Journal of Nonlinear and Convex Analysis
Volume 12, Number 3, 2011, 455–471

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ON S.-Y. CHANG'S INEQUALITIES AND NASH EQUILIBRIA

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Abstract. In a recent paper [2], S.-Y. Chang concentrated on the problem of the existence of equilibrium points for noncooperative generalized n-person games, n-person games of normal form and their related inequalities. She utilized the KKM lemma to obtain a theorem and then used it to obtain a new Fan type inequality and minimax theorems. Various new equilibrium point theorems were derived, with the necessary and sufficient conditions and with strategy spaces with no fixed point property. In this paper, Chang's 0-pair-concaveness is generalized and the 0-transfer continuity is extended to the intersectionally closed-valuedness of a corresponding multimap. Applying one of our new KKM type theorems, we obtain more refined generalizations of the Fan type inequalities, minimax theorems, and various equilibrium point theorems in [2,17,18].

1. Introduction

The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In [17], we clearly underlined that a sequence of statements from the partial KKM principle to the Nash equilibria can be obtained for any abstract convex space satisfying the partial KKM principle. This unifies previously known several proper examples of such sequences for particular types of KKM spaces.

Recently in [18], we clearly derived a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we added more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for arbitrary KKM spaces. Consequently, the paper [18] unified and enlarged previously known several proper examples of such statements for particular types of KKM spaces.

In a recent paper [2], S.-Y. Chang concentrated on the existence of equilibrium points for noncooperative generalized n-person games, n-person games of normal form and their related inequalities. She utilized the KKM lemma to obtain an inequality theorem and then used it to obtain a new Fan type inequality and minimax theorems. Various new equilibrium point theorems were also derived, with the necessary and sufficient conditions and with strategy spaces with no fixed point property. Chang's results generalized corresponding ones in previous works [1,5-9]. Moreover, in [2], examples were given to demonstrate that these existence theorems cover areas where other existence theorems break down.

2010 *Mathematics Subject Classification.* 47H04, 47H10, 49J40, 52A99, 54C65, 54H25, 58E35, 90D13, 91B50.
Key words and phrases. KKM type theorem, abstract convex space, generally 0-pair-concavity, generalized n-person game; Nash equilibrium, S-Nash equilibrium, pure strategy Nash equilibrium.

In this paper, from a new generalized KKM theorem for generalized KKM maps having intersectionally closed values in the sense of Luc et al. [10], we show that Chang's results can be generalized. Actually, Chang's 0-pair-concaveness is properly generalized and 0-transfer continuity is extended to the intersectionally closed-valuedness of a corresponding multimap. Consequently, we obtain a more refined generalization of Chang's inequality and apply this to obtain a new Fan type inequality and a minimax theorem. Furthermore, from Chang's key Lemma, we establish new equilibrium theorems such as Nash, S-Nash, pure-strategy Nash, and J-dominant-strategy Nash equilibrium theorems for generalized games or normal games. These have been established, as in [2], with the necessary and sufficient conditions and with topological strategy spaces that do not have the fixed point property. Our new results generalize those in [2,17,18] and some others.

In Section 2, the recent concepts on abstract convex spaces are introduced as in [18] and the references therein. Section 3 deals with new generalized KKM type theorems for generalized KKM maps having intersectionally closed values in the sense of Luc et al. [10]. We add a Ky Fan type minimax inequality as a direct consequence of the KKM theorem. In Section 4, Chang's 0-pair-concaveness is properly generalized and the 0-transfer continuity is extended to the intersectionally closed-valuedness of a corresponding multimap. Consequently, we obtain a more refined generalization of Chang's inequality and apply this to obtain a new Fan type inequality and a minimax theorem. Finally, in Section 5, from Chang's key lemma, we establish new equilibrium theorems such as Nash, S-Nash, pure-strategy Nash, and J-dominant-strategy Nash equilibrium theorems for generalized games or normal games.

In this paper, topological spaces are not necessarily Hausdorff. Multimaps are also called simply maps.

2. Abstract convex spaces

Let A be a subset of a topological space X. We denote by $\overline{A}$ or $\operatorname{cl} A$ the closure of A in X and, by $\operatorname{Int} A$ the interior of A. Let Δ_n be the standard n-dimensional simplex in $\mathbb{R}^{n+1}$. Let $\langle D\rangle$ be the set of all nonempty finite subsets of a set D.

For the concepts of abstract convex spaces and KKM spaces, the reader may consult our previous work [18] and the references therein.

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any nonempty $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$

satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

In our previous works [12-14], we gave many examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps.

The abstract convex subspaces are introduced by means of the following simple observation:

Proposition 2.3 ([15]). *For an abstract convex space* $(E, D; \Gamma)$ *and a nonempty subset* D' *of* D, *let* X *be a* Γ-*convex subset of* E *relative to* D' *and* $\Gamma' : \langle D' \rangle \multimap X$ *a map defined by*

$$\Gamma'_A := \Gamma_A \subset X \ \textit{for} \ A \in \langle D' \rangle.$$

Then $(X, D'; \Gamma')$ *itself is an abstract convex space called a subspace relative to* D'.

Proposition 2.4 ([15]). *Let* $(E, D; \Gamma)$ *be an abstract convex space,* $(X, D'; \Gamma')$ *a subspace, and* Z *a topological space. If* $F \in \mathfrak{KC}(E, D, Z)$, *then*

$$F|_X \in \mathfrak{KC}(X, D', \overline{F(X)}).$$

Definition 2.5. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

We have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Space satisfying the partial KKM principle
$\Longrightarrow$ Abstract convex space.

Example 2.6. There are plenty of examples of abstract convex spaces; see [18] and the references therein. Here we need only three classes of them:

(I) A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, for Δ_n with vertices $\{e_i\}_{i=0}^n$, Δ_J is its face corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

(II) A ϕ_A-*space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous maps $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D\rangle$ with $|A| = n+1$. Every ϕ_A-space can be made into a G-convex space; see [16]. Some authors' GFC-spaces or FC-spaces are ϕ_A-spaces or particular forms of them, resp.

