence it is very important to investigate an optimality condition for an efficient solution of a vector optimization problem which holds without any constraint qualification.

Jeyakumar et al. ([8]) and Jeyakumar et al. ([9]) gave optimality conditions for convex (scalar) optimization problems, which hold without any constraint qualification.

Recently, many authors have paid their attention to investigate properties of (weakly) ϵ-efficient solutions, ϵ-optimality conditions and ϵ-duality theorems for vector optimization problems([2, 3, 4, 12, 13, 16, 17, 18, 20]).

In this paper, a convex vector optimization probelm, which consists of more than two convex objective functions and finitely many convex constraint functions, is investigated. We consider ϵ-efficient solutions and weakly ϵ-efficient solutions for

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the convex vector optimization problem and obtain sequential ϵ-optimality theorems for such solutions of the convex vector optimization problem which hold without any constraint qualifications and are expressed by sequences. Futhermore, we give constraint qualifications expressed with epigraphs of conjugate functions and obtain ϵ-optimality theorems for the convex vector optimization problem which hold under the constraint qualifications.

2. Preliminaries

Now we give some definitions and preliminary results. The definitions can be found in [5, 15, 21].

Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The subdifferential of g at a is given by

$$\partial g(a) := \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a\rangle, \quad \forall x \in \mathrm{dom} g\},$$

where $\mathrm{dom} g := \{x \in \mathbb{R}^n \mid g(x) < \infty\}$ and $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$.

Let $\epsilon \geqq 0$. The ϵ-subdifferential of g at $a \in \mathrm{dom} g$ is defined by

$$\partial_\epsilon g(a) := \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a\rangle - \epsilon, \quad \forall x \in \mathrm{dom} g\}.$$

The conjugate function of $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$g^*(v) = \sup\{\langle v, x\rangle - g(x) \mid x \in \mathbb{R}^n\}.$$

The epigraph of g, $\mathrm{epi} g$, is defined by

$$\mathrm{epi} g = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid g(x) \leqq r\}.$$

For a nonempty closed convex subset C of $\mathbb{R}^n$, $\delta_C : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called the indicator of C if $\delta_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise,} \end{cases}$ and for a point $\bar{x} \in C$, the normal cone to C at $\bar{x}$ is defined as

$$N_C(\bar{x}) := \partial \delta_C(\bar{x}) = \{v \in \mathbb{R}^n \mid v^T(x - \bar{x}) \leqq 0 \quad \forall x \in C\}.$$

Lemma 2.1 ([7]). *If $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and if $a \in \mathrm{dom} h$, then*

$$\mathrm{epi} h^* = \bigcup_{\epsilon \geqq 0} \{(v, \langle v, a\rangle + \epsilon - h(a)) \mid v \in \partial_\epsilon h(a)\}.$$

Lemma 2.2 ([6]). *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

$$\mathrm{epi}(h + u)^* = \mathrm{epi} h^* + \mathrm{epi} u^*.$$

Lemma 2.3 ([1]). *Let $h_i : \mathbb{R}^n \to \mathbb{R}^m$, $i = 0, 1, \ldots, l$, be convex functions and let C be a closed convex subset of $\mathbb{R}^n$. Suppose that $\{x \in \mathbb{R}^n \mid h_i(x) \leqq 0, \ i = 1, \ldots, l\} \neq \emptyset$. Then the following statements are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid h_i(x) \leqq 0, \ i = 1, \ldots, l\} \subseteq \{x \in \mathbb{R}^n \mid \ h_0(x) \geqq 0\}$

(ii) $0 \in \mathrm{epi} h_0^* + \mathrm{cl}\Big(\bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i h_i)^* + \mathrm{epi}\delta_C^*\Big)$.

3. ϵ-OPTIMALITY THEOREMS

Consider the following convex vector optimization problem (**CVP**):

$$(\mathbf{CVP}) \qquad \text{Minimize} \quad f(x) := (f_1(x), \dots, f_p(x))$$
$$\text{subject to} \quad x \in Q := \{x \in C \mid g_j(x) \leqq 0, \quad j = 1, \dots, m\}.$$

Let $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \dots, p$, and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \dots, m$ be convex functions, C a closed convex set and $\epsilon = (\epsilon_1, \dots \epsilon_p)$, where $\epsilon_i \geqq 0$, $i = 1, \dots, p$.

Let for any $z \in \mathbb{R}^n$, $S(z) = \{x \in \mathbb{R}^n \mid f_i(x) \leqq f_i(z) - \epsilon_i, \quad \text{for all } i = 1, \dots, p\}$.

Now we give the definition of ϵ-efficient solution of (**CVP**) which can be found in ([14]).

Definition 3.1. The point $\bar{x} \in Q$ is said to be an ϵ-efficient solution of (**CVP**) if there does not exist $x \in Q$ such that

$$f_i(x) \leqq f_i(\bar{x}) - \epsilon_i, \text{ for all } i = 1, \dots, p,$$
$$f_j(x) < f_j(\bar{x}) - \epsilon_j, \text{ for some } j.$$

When $\epsilon = 0$, then the ϵ-efficiency becomes the efficiency for (**CVP**)(see the definition of efficient solution of (**CVP**) in [19]).

Now we give the definition of weakly ϵ-efficient solution of (**CVP**) which is weaker than ϵ-efficient solution of (**CVP**).

Definition 3.2. A point $\bar{x} \in Q$ is said to be a weakly ϵ-efficient solution of (**CVP**) if there does not exist $x \in Q$ such that

$$f_i(x) < f_i(\bar{x}) - \epsilon_i, \text{ for all } i = 1, \dots, p.$$

When $\epsilon = 0$, then the weak ϵ-efficiency becomes the weak efficiency for (**CVP**)(see the definition of efficient solution of (**CVP**) in [19]). Note that even though $\bar{x}$ is an ϵ-efficient solution of (**CVP**), $Q \cap S(\bar{x})$ may be empty.

Modifying Proposition 3.1 in [20], we can obtain the following proposition:

Proposition 3.3. *$\bar{x}$ is an ϵ-efficient solution of* (**CVP**) *if and only if*

$$Q \cap S(\bar{x}) = \emptyset \quad \text{or}$$
$$\sum_{i=1}^{p} f_i(x) \geqq \sum_{i=1}^{p} f_i(\bar{x}) - \sum_{i=1}^{p} \epsilon_i, \quad \text{for any} \quad x \in Q \cap S(\bar{x}).$$

We can easily obtain the following proposition:

Proposition 3.4. *Let $\epsilon \geqq 0$ and $\bar{x} \in Q$ then $\bar{x}$ is a weakly ϵ-efficient solution of* (**CVP**) *if and only if there exist $\mu_i \geqq 0$ $i = 1, \dots, p$, $\sum_{i=1}^{p} \mu_i = 1$ such that*

$$\sum_{i=1}^{p} \mu_i f_i(x) \geqq \sum_{i=1}^{p} \mu_i f_i(\bar{x}) - \sum_{i=1}^{p} \mu_i \epsilon_i, \quad \text{for any} \quad x \in Q.$$

Now we give a necessary and sufficient ϵ-optimality theorem for the ϵ-efficient solution of (**CVP**) which holds without any constraint qualification.

Theorem 3.5. *Let $\bar{x} \in Q$. Suppose that $Q \cap S(\bar{x}) \neq \emptyset$. Then $\bar{x}$ is an ϵ-efficient solution of* (**CVP**) *if and only if there exist $\alpha_i \geqq 0, u_i \in \partial_{\alpha_i} f_i(\bar{x}), i = 1, \ldots, p$, $\lambda_j^n \geqq 0$, $\beta_j^n \geqq 0, \delta^n \geqq 0$, $v_j^n \in \partial_{\beta_j^n}(\lambda_j^n g_j^n)(\bar{x}), j = 1, \ldots, m$, $\mu_k^n \geqq 0, \gamma_k^n \geqq 0, w_k^n \in \partial_{\gamma_k^n}(\mu_k f_k)(\bar{x})$, $k = 1, \ldots, p$, $\delta^n \geqq 0$, $z^n \in N_C^{\delta^n}(\bar{x})$ such that*

$$-\sum_{i=1}^{p} u_i = \lim_{n\to\infty} \Big(\sum_{j=1}^{m} v_j^n + \sum_{k=1}^{p} w_k^n + z^n \Big)$$

and

$$\sum_{i=1}^{p} \epsilon_i = \sum_{i=1}^{p} \alpha_i + \lim_{n\to\infty} \Big\{ \sum_{j=1}^{m} (\beta_j^n - (\lambda_j^n g_j^n)(\bar{x})) + \sum_{k=1}^{p} (\gamma_k^n - \mu_k^n \epsilon_k) + \delta^n \Big\}.$$