Here we give a new usage of ϕ_A-spaces: In [5], its author gave a necessary and sufficient condition for the existence of a pure-strategy Nash equilibrium for non-cooperative games in topological spaces. He adopted the following concept:

Definition 2.7. Let X be a topological space, and $D, Y \subset X$. A real function $f : X \times Y \to \mathbb{R}$ is said to be $\mathcal{C}$-*quasiconcave* on D if, for any $N = \{x^0, x^1, \dots, x^n\} \in \langle D\rangle$, there exists a continuous map $\phi_N : \Delta_n \to Y$ such that

$$\min\{f(x^i, \phi_N(\lambda)) \mid i \in J\} \le f(\phi_N(\lambda), \phi_N(\lambda))$$

for all $\lambda := (\lambda_0, \lambda_1, \dots, \lambda_n) \in \Delta_n$, where $J := \{i \mid \lambda_i \neq 0\}$.

Note that $(Y, D; \{\phi_N\}_{N\in\langle D\rangle})$ is a ϕ_A-space.

By Propositions 1 and 2 in [5], the $\mathcal{C}$-quasiconcavity unifies the diagonal transfer quasiconcavity (weaker than quasiconcavity) [1] and the $\mathcal{C}$-concavity (weaker than concavity) [7].

(III) We give another example of spaces satisfying the partial KKM principle; see [19].

Definition 2.8. A Φ_A-*space*

$$(X, D; \{\Phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of l.s.c. maps $\Phi_A : \Delta_n \multimap X$ for $A \in \langle D\rangle$ with $|A| = n+1$.

Note that any Φ_A-space is an abstract convex space $(X, D; \Gamma)$ with $\Gamma_A := \operatorname{Im}\Phi_A$ for $A \in \langle D\rangle$.

Definition 2.9. For a Φ_A-space $(X, D; \{\Phi_A\}_{A\in\langle D\rangle})$, any map $T : D \multimap X$ satisfying

$$\Phi_A(\Delta_J) \subset T(J) \quad \text{for each} \quad A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map*.

Proposition 2.10. *A KKM map $T : D \multimap X$ on a Φ_A-space $(X, D; \{\Phi_A\})$ is a KKM map on a new abstract convex space $(X, D; \Gamma^T)$.*

The following is a KKM theorem for Φ_A-spaces:

Proposition 2.11. *For a Φ_A-space $(X, D; \{\Phi_A\}_{A\in\langle D\rangle})$, let $G : D \multimap X$ be a KKM map with closed values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D\rangle$ with $|N| = n+1$, we have $\Phi_N(\Delta_n) \cap \bigcap_{z\in N} G(z) \neq \emptyset$.)*

Further, if

$(*)$ *$\bigcap_{z\in M} G(z)$ is compact for some $M \in \langle D\rangle$,* *then we have* $\bigcap_{z\in D} G(z) \neq \emptyset$.

3. Basic KKM theorems

The following whole intersection property for the map values of a KKM map is a standard form of the KKM type theorems:

Theorem 3.1. *Let* $(E, D; \Gamma)$ *be an abstract convex space, the identity map* $1_E \in \mathfrak{KC}(E, D, E)$ [*resp.,* $1_E \in \mathfrak{KO}(E, D, E)$], *and* $G : D \multimap E$ *a multimap satisfying*

(1) *G has closed* [*resp., open*] *values; and*

(2) $\Gamma_N \subset G(N)$ *for any* $N \in \langle D\rangle$ (*that is, G is a KKM map*).

Then $\{G(z)\}_{z\in D}$ *has the finite intersection property.*

Further, if

(3) $\bigcap_{z\in M} \overline{G(z)}$ *is compact for some* $M \in \langle D\rangle$,

then we have

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Proof. The first part is a simple consequence of definition. For the second part, let $K := \bigcap_{z\in M} \overline{G(z)}$. Since $\{G(z) \mid z \in D\}$ has the finite intersection property, so does $\{K \cap \overline{G(z)} \mid z \in D\}$ in the compact set K. Hence it has the whole intersection property. □

Recall that Theorem 3.1 is a simple consequence of the definitions of the partial KKM principle or the KKM space.

Recall that the main conclusions of KKM type theorems are of the form

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$$

for a multimap $G : D \multimap E$.

Consider the following related four conditions:

(a) $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z\in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z\in D} \overline{G(z)} = \overline{\bigcap_{z\in D} G(z)}$ (G is *intersectionally closed-valued* [10]).

(c) $\bigcap_{z\in D} \overline{G(z)} = \bigcap_{z\in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

In [10], its authors noted that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and gave examples of multimaps satisfying (b) but not (c). Therefore it is a proper time to deal with condition (b) instead of (c) in the KKM theory.

Example 3.2. The following maps G are intersectionally closed-valued, but not transfer closed-valued:

(1) $G(z) = (0, 1)$ for every $z \in [0, 1]$ is a constant multimap from $D = [0, 1]$ to $E = [0, 1]$; see [10].

(2) $G(z)$ is a convex set in a Euclidean space having a relative interior point in common; see Rockafellar [21, Theorem 6.5].

(3) For a given subset E of a topological vector space with $x^* \in E$, each $G(z), z \in D$, is a nicely star-shaped at x^*; see [10].

From Theorem 3.1, we have the following form of the KKM type theorems:

Theorem 3.3. *Let* $(E, D; \Gamma)$ *be an abstract convex space,* Z *a topological space,* $F \in \mathfrak{KC}(E, D, Z)$*, and* $G : D \multimap Z$ *a map such that*

(1) $\overline{G}$ *is a KKM map w.r.t.* F*; and*

(2) *there exists a nonempty compact subset* K *of* Z *such that either*

(i) $\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ *for some* $M \in \langle D\rangle$*; or*

(ii) *for each* $N \in \langle D\rangle$*, there exists a* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$*,* $\overline{F(L_N)}$ *is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if* G *is transfer closed-valued, then* $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$*;*

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$*.*

Proof. Case (i): Since $F(\Gamma_N) \subset \overline{G}(N)$ for each $N \in \langle D\rangle$ by (1), we have

$$F(\Gamma_N) \subset F(E) \cap \overline{G}(N) \subset \overline{F(E)} \cap \overline{G}(N) = G'(N),$$

where $G'(y) := \overline{F(E)} \cap \overline{G(y)}$ is closed for each $y \in D$. Then, by Proposition 2.4 on $(E, D', \overline{F(E)})$, $\{G'(y) \mid y \in D\}$ has the finite intersection property. Since the requirement (i) implies

$$\overline{F(E)} \cap K \supset \overline{F(E)} \cap \bigcap_{y \in M} \overline{G(y)} = \bigcap_{y \in M} G'(y),$$

$\bigcap_{y \in M} G'(y)$ is compact. Therefore $\bigcap\{G'(y) \mid y \in D\} \neq \emptyset$ by Theorem 3.1 and hence

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Case (ii): Suppose that

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} = \emptyset.$$

Since $\overline{F(E)} \cap K$ is compact, $\overline{F(E)} \cap K \subset \bigcup\{Z \setminus \overline{G(y)} \mid y \in N\}$ for some $N \in \langle D\rangle$. Let L_N be the Γ-convex subset of E in (ii). Define $G' : D' \multimap \overline{F(L_N)}$ by $G'(y) := \overline{G(y)} \cap \overline{F(L_N)}$ for $y \in D'$. For each $A \in \langle D'\rangle$, define $\Gamma'_A := \Gamma_A \cap L_N$. Then $(L_N, D'; \Gamma')$ is an abstract convex space. Moreover,

$$(F|_{L_N})(\Gamma'_A) \subset F(\Gamma_A) \cap F(L_N) \subset \overline{G}(A) \cap \overline{F(L_N)} = G'(A)$$

for each $A \in \langle D' \rangle$ by (2); and hence $G' : D' \multimap \overline{F(L_N)}$ is a KKM map w.r.t. $F|_{L_N}$ on the abstract convex space $(L_N, D'; \Gamma')$ with closed values in $\overline{F(L_N)}$. Since $F \in \mathfrak{KC}(E, D, Z)$, by Proposition 2.4, we have $F|_{L_N} \in \mathfrak{KC}(L_N, D', \overline{F(L_N)})$ and hence, $\{G'(y) \mid y \in D'\} = \{\overline{G(y)} \cap \overline{F(L_N)} \mid y \in D'\}$ has the finite intersection property. Since we assumed that $\overline{F(L_N)}$ is compact, each $G'(y)$ is compact. Hence $\bigcap\{G'(y) \mid y \in D'\} \neq \emptyset$ by Theorem 3.1 and there exists a

$$z \in \bigcap_{y \in D'} G'(y) = \overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K$$

by (ii). Since $z \in K$ and $z \in \overline{F(L_N)}$, we have $z \in \bigcup\{Z \setminus \overline{G(y)} \mid y \in N\}$ by our assumption. So $z \notin G(y)$ for some $y \in N \subset D'$, and hence $z \notin \bigcap\{\overline{G(y)} \mid y \in D'\}$. This contradicts $z \in \bigcap\{G'(y) \mid y \in D'\}$. Therefore, we must have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

(α) Since G is transfer closed-valued, we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} G(y) = \overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

(β) Since G is intersectionally closed-valued, we have

$$\overline{\bigcap_{y \in D} G(y)} = \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

This implies the conclusion. □

Note that Theorem 3.3 can be reformulated to the equivalent forms of coincidence theorems, matching theorems, analytic alternatives, minimax inequalities, geometric and section properties as in our previous work [20].

For a multimap $G : D \multimap E$, consider the following related four conditions:

(a) $\bigcup_{z \in D} G(z) = E$ implies $\bigcup_{z \in D} \operatorname{Int} G(z) = E$.

(b) $\operatorname{Int} \bigcup_{z \in D} G(z) = \bigcup_{z \in D} \operatorname{Int} G(z)$ (G is *unionly open-valued* [10]).

(c) $\bigcup_{z \in D} G(z) = \bigcup_{z \in D} \operatorname{Int} G(z)$ (G is *transfer open-valued*).

(d) G is open-valued.

Proposition 3.4 ([10]). *The multimap G is intersectionally closed-valued* (*resp., transfer closed-valued*) *if and only if its complement G^c is unionly open-valued* (*resp., transfer open-valued*).

In view of this proposition, we have proper examples of unionly open-valued maps by applying the preceding examples.

From the KKM Theorem 3.3, we obtain Ky Fan type minimax inequalities. The following are some examples:

Theorem 3.5. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle. Let $f : D \times E \to \overline{\mathbb{R}}$ be an extended real-valued function and $\gamma \in \overline{\mathbb{R}}$ such that*

(1) *for each $z \in D$, $\{y \in E \mid f(z, y) \le \gamma\}$ is intersectionally closed [resp., transfer closed];*

(2) *for each $N \in \langle D\rangle$ and $y \in \Gamma_N$, $\min\{f(z, y) \mid z \in N\} \le \gamma$; and*

(3) *the coercivity condition (2) of Theorem 3.3 with $E = Z$ and $F = 1_E$ holds.*

Then (a) *there exists a $\hat{y} \in E$ [resp., $\hat{y} \in K$] such that*

$$f(z, \hat{y}) \le \gamma \quad \textit{for all} \;\; z \in D; \textit{ and}$$

(b) *if $E = D$ and $\gamma = \sup\{f(x, x) \mid x \in E\}$, then we have the minimax inequality:*

$$\inf_{y\in E} \sup_{x\in E} f(x, y) \le \sup_{x\in E} f(x, x).$$

Recall that if $(E, D; \Gamma)$ is a G-convex space, then for any $N \in \langle D\rangle$, there exists a continuous function $\phi_N : \Delta_{|N|-1} \to \Gamma_N$. In such case, the following holds:

Theorem 3.6. *In Theorem 3.5, the requirement (2) can be replaced by the following without affecting its conclusion:*

$(2)'$ *for each $N \in \langle D\rangle$, each continuous map $\phi_N : \Delta_{|N|-1} \to \Gamma_N$, and each $y \in \phi_N(\Delta_{|N|-1})$, we have $\min\{f(z, y) \mid z \in N\} \le \gamma$.*

Lemma 3.7. *Under the hypothesis of Theorem 3.5, condition (2) or $(2)'$ holds if and only if the map $G : D \multimap E$ defined by*

$$G(z) := \{y \in E \mid f(z, y) \le \gamma\} \quad \textit{for } z \in D$$

is a KKM map.

Proof. We give the proof for the case $(2)'$. The proof of the case (2) follows from this one, via slight modifications.