Proof. Let $h_0(x) = \sum_{i=1}^{p} f_i(x) - \sum_{i=1}^{p} f_i(\bar{x}) + \sum_{i=1}^{p} \epsilon_i$. Then

$$\mathrm{epi} h^* = \sum_{i=1}^{p} \mathrm{epi} f_i{}^* + \begin{pmatrix} 0 \\ \sum_{i=1}^{p} f_i(\bar{x}) - \sum_{i=1}^{p} \epsilon_i \end{pmatrix}^T.$$

So, we have,

$\bar{x}$ is an ϵ-efficient solution of (**CVP**).

$\Longleftrightarrow$ (by Proposition 3.3) $h_0(x) \geqq 0, \forall x \in Q \cap S(\bar{x})$.

$\Longleftrightarrow$ $\{x|\, g_i(x) \leqq 0\,, i = 1, \ldots m, f_j(x) - f_j(\bar{x}) + \epsilon_j \leqq 0, j = 1, \ldots, p\} \subset \{x|\, h_0(x) \geqq 0\}$.

$\Longleftrightarrow$ (by Lemma 2.3)

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}^T \in \sum_{i=1}^{p} \mathrm{epi} f_i^* + \begin{pmatrix} 0 \\ \sum_{i=1}^{p} f_i(\bar{x}) - \sum_{i=1}^{p} \epsilon_i \end{pmatrix}^T + \mathrm{cl} \Bigg\{ \Bigg(\bigcup_{\lambda_j \geqq 0} \sum_{j=1}^{m} \mathrm{epi}(\lambda_j g_j)^* \Bigg) + \bigcup_{\mu_j \geqq 0} \sum_{j=1}^{p} \Bigg[\mathrm{epi}(\mu_j f_j)^* + \begin{pmatrix} 0 \\ \mu_j f_j(\bar{x}) - \mu_j \epsilon_j \end{pmatrix}^T \Bigg] + \mathrm{epi}\delta_C^* \Bigg\}.$$

$\Longleftrightarrow$ (by Lemma 2.1) there exist $\alpha_i \geqq 0, u_i \in \partial_{\alpha_i} f_i(\bar{x}), i = 1, \ldots, p, \lambda_j^n \geqq 0, \beta_j^n \geqq 0, v_j^n \in \partial_{\beta_j^n}(\lambda_j^n g_j^n)(\bar{x}), j = 1, \ldots, m, \mu_k^n \geqq 0, \gamma_k^n \geqq 0, w_k^n \in \partial_{\gamma_k^n}(\mu_k f_k)(\bar{x}), k = 1, \ldots, p$, $\delta^n \geqq 0$, $z^n \in N_C^{\delta^n}(\bar{x})$ such that

$$\begin{pmatrix} 0 \\ \sum_{i=1}^{p} \epsilon_i - \sum_{i=1}^{p} f_i(\bar{x}) \end{pmatrix}^T = \sum_{i=1}^{p} \begin{pmatrix} u_i \\ u_i^T \bar{x} + \alpha_i - f_i(\bar{x}) \end{pmatrix}^T + \lim_{n\to\infty} \Bigg\{ \sum_{j=1}^{m} \begin{pmatrix} v_j^n \\ v_j^{nT} \bar{x} + \beta_j^n - (\lambda_j^n g_j)(\bar{x}) \end{pmatrix}^T$$

$$+\sum_{k=1}^{p}\left[\begin{pmatrix} \omega_k^n \\ \omega_k^{nT}\bar{x}+\gamma_k^n-(\mu_k^n f_k)(\bar{x}) \end{pmatrix}^T+\begin{pmatrix} 0 \\ \mu_k^n f_k(\bar{x})-\mu_k^n\epsilon_k \end{pmatrix}^T\right]$$
$$\left.+\begin{pmatrix} z^n \\ z^{nT}\bar{x}+\delta^n \end{pmatrix}^T\right\}.$$