(Necessity) Suppose, on the contrary, that there exists an $N \in \langle D\rangle$ such that $\Gamma_N \not\subset G(N)$. Choose a $y \in \phi_N(\Delta_{|N|-1}) \subset \Gamma_N$ such that $y \notin G(N)$, whence $f(z, y) > \gamma$ for all $z \in N$. Then $\min_{z\in N} f(z, y) > \gamma$, which contradicts $(2)'$. Therefore, G is a KKM map.

(Sufficiency) Since G is a KKM map, for any $N \in \langle D\rangle$, we have $\Gamma_N \subset G(N)$. If $y \in \phi_N(\Delta_{|N|-1}) \subset \Gamma_N$, then $y \in G(z)$ or $f(z, y) \le \gamma$ for some $z \in N$. Therefore, $\min\{f(z, y) \mid y \in N\} \le \gamma$. □

Proof of Theorems 3.5 and 3.6. Let $G(z) := \{y \in E \mid f(z, y) \le \gamma\}$ for $z \in D$. Then G is an intersectionally closed-valued [resp., a transfer closed-valued] KKM map by Lemma 3.7. Note that G satisfies all requirements of Theorem 3.3 with $F = 1_E$ and hence there exists a $\hat{x} \in E$ [resp., $\hat{x} \in K$] such that $\hat{x} \in G(z)$ for all $z \in D$; that is, $f(z, \hat{x}) \le \gamma$ for all $z \in D$. This completes the proof of (a). Note that (b) clearly follows from (a). □

4. Generalizations of S.-Y. Chang's inequalities

Chang [2] extended the $\mathcal{C}$-quasiconcavity [5] to 0-pair-concavity, which can be further generalized as follows:

Definition 4.1. Let X be a nonempty set and Y be a topological space, and $D \subset X$. A function $f : X \times Y \to \mathbb{R}$ is said to be *generally 0-pair-concave* on D, if for any $\{x^0, \dots, x^n\} \in \langle D \rangle$, there is a multimap $\Phi_n \in \mathfrak{KC}(\Delta_n, V, Y)$, where $V = \{e_i\}_{i=0}^n$ is the standard base in Δ_n, such that

$$\min_{i \in I(\lambda)} f(x^i, y) \leq 0$$

for all $\lambda = \{\lambda_0, \dots, \lambda_n\} \in \Delta_n$ and $y \in \Phi_n(\lambda)$, where $I(\lambda) = \{i \mid \lambda_i \neq 0\}$.

When $\Phi_n : \Delta_n \to Y$ is a single-valued continuous map, then f is 0-pair-concave in the sense of Chang.

Remark 4.2. 1. Note that, in the above definition, an abstract convex space $(Y, D; \Gamma)$ can be obtained by defining $\Gamma_N = \Phi_n(\Delta_n)$ for each $N \in \langle D \rangle$ with $|N| = n + 1$.

2. In the above definition, we may adopt an l.s.c. map $\Phi_n : \Delta_n \multimap Y$ for $A \in \langle D \rangle$ instead of $\Phi_n \in \mathfrak{KC}(\Delta_n, V, Y)$.

For the 0-pair-concavity, Chang had the following proposition that the 0-pair-concavity includes the $\mathcal{C}$-quasiconcavity [5].

Proposition 4.3 ([2, Proposition 3.1])**.** *Let X be a topological space, and $D, Y \subset X$. A function $f : X \times Y \to \mathbb{R}$ is $\mathcal{C}$-quasiconcave on A. Define $U : X \times Y \to \mathbb{R}$ by $U(x, y) = f(x, y) - f(y, y)$ for all $(x, y) \in X \times Y$. Then U is 0-pair-concave on A.*

Chang also extended the diagonally transfer continuity of Baye et al. [1] as follows:

Definition 4.4 ([2])**.** Let X be a nonempty set and Y be a topological space, $D \subset X$, $C \subset Y$, and a function $f : X \times Y \to \mathbb{R}$. $f|_{D \times C}(x, y)$ is said to be *0-transfer-continuous* in y, if for every $(x, y) \in D \times C$, $f(x, y) > 0$ implies that there exists some $x' \in D$ and some neighborhood N_y of y in Y such that $f(x', z) > 0$ for all $z \in N_y \cap C$.

We show the following:

Proposition 4.5. *If $f|_{D \times C}(x, y)$ is 0-transfer-continuous in y, then the map $G : D \multimap C$ defined by $G(x) = \{y \in C \mid f(x, y) > 0\}$ for $x \in D$ is transfer open-valued and, hence, unionly open-valued.*

Proof. It suffices to show that $\bigcup_{x \in D} G(x) \subset \bigcup_{x \in D} \operatorname{Int} G(x)$. Let $y \in G(x)$ for $(x, y) \in D \times C$. Since $f(x, y) > 0$ and $y \mapsto f(x, y)$ is 0-transfer-continuous, there exist $x' \in D$ and N_y such that $f(x', z) > 0$ for $z \in N_y \cap C$. Then $z \in G(x')$ and hence $y \in N_y \cap C \subset \operatorname{Int} G(x')$. This completes our proof. □

Recall that, for an abstract convex space $(E \supset D; \Gamma)$, a function $f : E \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in E \mid f(x) > r\}$ [resp., $\{x \in E \mid f(x) < r\}$] is Γ-convex for each $r \in \overline{\mathbb{R}}$.

We define new concepts as follows:

Definition 4.6. An extended real-valued function $f : D \times E \to \overline{\mathbb{R}}$ is said to be *generally lower* [resp., *upper*] *semicontinuous* (g.l.s.c.) [resp., g.u.s.c.] for each $z \in D$, $\{y \in E \mid f(z,y) \leq r\}$ [resp., $\{y \in E \mid f(z,y) \geq r\}$] is intersectionally closed for each $r \in \overline{\mathbb{R}}$.

Example 4.7. 1. If the intersectionally closed sets are replaced by mere closed sets, then f is said to be l.s.c. [resp., u.s.c.].

2. If the intersectionally closed sets are replaced by transfer closed sets for a particular $\gamma \in \overline{\mathbb{R}}$ instead of arbitrary r, then f is said to be γ-transfer l.s.c. in y [that is, for each $x \in X$, $\{y \in Y \mid \phi(x,y) \leq \gamma\}$ is transfer closed]; see Tian [22].

3. Similarly, we can adopt the term γ-g.l.s.c. The 0-transfer-continuity of $y \mapsto f(x,y)$ is properly generalized by the 0-g.l.s.c. of G in y.

4. Note that these concepts can be extended to any simply ordered set $\mathbb{S}$ instead of $\overline{\mathbb{R}}$.

Theorem 4.8. *Let D be a nonempty set and Y be a topological space. A function $U : D \times Y \to \mathbb{R}$ satisfies the following conditions:*

(1) *the map $G : D \multimap Y$ defined by $G(x) = \{y \in Y \mid U(x,y) \leq 0\}$ for $x \in D$ is intersectionally closed-valued [resp., transfer closed-valued]; and*

(2) *there exists a nonempty compact subset $K = \bigcap\{\overline{G(x)} \mid x \in M\}$ of Y for some $M \in \langle D\rangle$.*

Then there exists $\tilde{z} \in Y$ [resp., $\tilde{z} \in K$] such that

$$\sup_{x \in D} U(x, \tilde{z}) \leq 0$$

if and only if U is generally 0-pair-concave on D.

Proof. Suppose that U is generally 0-pair-concave on D. For any finite subset $\{x_0, x_1, \ldots, x_m\} \in \langle D\rangle$, there exists an abstract convex space $(\Delta_m, V; \mathrm{co})$, where $V = \{e_i\}_{i=0}^m$ is the vertices of Δ_m. Since there exists a map $\Phi_m \in \mathfrak{KC}(\Delta_m, V, Y)$ such that

$$\min_{i \in I(\lambda)} U(x_i, y) \leq 0$$

for all $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m) \in \Delta_m$ and $y \in \Phi_m(\lambda)$, where $I(\lambda) = \{i \mid \lambda_i \neq 0\}$.

Let $H(e_i) = G(x_i) = \{y \in Y \mid U(x_i, y) \leq 0\}$ for $i = 0, 1, \ldots, m$. From the above inequality, we have

$$\Phi_m(\mathrm{co}\{e_i \mid i \in S\}) \subset \bigcup_{i \in S} H(e_i)$$

for each $S \subset \{0, 1, \ldots, m\}$. Hence, $H : V \multimap Y$ is a KKM map w.r.t. Φ_m. Since Δ_m is compact, by Theorem 3.3, we have $\bigcap_{i=0}^m \overline{H(e_i)} = \bigcap_{i=0}^m \overline{G(x_i)} \neq \emptyset$. Therefore $\{\overline{G(x)}\}_{x \in D}$ has the finite intersection property, and so does $\{K \cap \overline{G(z)} \mid z \in D\}$ in the compact set K in (2). Hence it has the whole intersection property and we have $K \cap \bigcap_{x \in D} \overline{G(x)} \neq \emptyset$.

Since G is intersectionally closed-valued [resp., transfer closed-valued], we have $\bigcap_{x \in D} G(x) \neq \emptyset$ [resp., $K \cap \bigcap_{x \in D} G(x) \neq \emptyset$]. Therefore, we have $\tilde{z} \in Y$ [resp., $\tilde{z} \in K$] such that $\sup_{x \in D} U(x, \tilde{z}) \leq 0$.

Conversely, if there exists $\tilde{z} \in Y$ such that $\sup_{x \in D} U(x, \tilde{z}) \leq 0$. For any finite points $x_0, x_1, \ldots, x_k \in D$, define $\phi_k : \Delta_k \to Y$ by $\phi_k(\lambda) = \tilde{z}$ for all $\lambda =$

$\{\lambda_0, \lambda_1, \ldots, \lambda_k\} \in \Delta_k$. We see that U is 0-pair-concave on D. This completes the proof. □

Remark 4.9. 1. Actually, (1) states that G is 0-g.l.s.c. [resp., 0-transfer l.s.c.].

2. When U is generally 0-pair-concave on D, we can define an abstract convex space $(Y, D; \Gamma)$, and condition (2) can be replaced by another compactness condition corresponding to (ii) in the KKM Theorem 3.1 without affecting the conclusion of Theorem 4.8.

Corollary 4.10 ([2, Theorem 3.1])**.** *Let D be a nonempty subset of a set X and Y be a topological space. A function $U : X \times Y \to \mathbb{R}$ satisfies the following conditions:*

(1) *there exist $\{x^0, x^1, \ldots, x^n\} \subset D$ such that $K = \bigcap_{i=0}^n \overline{G(x^i)}$ is compact where $G(x) = \{y \in Y \mid U(x, y) \le 0\}$;*

(2) *$U|_{D\times K}(x, y)$ is 0-transfer-continuous in y.*

Then there exists $\tilde{z} \in K$ such that

$$\sup_{x\in A} U(x, \tilde{z}) \le 0$$

if and only if U is 0-pair-concave on A.

From Theorem 4.8, we have the following generalization of the Fan minimax inequality [3]:

Theorem 4.11. *Let X be a topological space, D a nonempty subset of X, and $f, g : X \times X \to \mathbb{R}$. Assume that:*

(1) *$f \le g$ on $X \times X$;*

(2) *there exist $x^1, \ldots, x^n \in D$ such that $K = \bigcap_{i=1}^n \overline{G(x^i)}$ is compact where $G(x) = \{z \in X \mid f(x, z) \le \mu\}$ and $\mu = \sup_{y\in X} g(y, y)$;*

(3) *$g|_{D\times X}$ is $\mathcal{C}$-quasiconcave on D; and*

(4) *for each $x \in D$, $\{y \in X \mid f(x, y) \le \mu\}$ is intersectionally closed [resp., transfer closed].*

Then there exists $\tilde{z} \in X$ [resp., $\tilde{z} \in K$] such that

$$\sup_{x\in D} f(x, \tilde{z}) \le \sup_{y\in X} g(y, y)$$

holds.