$\iff$ there exist $\alpha_i \geqq 0, u_i \in \partial_{\alpha_i} f_i(\bar{x}), i=1,\ldots,p, \lambda_j^n \geqq 0, \beta_j^n \geqq 0, v_j^n \in \partial_{\beta_j^n}(\lambda_j^n g_j^n)(\bar{x}), j=1,\ldots,m, \mu_k^n \geqq 0, \gamma_k^n \geqq 0, w_k^n \in \partial_{\gamma_k^n}(\mu_k f_k)(\bar{x}), k=1,\ldots,p$ such that

$$0=\sum_{i=1}^{p}u_i+\lim_{n\to\infty}\Big(\sum_{j=1}^{m}v_j^n+\sum_{k=1}^{p}w_k^n+z^n\Big)$$

and

$$\sum_{i=1}^{p}\epsilon_i=\sum_{i=1}^{p}\alpha_i+\lim_{n\to\infty}\Big\{\Big(\sum_{j=1}^{m}(\beta_j^n-(\lambda_j^n g_j^n)(\bar{x})\Big)+\sum_{k=1}^{p}(\gamma_k^n-\mu_k^n\epsilon_k)+\delta^n\Big\}.$$

□

Following the first part of the proof of Theorem 3.5, we can easily obtain the following necessary and sufficient ϵ-optimality theorem for ϵ-efficient solution of (**CVP**) under a constraint qualification, which is called the closedness assumption for ϵ-efficient solution of (**CVP**).

Theorem 3.6. *Let $\bar{x} \in Q$ and assume that $Q \cap S(\bar{x}) \neq \emptyset$. Suppose that*

$$\bigcup_{\lambda_j\geqq 0}\sum_{j=1}^{m}\mathrm{epi}(\lambda_j g_j)^*+\bigcup_{\mu_j\geqq 0}\sum_{j=1}^{p}\left[\mathrm{epi}(\mu_j f_j)^*+\begin{pmatrix} 0 \\ \mu_j f_j(\bar{x})-\mu_j\epsilon_j \end{pmatrix}^T\right]+\mathrm{epi}\delta_C^*$$

is closed. Then the following are equivalent:

(i) $\bar{x}$ is an ϵ-efficient solution of (**CVP**).

(ii) $$\begin{pmatrix} 0 \\ 0 \end{pmatrix}^T\in\sum_{i=1}^{p}\mathrm{epi}f_i^*+\begin{pmatrix} 0 \\ \sum_{i=1}^{p}f_i(\bar{x})-\sum_{i=1}^{p}\epsilon_i \end{pmatrix}^T+\left(\bigcup_{\lambda_j\geqq 0}\sum_{j=1}^{m}\mathrm{epi}(\lambda_j g_j)^*\right)$$
$$+\bigcup_{\mu_j\geqq 0}\sum_{j=1}^{p}\left[\mathrm{epi}(\mu_j f_j)^*+\begin{pmatrix} 0 \\ \mu_j f_j(\bar{x})-\mu_j\epsilon_j \end{pmatrix}^T\right]+\mathrm{epi}\delta_C^*.$$

Now we give an example illustrating Theorem 3.6.

Example 3.7. Consider the following convex vector optimization problem:

$(\mathbf{CVP})_1$	Minimize	(x_1, x_2)
	subject to	$(x_1,x_2)\in Q:=\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1^2+x_2^2\leqq 1\}$.

Let $\epsilon=(\epsilon_1,\epsilon_2)=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ and $f(x_1,x_2)=(f_1(x_1,x_2),f_2(x_1,x_2))=(x_1,x_2)$. Then $(0,0)$ is an ϵ-efficient solution of $(\mathbf{CVP})_1$, $f_1(0,0)-\epsilon_1=-\frac{1}{\sqrt{2}}$, *and* $f_2(0,0)-\epsilon_2=$

$-\frac{1}{\sqrt{2}}$. Then we have,

$$\begin{aligned}&Q\cap S(0,0)\\&=Q\cap\{(x_1,x_2)\in\mathbb{R}^2\mid f_1(x_1,x_2)\leqq f_1(0,0)-\epsilon_1,\ f_2(x_1,x_2)\leqq f_2(0,0)-\epsilon_2\}\\&=Q\cap\Big\{(x_1,x_2)\in\mathbb{R}^2\mid x_1\leqq-\frac{1}{\sqrt{2}},\ x_2\leqq-\frac{1}{\sqrt{2}}\Big\}\\&=\Big\{\Big(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\Big)\Big\}.\end{aligned}$$