Proof. Clearly we may assume that $\mu < \infty$. Define $U : D \times X \to \mathbb{R}$ by $U(x, z) = f(x, z) - \mu$. Then, from assumption (4), U satisfies assumption (1) of Theorem 4.8. For arbitrary $\{\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^k\} \in \langle D\rangle$, by (3), there is a continuous function $\phi_k : \Delta_k \to X$ such that

$$\min_{i\in I(\lambda)} g(\hat{x}^i, \phi_k(\lambda)) - g(\phi_k(\lambda), \phi_k(\lambda)) \le 0.$$

From assumption (1), $f(\hat{x}^i, \phi_k(\lambda)) - \mu \le g(\hat{x}^i, \phi_k(\lambda)) - g(\phi_k(\lambda), \phi_k(\lambda))$, so

$$\min_{i\in I(\lambda)} U(\hat{x}^i, \phi_k(\lambda)) \le 0$$

for all $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Delta_k$, where $I(\lambda) = \{i \mid \lambda_i \ne 0\}$. Hence U is 0-pair-concave. Thus according to Theorem 4.8, there exists $\tilde{z} \in Y$ [resp., $\tilde{z} \in K$] such that $\sup_{x\in D} U(x,\tilde{z}) \le 0$. Then

$$\sup_{x\in D} f(x,\tilde{z}) \le \sup_{y\in X} g(y,y).$$

This completes our proof. □

Remark 4.12. 1. Instead of (4), we can choose

(4)′ *for each* $x \in D$, $y \mapsto f(x,y)$ *is* μ*-g.l.s.c. on* K.

Then Theorem 4.11 still improves [2, Theorem 3.2].

2. Theorems 4.8 and 4.11 seem to be not directly related to abstract convex spaces, but they are consequences of our theory on G-convex spaces. For example, Theorem 4.11 follows from Theorem 3.6 as follows:

Proof of Theorem 4.11 using Theorem 3.6. Since $g|_{D\times X}$ is $\mathcal{C}$-quasiconcave on D by (3), for each $A \in \langle D\rangle$, there exists a continuous map $\phi_A : \Delta_{|A|-1} \to X$. Therefore, $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ is a ϕ_A-space and can be made into a G-convex space. Note that every G-convex space satisfies the partial KKM principle. Now we apply Theorem 3.6.

(a) By (4), for each $x \in D$, $\{y \in X \mid f(x,y) \le \mu\}$ is intersectionally closed [resp., transfer closed].

(b) We have to show that, for each $N \in \langle D\rangle$ and $y \in \phi_N(\Delta_{|N|-1}) \subset \Gamma_N$, we have $\min\{f(z,y) \mid z \in N\} \le \mu$.

In fact, since $g|_{D\times X}$ is $\mathcal{C}$-quasiconcave on D by (3), we have

$$\min\{g(x^i, \phi_N(\lambda)) \mid i \in J\} \le g(\phi_N(\lambda), \phi_N(\lambda))$$

as in the definition in [5]. Therefore, for any $y = \phi_N(\lambda) \in \phi_N(\Delta_{|N|-1}) \subset \Gamma_N$, we have

$$\min\{f(x,y) \mid x \in N\} \le \min\{g(x,y) \mid x \in N\} \le \min\{g(x^i, \phi_N(\lambda)) \mid i \in J\}$$
$$\le g(\phi_N(\lambda), \phi_N(\lambda)) \le \sup_{x\in X} g(x,x).$$

(c) The compactness condition holds by (2).

Since (a)-(c) imply requirements (1)-(3) of Theorem 3.6, resp., its conclusion holds, that is, there exists a $\hat{y} \in X$ [resp., $\hat{y} \in K$] such that

$$f(x,\hat{y}) \le \gamma \quad \text{for all } x \in D.$$

This completes our proof. □

From Theorem 4.11, we have the following useful result:

Corollary 4.13 ([2, Corollary 3.1])**.** *Let X be a compact topological space, D be a nonempty subset of X, and $g : X \times X \to \mathbb{R}$. Suppose $g|_{D\times X}$ is $\mathcal{C}$-quasiconcave on D, and the function $y \mapsto g(x,y)$ is l.s.c. for each $x \in D$. Then*

$$\min_{y\in X} \sup_{x\in D} g(x,y) \le \sup_{y\in X} g(y,y).$$

The equilibrium for a two-person zero-sum game is a minimax problem. Applying Theorem 4.8 as in [2], we obtain the following slight generalization of [2, Theorem 3.3]:

Theorem 4.14. *Let X and Y be topological spaces, D, C be nonempty compact subsets of X, Y, resp., $f, g : X \times Y \to \mathbb{R}$, and $U : (D \times C) \times (X \times Y) \mapsto \mathbb{R}$ be defined by $U((x, y), (u, v)) = f(u, y) - g(x, v)$. Assume that:*
(1) *the function $x \mapsto \sup_{y \in Y} f(x, y)$ is l.s.c. on D;*
(2) *the function $y \mapsto \inf_{x \in X} g(x, y)$ is u.s.c. on C;*
(3) *U is generally 0-pair-concave on $X \times Y$.*
Then the minimax inequality

$$\min_{x \in D} \sup_{y \in Y} f(x, y) \leq \max_{y \in C} \inf_{x \in X} g(x, y)$$

holds. Furthermore, if $f = g$, then

$$\inf_{x \in X_1} \sup_{y \in Y_1} f(x, y) = \sup_{y \in Y_2} \inf_{x \in X_2} f(x, y);$$

where X_i is either D or X and Y_i is either C or Y for $i = 1, 2$.

Proof. From assumption (3), by Theorem 4.8, there exists $(\tilde{x}, \tilde{y}) \in D \times C$ such that

$$\sup_{(u,v) \in X \times Y} [f(\tilde{x}, v) - g(u, \tilde{y})] \leq 0.$$

Then

$$\sup_{v \in Y} f(\tilde{x}, v) \leq \inf_{u \in X} g(u, \tilde{y}).$$

From assumptions (1) and (2), we have

$$\min_{x \in D} \sup_{y \in Y} f(x, y) \leq \max_{y \in C} \inf_{x \in X} g(x, y).$$

For the case $f = g$, just follow the proof of [2, Theorem 3.3]. □

Remark 4.15. 1. When U is 0-pair-concave on $X \times Y$ in (3), Theorem 4.14 reduces to [2, Theorem 3.3].