We will show that closedness assumption and the condition (ii) in Theorem 3.6 hold for $(\mathbf{CVP})_1$ at $(\bar{x}_1,\bar{x}_2)=(0,0)$ and $(\epsilon_1,\epsilon_2)=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$. We can check that

$$(1,1,-\sqrt{2})\in\sum_{i=1}^{2}\text{epi}f_i^*+\begin{pmatrix}0\\ \sum_{i=1}^2 f_i(\bar{x})-\sum_{i=1}^2\epsilon_i\end{pmatrix}^T.$$

Indeed,

$$f_1^*(v_1,v_2)=\sup_{(x_1,x_2)\in\mathbb{R}^2}\{v_1x_1+v_2x_2-x_1\}=\sup_{(x_1,x_2)\in\mathbb{R}^2}\{(v_1-1)x_1+v_2x_2\}$$

$$=\begin{cases}0 & \text{if } v_1=1,v_2=0\\ +\infty & \textit{otherwise},\end{cases}$$

$$\begin{aligned}f_2^*(v_1,v_2)&=\sup_{(x_1,x_2)\in\mathbb{R}^2}\{v_1x_1+v_2x_2-x_2\}=\sup_{(x_1,x_2)\in\mathbb{R}^2}\{(v_2-1)x_2+v_1x_1\}\\&=\begin{cases}0 & \text{if } v_2=1,v_1=0\\ +\infty & \textit{otherwise}.\end{cases}\end{aligned}$$

Thus $\text{epi}f_1^*=\{(1,0)\}\times[0,\infty)$ and $\text{epi}f_2^*=\{(0,1)\}\times[0,\infty)$. Hence we have,

$$(1,1,-\sqrt{2})\in\sum_{i=1}^{2}\text{epi}f_i^*+\left(0,0,\sum_{i=1}^2 f_i(\bar{x})-\sum_{i=1}^2\epsilon_i\right).$$

Let $g(x_1,x_2)=x_1^2+x_2^2-1$. Then we can easily see that
$\bigcup_{\lambda\geqq 0}\text{epi}(\lambda g)^*=\{(0,0)\}\times\mathbb{R}_+\cup\bigcup_{\lambda\geqq 0}\{(\frac{v_1}{2\lambda},\frac{v_2}{2\lambda},\frac{v_1^2}{4\lambda}+\frac{v_2^2}{4\lambda}+\lambda+\alpha)\mid(v_1,v_2)\in\mathbb{R}^2,\alpha\geqq 0\}$.

Moreover, we can check that $\bigcup_{\lambda_i\geqq 0}\text{epi}(\lambda_i g_i)^*=$ epi h, where $h(v_1,v_2)=\sqrt{v_1^2+v_2^2}$. Indeed, for any $\lambda>0$ and for any $v_1,v_2\in\mathbb{R}$,

$$\begin{aligned}\frac{v_1^2}{4\lambda}+\frac{v_2^2}{4\lambda}+\lambda-\sqrt{v_1^2+v_2^2}&=\frac{1}{4\lambda}\Big[\sqrt{(v_1^2+v_2^2)^2}-4\lambda\sqrt{v_1^2+v_2^2}+4\lambda^2\Big]\\&=\frac{1}{4\lambda}\Big(\sqrt{(v_1^2+v_2^2)^2}-2\lambda\Big)^2\geqq 0\end{aligned}$$

and hence $\frac{v_1^2}{4\lambda}+\frac{v_2^2}{4\lambda}+\lambda\geqq\sqrt{v_1^2+v_2^2}$. It is clear that $\{(0,0)\}\times\mathbb{R}_+\subset\text{epi}h$. Thus $\bigcup_{\lambda\geqq 0}\text{epi}(\lambda g)^*\subset\text{epi}h$.