2. Theorems 4.11 and 4.14 also generalize Theorem 4 and Theorem 3 in [8, p.1211–1214].

5. Existence of Nash equilibria

We follow [2]. Let $I = \{1, \ldots, n\}$ be a set of players. A non-cooperative generalized n-person game is an ordered $3n$-tuple

$$\mathcal{G} = \{X_1, \ldots, X_n;\ T_1, \ldots, T_n;\ u_1, \ldots, u_n\},$$

and for each player $i \in I$, the nonempty set X_i is the strategy set, $T_i : X = \prod_{i \in I} X_i \multimap X_i$ is the player's constraint correspondence (multimap), and $u_i : X \to \mathbb{R}$ is the i-th player's payoff function. Whenever the player's constraint correspondence $T_i(x) = X_i$ for all $x \in X$ and all $i \in I$, the generalized game reduces to $2n$-tuple $\mathcal{G} = \{X_1, \ldots, X_n; u_1, \ldots, u_n\}$ and is called an n-person game of normal form. The set X is the Cartesian product of the individual strategy spaces. Denote by $X_{-i} = \prod_{j \in I \setminus \{i\}} X_j$. Denote by x_i and x_{-i} an element of X_i and X_{-i}, resp.

Denote an arbitrary point of X by $x = (x_i, x_{-i})$, with x_i in X_i and x_{-i} in X_{-i}. Let J be a nonempty subset of I. Denote $X_J = \prod_{i\in J} X_i$ and $X_{-J} = \prod_{i\in I\setminus J} X_i$. Denote by x_J and x_{-J} an element of X_J and X_{-J}, resp. Denote an arbitrary point of X by $x = (x_J, x_{-J})$, with x_J in X_J and x_{-J} in X_{-J}. Denote $T : X \multimap X$ by $T(x) = \prod_{i=1}^n T_i(x)$ for all $x \in X$.

A strategy vector $\tilde{x} \in X$ is said to be a *Nash equilibrium* for the generalized n-person game $\mathcal{G}$ if for each $i \in I$

$$\tilde{x}_i \in T_i(\tilde{x}) \text{ and } u_i(\tilde{x}_i, \tilde{x}_{-i}) \geq u_i(x_i, \tilde{x}_{-i}) \text{ for all } x_i \in T_i(\tilde{x}).$$

A strategy vector $\tilde{x} \in X$ is said to be an *S-Nash equilibrium* for the generalized n-person game $\mathcal{G}$ if $\tilde{x}$ is a Nash equilibrium for $\mathcal{G}$ and

$$\sum_{i=1}^n u_i(x) \leq \sum_{i=1}^n u_i(\tilde{x}_i, x_{-i}) \text{ for all } x \in T(\tilde{x}).$$

Whenever the player's constraint correspondence $T_i(x) = X_i$ for all $x \in X$ and $i \in I$, a Nash equilibrium $\tilde{x} \in X$ is said to be a *pure-strategy Nash equilibrium* for the n-person game $\mathcal{G}$ of normal form; an S-Nash equilibrium is said to be an *S-Nash-strategy equilibrium* for the n-person game $\mathcal{G}$ of normal form.

Let $J \subset I$. A strategy vector $\tilde{x}_J \in X_J$ is said to be a *J-dominant-strategy* if

$$u_i(\tilde{x}_i, x_{-i}) \geq u_i(x_i, x_{-i}) \text{ for all } x \in X \text{ and } i \in J.$$

A strategy vector $\tilde{x} \in X$ is said to be a *J-dominant-strategy Nash equilibrium* for the n-person game $\mathcal{G}$ of normal form when $\tilde{x}$ is a pure-strategy Nash equilibrium and $\tilde{x}_J$ is a J-dominant-strategy.

Let $\mathcal{G} = \{X_1, \ldots, X_n; T_1, \ldots, T_n; u_1, \ldots, u_n\}$ be a non-cooperative generalized n-person game. By following the method introduced by Nikaido-Isoda [11], let us define the aggregate payoff function $U : X \times X \to \mathbb{R}$ associated as follows:

$$U(x, y) = \sum_{i=1}^n [u_i(y_i, x_{-i}) - u_i(x)],$$

for every $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X = \prod_{i=1}^n X_i$. Also, we denote $\mathcal{F}(T) = \{x \in X \mid x_i \in T_i(x),\ i \in I\}$ and $T(x) = \prod_{i=1}^n T_i(x_i)$ for all $x \in X$. Then we shall need the following which is a general form of Proposition 1 in [9]:

Lemma 5.1 ([2, Lemma 4.1])**.** *Let $\mathcal{G}$ be a non-cooperative generalized n-person game and $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in T(\hat{x})$.*

(1) *$\hat{x}$ is a Nash equilibrium if and only if $U(\hat{x}, y) \leq 0$ for all $y \in T(\hat{x})$.*

(2) *$\hat{x}$ is an S-Nash equilibrium if and only if $U(x, \hat{x}) \geq 0$ for all $x \in T(\hat{x})$.*

Using Theorem 4.8 and Lemma 5.1, we can now prove the following existence theorems of equilibria in generalized n-person games and normal forms of n-person games:

Theorem 5.2. *Let $\mathcal{G}$ be a non-cooperative generalized n-person game, and $U : X \times X \to \mathbb{R}$ be the aggregate payoff function. Then $\mathcal{G}$ has a Nash equilibrium $\hat{x} \in X$ [resp., $\hat{x} \in K$] if and only if the following conditions are fulfilled:*

(1) *the set K is a nonempty compact subset of $\mathcal{F}(T)$;*

(2) *the map* $G : K \multimap T(K)$ *defined by* $G(x) = \{y \in T(K) \mid U(x, y) \le 0\}$ *for* $x \in K$ *is intersectionally closed-valued [resp., transfer closed-valued];*
(3) $U|_{K\times T(K)}$ *is generally 0-pair-concave on* $T(K)$.

Proof. Suppose the three conditions hold. Let $A = T(K)$. From Theorem 4.8, there exists $\hat{z}$ such that

$$\sup_{y\in T(K)} U(\hat{z}, y) \le 0.$$

Hence from Lemma 5.1, $\hat{z}$ is a Nash equilibrium for the generalized game $\mathcal{G}$.