Conversely, let $(\bar{v}_1,\bar{v}_2,\bar{\beta})\in\text{epi}h$. Then there exists $\bar{\alpha}\geqq 0$ such that $\bar{\beta}=$

$\sqrt{\bar{v}_1^2+\bar{v}_2^2}+\bar{\alpha}$. If $(\bar{v}_1,\bar{v}_2)=(0,0)$, then $(\bar{v}_1,\bar{v}_2,\bar{\beta})\in\{(0,0)\}\times\mathbb{R}_+$ and hence $(\bar{v}_1,\bar{v}_2,\bar{\beta})\in\bigcup\limits_{\lambda_i\geqq 0}\text{epi}(\lambda_i g_i)^*$. Assume that $(\bar{v}_1,\bar{v}_2)\neq(0,0)$ and let $\frac{1}{2}\sqrt{\bar{v}_1^2+\bar{v}_2^2}=\bar{\lambda}$. Then $\frac{\bar{v}_1^2}{4\bar{\lambda}}+\frac{\bar{v}_1^2}{4\bar{\lambda}}+\bar{\lambda}=\frac{\bar{v}_1^2+\bar{v}_2^2}{2\sqrt{\bar{v}_1^2+\bar{v}_2^2}}+\frac{1}{2}\sqrt{\bar{v}_1^2+\bar{v}_2^2}=\sqrt{\bar{v}_1^2+\bar{v}_2^2}$. Hence $(\bar{v}_1,\bar{v}_2,\bar{\beta})=(\bar{v}_1,\bar{v}_2,\sqrt{\bar{v}_1^2+\bar{v}_2^2}+\bar{\alpha})\in\{(v_1,v_2,\frac{v_1^2}{4\lambda}+\frac{v_1^2}{4\lambda}+\lambda+\alpha)|\,(v_1,v_2)\in\mathbb{R}^2,\,\alpha\geqq 0\}$. Thus $(\bar{v}_1,\bar{v}_2,\bar{\beta})\in\bigcup\limits_{\lambda\geqq 0}\text{epi}(\lambda g)^*$. Consequently, $\text{epi}h\subset\bigcup\limits_{\lambda\geqq 0}\text{epi}(\lambda g)^*$. Hence $\bigcup\limits_{\lambda\geqq 0}\text{epi}(\lambda g)^*=\{(x,y,z)\ |z\geqq\sqrt{x^2+y^2}\}$. For any $\mu=(\mu_1,\mu_2)\in\mathbb{R}^2_+$,

$$\begin{aligned}\Big(\sum_{i=1}^2\mu_i f_i\Big)^*(v_1,v_2) &= \sup_{(x_1,x_2)\in\mathbb{R}^2}\Big\{v_1x_1+v_2x_2-\Big(\sum_{i=1}^2\mu_i f_i\Big)(x_1,x_2)\Big\}\\ &= \sup_{(x_1,x_2)\in\mathbb{R}^2}\{v_1x_1+v_2x_2-(\mu_1x_1+\mu_2x_2)\}\\ &= \sup_{(x_1,x_2)\in\mathbb{R}^2}\{(v_1-\mu_1)x_1+(v_2-\mu_2)x_2\}\\ &= \begin{cases}0 & \text{if } \mu_1=v_1,\mu_2=v_2\\ +\infty & \text{otherwise}\end{cases}\end{aligned}$$

and hence $\text{epi}(\sum_{i=1}^2\mu_i f_i)^*=\{(\mu_1,\mu_2)\}\times[0,\infty)$. Thus

$$\begin{aligned}&\bigcup_{\mu_1\geqq 0,\mu_2\geqq 0}\text{epi}\Big(\sum_{i=1}^2\mu_i f_i\Big)^*+(0,0,-\mu_1\epsilon_1-\mu_2\epsilon_2)+\text{epi}\delta^*_{\mathbb{R}}\\ &=\bigcup_{\mu_1\geqq 0,\mu_2\geqq 0}\{(\mu_1,\mu_2,-\mu_1\epsilon_1-\mu_2\epsilon_2)+(0,0,\alpha)\ |\alpha\geqq 0\}\\ &=\{(x,y,-\epsilon_1x-\epsilon_2y+\alpha)\ |x\geqq 0,y\geqq 0,\alpha\geqq 0\}.\end{aligned}$$

So,

$$\begin{aligned}&\bigcup_{\lambda\geqq 0}\text{epi}(\lambda g)^*+\bigcup_{\lambda_j\geqq 0}\text{epi}\Big(\sum_{i=1}^2\mu_j f_j\Big)^*+(0,0,-\mu_1\epsilon_1-\mu_2\epsilon_2)+\text{epi}\delta^*_{\mathbb{R}^n}\\ &=\Big\{\Big(x,y,-y-\frac{x}{\sqrt{2}}+\alpha\Big)\ |x\geqq 0,y\leqq 0,\alpha\geqq 0\Big\}\cup\\ &\quad\Big\{\Big(x,y,-\frac{x+y}{\sqrt{2}}+\beta\Big)\ |x\geqq 0,y\geqq 0,\beta\geqq 0\Big\}\cup\\ &\quad\Big\{\Big(x,y,-x-\frac{y}{\sqrt{2}}+\gamma\Big)\ |x\leqq 0,y\geqq 0,\gamma\geqq 0\Big\}\cup\\ &\quad\{(x,y,z+\delta)\ |z\geqq\sqrt{x^2+y^2},x\leqq 0,y\leqq 0,\delta\geqq 0\}\end{aligned}$$

and hence this set is closed. Thus the closedness assumption in Theorem 3.6 holds.

Moreover, since $(-1,-1,\sqrt{2})\in\{(x,y,z+\delta)\ |z\geqq\sqrt{x^2+y^2},x\leqq 0,y\leqq 0,\delta\geqq 0\}$ and

$$(1,1,-\sqrt{2})\in\sum_{i=1}^2\text{epi}f_i^*+\begin{pmatrix}0\\ \sum_{i=1}^2 f_i(\bar{x})-\sum_{i=1}^2\epsilon_i\end{pmatrix}^T,$$

(ii) of Theorem 3.6 holds. □

We present a necessary and sufficient ϵ-optimality theorem for weakly ϵ-efficient solution of (**CVP**) which holds without any constraint qualification.

Theorem 3.8. *Let $\bar{x} \in Q$. Then $\bar{x}$ is a weakly ϵ-efficient solution of* (**CVP**) *if and only if there exist $\alpha_i \geqq 0, \mu_i \geqq 0,\ u_i \in \partial_{\alpha_i}(\mu_i f_i)(\bar{x})\,, i = 1, \dots, p, \lambda_j^n \geqq 0, \beta_j^n \geqq 0,\ v_j^n \in \partial_{\beta_j^n}(\lambda_j^n g_j)(\bar{x}), j = 1, \dots, m, \delta^n \geqq 0, z^n \in N_C{}^{\delta_n}(\bar{x})$ such that*

$$-\sum_{i=1}^{p} u_i = \lim_{n\to\infty} \Big\{ \sum_{j=1}^{m} v_j^n + z^n \Big\}$$

and

$$\sum_{i=1}^{p} \mu_i \epsilon_i = \sum_{i=1}^{p} \alpha_i + \lim_{n\to\infty} \Big\{ \sum_{j=1}^{m} (\beta_j^n - (\lambda_j^n g_j)(\bar{x})) + \delta^n \Big\}.$$

Proof. Let $h_0(x) = \sum_{i=1}^{p} \mu_i f_i(x) - \sum_{i=1}^{p} \mu_i f_i(\bar{x}) + \sum_{i=1}^{p} \mu_i \epsilon_i$. Then

$$\mathrm{epi} h_0^* = \sum_{i=1}^{p} \mathrm{epi}(\mu_i f_i)^* + \begin{pmatrix} 0 \\ \sum_{i=1}^{p} \mu_i f_i(\bar{x}) - \sum_{i=1}^{p} \mu_i \epsilon_i \end{pmatrix}^T.$$

Then we have,

$\bar{x}$ is a weakly ϵ-efficient solution of (**CVP**)

$\Longleftrightarrow$ (by Proposition 3.4) $\{x|\, g_i(x) \leqq 0\,, i = 1, \dots m, \mu_j f_j(x) - \mu_j f_j(\bar{x}) + \mu_j \epsilon_j \leqq 0, j = 1, \dots, p\} \subset \{x|\, h_0(x) \geqq 0\}$.