Suppose $\mathcal{G}$ has a Nash equilibrium $\hat{z}$. Let $K = \{\hat{z}\}$. Then K is a compact subset of $\mathcal{F}(T)$ and $U|_{K\times T(K)}(x, y)$ is 0-transfer continuous in x. Also, $U|_{K\times T(K)}$ is 0-pair-concave on $T(\{\tilde{z}\})$. This completes the proof. □

Remark 5.3. Chang [2, Theorem 4.1] obtained Theorem 5.2 under the assumptions
(2)′ $U|_{K\times T(K)}(x, y)$ is 0-transfer continuous in x, and
(3)′ $U|_{K\times T(K)}$ is 0-pair-concave on $T(K)$,
instead of (2) and (3), resp.

Theorem 5.4. *Let* $\mathcal{G}$ *be a non-cooperative generalized N-person game, and* $U : X \times X \to \mathbb{R}$ *be the aggregate payoff function. Then* $\mathcal{G}$ *has an S-Nash equilibrium* $\tilde{z} \in X$ *[resp.,* $\tilde{z} \in K$*] if and only if the following conditions are fulfilled:*
(1) *the set K is a nonempty compact subset of* $\mathcal{F}(T)$*;*
(2) *the map* $G : K \multimap T(K)$ *defined by* $G(x) = \{y \in T(K) \mid -U(x, y) \le 0\}$ *for* $x \in K$ *is intersectionally closed-valued [resp., transfer closed-valued];*
(3) $-U|_{K\times T(K)}$ *is generally 0-pair-concave on* $T(K)$.

Proof. Let $A = T(K)$. From Theorem 4.8, there exists $\tilde{z}$ such that

$$\sup_{x\in T(K)} -U(x, \tilde{z}) \le 0.$$

Hence from Lemma 5.1, $\tilde{z}$ is an S-Nash equilibrium for the generalized game $\mathcal{G}$. It is obvious that the converse is also true. □

Remark 5.5. Chang [2, Theorem 4.2] obtained Theorem 5.4 under the assumptions
(2)′ $-U|_{K\times T(K)}(x, y)$ is 0-transfer continuous in x, and
(3)′ $-U|_{K\times T(K)}$ is 0-pair-concave on $T(K)$,
instead of (2) and (3), resp.

Also, from Theorem 4.8 and Lemma 5.1, we have the following theorems:

Theorem 5.6. *Let* $\mathcal{G} = \{X_1, \dots, X_n; u_1, \dots, u_n\}$ *be an n-person game of normal form and* $J \subset \{1, 2, \dots, n\}$*. Suppose that for each* $i \in J$ *the following conditions are fulfilled:*
(1) *there exist* $y^{i,0}, y^{i,1}, \dots, y^{i,n_i} \in X_i$ *such that* $K_i = \bigcap_{j=0}^{n_i} \overline{G_i(y^{i,j})}$ *is compact where* $G_i(y) = \{x_i \in X_i \mid U_i(x_i, y) \le 0\}$ *and* $U_i : X_i \times X \to \mathbb{R}$ *is defined by* $U_i(x_i, y) = u_i(y) - u_i(x_i, y_{-i})$*;*
(2) *the map* $H_i : K_i \multimap X$ *defined by* $H(x_i) = \{y \in X \mid U_i(x_i, y) \le 0\}$ *for* $x_i \in K_i$ *is intersectionally closed-valued [resp., transfer closed-valued].*

Then there exists a J-dominant-strategy $\tilde{z}_J$ *in* $\prod_{i\in J} X_i$ *[resp., in* $\prod_{i\in J} K_i$*] for the game* $\mathcal{G}$ *if and only if* $U_i|_{X_i\times X}$ *is generally 0-pair-concave on X for each* $i \in J$.

Theorem 5.7. *Let $\mathcal{G} = \{X_1, \dots, X_n; u_1, \dots, u_n\}$ be an n-person game of normal form, $J \subset \{1, 2, \dots, n\}$, and $U : X \times X \to \mathbb{R}$ be the aggregate payoff function. Suppose that the following conditions are fulfilled:*

(1) *$E_J = \{x_J \in X_J \mid x_J$ is a J-dominant-strategy$\}$;*

(2) *there exist $y^0, y^1, \dots, y^n \in X$ such that $K = \bigcap_{i=0}^n \mathrm{cl}_Y G(y_i)$ is compact where $G(y) = \{x \in E_J \times X_{-J} \mid U(x, y) \leq 0\}$ and $Y = E_J \times X_{-J}$;*

(3) *the map $H : K \multimap X$ defined by $H(x) = \{y \in X \mid U(x, y) \leq 0\}$ for $x \in K$ is intersectionally closed-valued [resp., transfer closed-valued].*

Then there exists a J-dominant-strategy Nash equilibrium $\tilde{z} \in X$ [resp., $\tilde{z} \in K$] for the game $\mathcal{G}$ if and only if $U|_{Y \times X}$ is generally 0-pair-concave on X.

Theorem 5.8. *Let $\mathcal{G} = \{X_1, \dots, X_n; u_1, \dots, u_n\}$ be an n-person game of normal form and $U : X \times X \to \mathbb{R}$ be the aggregate payoff function. Then there exists a pure-strategy Nash equilibrium for the game $\mathcal{G}$ if and only if the following conditions are fulfilled:*

(1) *there exists a subset $Z \subset X$ and $y^0, y^1, \dots, y^n \in X$ such that $K = \bigcap_{i=0}^n \mathrm{cl}_Z G(y_i)$ is compact where $G(y) = \{x \in Z \mid U(x, y) \leq 0\}$;*

(2) *the map $H : K \multimap X$ defined by $H(x) = \{y \in X \mid U(x, y) \leq 0\}$ for $x \in K$ is intersectionally closed-valued [e.g., $U|_{K \times X}(x, y)$ is 0-transfer continuous in x];*

(3) *$U|_{Z \times X}$ is generally 0-pair-concave on X.*

Remark 5.9. 1. Theorems 5.6 - 5.8 generalize Theorems 4.3 - 4.5 of [2], resp.

2. According to [2], Theorem 5.2 generalizes Theorem 1 in [6]. Theorem 5.8 generalizes Theorems 1 and 3 in [1], Theorem 1 [8], and Theorem 1 of [5] without the fixed property. Theorems 5.6 and 5.7 generalize Theorems 4 and 5 in [1].

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Manuscript received March 15, 2011
revised October 25, 2011

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