$\Longleftrightarrow$ (by Lemma 2.3) there exist $\mu_i \geqq 0\,,\ i = 1, \dots, p,\ \sum_{i=1}^{p} \mu_i = 1$ such that

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^T &\in \sum_{i=1}^{p} \mathrm{epi}(\mu_i f_i)^* + \begin{pmatrix} 0 \\ \sum_{i=1}^{p} \mu_i f_i(\bar{x}) - \sum_{i=1}^{p} \mu_i \epsilon_i \end{pmatrix}^T \\ &+ \ \mathrm{cl} \left(\bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i g_i)^* + \mathrm{epi} \delta_C^* \right) \end{aligned}$$

$\Longleftrightarrow$ (by Lemma 2.1) there exist $\alpha_i \geqq 0, \mu_i \geqq 0,\ i = 1, \dots, p\ \sum_{i=1}^{p} \mu_i = 1\,, u_i \in \partial_{\alpha_i}(\mu_i f_i)(\bar{x})\,, \lambda_j^n \geqq 0, \beta_j^n \geqq 0,\ v_j^n \in \partial_{\beta_j^n}(\lambda_j^n g_j)(\bar{x}), j = 1, \dots, m, \delta^n \geqq 0, z^n \in N_C{}^{\delta_n}(\bar{x})$ such that

$$\begin{aligned} \begin{pmatrix} 0 \\ \sum_{i=1}^{p} \mu_i \epsilon_i - \sum_{i=1}^{p} \mu_i f_i(\bar{x}) \end{pmatrix}^T &= \sum_{i=1}^{p} \begin{pmatrix} u_i \\ u_i{}^T \bar{x} + \alpha_i - \mu_i f_i(\bar{x}) \end{pmatrix}^T \\ + \lim_{n\to\infty} &\left\{ \sum_{j=1}^{m} \begin{pmatrix} v_j^n \\ v_j^{nT} \bar{x} + \beta_j^n - \sum_{j=1}^{m} \lambda_j^n g_j(\bar{x}) \end{pmatrix}^T + \begin{pmatrix} z^n \\ z^{nT} \bar{x} + \delta^n \end{pmatrix}^T \right\}. \end{aligned}$$

$\Longleftrightarrow$ there exist $\alpha_i \geqq 0, \mu_i \geqq 0,\ \sum_{i=1}^{p} \mu_i = 1\,, u_i \in \partial_{\alpha_i}(\mu_i f_i)(\bar{x})\,,\ i = 1, \dots, p\,, \lambda_j^n \geqq 0, \beta_j^n \geqq 0,\ v_j^n \in \partial_{\beta_j^n}(\lambda_j^n g_j)(\bar{x})\,,\ j = 1, \dots, m$, such that

$$-\sum_{i=1}^{p} u_i = \lim_{n\to\infty} \Big\{ \sum_{j=1}^{m} v_j^n + z^n \Big\},$$

$$\sum_{i=1}^{p} \mu_i \epsilon_i = \sum_{i=1}^{p} \alpha_i + \lim_{n\to\infty} \Big\{ \sum_{j=1}^{m} (\beta_j^n - \lambda_j^n g_j(\bar{x})) + \delta^n \Big\}.$$

□

From the proof of Theorem 3.8, we can easily obtain the following necessary and sufficient ϵ-optimality theorem for weakly ϵ-efficient solution of (**CVP**) under a constraint qualification, which is called the closedness assumption for weakly ϵ-efficient solution of (**CVP**).

Theorem 3.9. *Let $\bar{x} \in Q$ and assume that $\bigcup_{\lambda_j \geqq 0} \mathrm{epi}(\sum_{j=1}^{m} \lambda_j g_j)^* + \mathrm{epi}\delta_C^*$ is closed. Then the following are equivalent;*
(i) *$\bar{x}$ is a weakly ϵ-efficient solution of* (**CVP**).
(ii) *there exist $\mu_i \geqq 0$, $i = 1, \dots, p$, $\sum_{i=1}^{p} \mu_i = 1$ such that*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}^T \in \sum_{i=1}^{p} \mathrm{epi}(\mu_i f_i)^* + \begin{pmatrix} 0 \\ \sum_{i=1}^{p} \mu_i f_i(\bar{x}) - \sum_{i=1}^{p} \mu_i \epsilon_i \end{pmatrix}^T + \bigcup_{\lambda_j \geqq 0} \sum_{j=1}^{m} \mathrm{epi}(\lambda_j g_j)^* + \mathrm{epi}\delta_C^*.$$

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