Journal of Nonlinear and Convex Analysis
Volume 12, Number 3, 2011, 441–453

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ON THE RADIUS OF SURJECTIVITY FOR RECTANGULAR MATRICES AND ITS APPLICATION TO MEASURING STABILIZABILITY OF LINEAR SYSTEMS UNDER STRUCTURED PERTURBATIONS

NGUYEN KHOA SON* AND DO DUC THUAN

Dedicated to 70th birthday of Professor Pham Huu Sach

Abstract. The main purpose of this paper is to get generalizations of the well-known Eckart-Young theorem which identifies the distance to singularity of a non-singular square matrix to the case of surjective rectangular matrices, subjected to arbitrary affine purturbations. As an application of these results, we shall derive the formulas of structured stabilizability radius for the linear control system $[A, B] : \dot{x} = Ax + Bu, t \geq 0$.

1. Introduction

Let $W \in \mathbb{C}^{n\times n}$ be a non-singular complex matrix (i.e. $\operatorname{rank}(W) = n$) then the classical Eckart-Young theorem (see [4]) confirmed that the distance from W to the set Σ_n of all singular $(n \times n)$-matrices is

$$\operatorname{dist}(W, \Sigma_n) = \inf\{\|\Delta\| : \Delta \in \mathbb{C}^{n\times n},\ W + \Delta \in \Sigma_n\} = \frac{1}{\|W^{-1}\|}, \tag{1.1}$$

where the matrix norm in (1.1) is the operator norm induced by a given vector norm of the corresponding vector space $\mathbb{C}^n$.

The quantity $\operatorname{dist}(W, \Sigma)$ is called the *radius of non-singularity* (or *distance to singularity*) of matrix W. It characterizes the extent to which non-singularity of W is preserved under perturbations of the form: $W \rightsquigarrow W + \Delta,\ \Delta \in \mathbb{C}^{n\times n}$. In the case of rectangular matrices, i.e. $W \in \mathbb{C}^{n\times m}, n \leq m$ with $\operatorname{rank} W = n$ (or equivalently $W\mathbb{C}^m = \mathbb{C}^n$), a natural generalization of the above definition is the notion of *radius of surjectivity*:

$$\operatorname{dist}(W, \Sigma_{n,m}) = \inf\{\|\Delta\| : \Delta \in \mathbb{C}^{n\times m},\ \operatorname{rank}(W + \Delta) < n\},$$

where $\Sigma_{n,m}$ denotes the set of all rank deficient $(n \times m)$-matrices. Obviously, due to calculation errors, the above mentioned radii play a crucial role in any numerical algorithm of solving mathematical problems which involve inverse or generalized inverse matrices. In particular, the problem of calculation of the radius of surjectivity

2010 *Mathematics Subject Classification.* 06B99, 34D99, 47A10, 47A99, 65P99.
Key words and phrases. Linear multi-valued operator, radius of surjectivity, radius of stabilizability, structured perturbations .
*Corresponding author.
This work was supported financially by NAFOSTED (Vietnam National Foundation for Science and Technology Development) under the research Project "Qualitative properties of nonlinear control systems under perturbations and their applications".

of matrices is a topic of significant interest in mathematical control and optimization theory and has attracted thereby a good deal of attention from researchers over several last decades, see [2, 3, 11, 12, 13, 14, 15, 16, 17, 18, 19]. It worth noticing that in most of papers only the case of "unstructured radius" was studied, where the perturbation model is assumed to be "component-wise" as $w_{ij} \rightsquigarrow w_{ij} + \delta_{ij}$, $\forall i, j$. In many cases, however, the perturbations are restricted to some specific structure (for example, only one row or column of W or even only one entry of W is perturbed) and ignoring such structures may lead to substantial underestimation of the radius under consideration.

The aim of this paper is to generalize the classical result of Eckart and Young to the case where the matrix $W \in \mathbb{C}^{n\times m}$ is subjected to the structured perturbations of the form

$$W \rightsquigarrow \tilde{W} = W + D\Delta E$$

and the multi-perturbations of the form

$$W \rightsquigarrow \tilde{W} = W + \sum_{i=1}^{N} D_i \Delta_i E_i,$$

where D, E and $D_i, E_i, i = 1, \dots, N$ are given matrices defining the structure of perturbations. The above structured perturbation model allows us, by choosing the apropriate structuring matrices D and E, to describe the case where only one row or column of W or even only one entry of W is perturbed, whereas the more general class of multi-perturbations will cover all cases of affine perturbations (for instance, the case where the diagonal entries of W are perurbed). We shall then apply our result to calculating the stabilizability radius of a linear system. The key technique is to make use of some well-known facts from the theory of linear multi-valued operators in representing equations and evaluating the norms of matrices involved in the calculation. A similar technique has been used in our recent paper [19] for calculating structured controllability radii.

The organization of the paper is as follows. In the next Section, for the reader's convenience, we shall recall some known notions and results from the theory of linear multi-valued operators (see, e.g. [1, 19]). In Section 3 we prove the main results of the paper which give formulas for calculating the complex radius of surjectivity of a rectangular matrix subjected to structured perturbations and to multi-perturbations. In Section 4, we apply the results of the previous section to establish formulas for the stabilizability radius and give an illustrating example. In Conclusion we summarize the obtained results and give some remarks of further investigation.

2. Preliminaries

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, the set of real or complex numbers. If $A \in \mathbb{K}^{m\times n}$, then $A^* \in \mathbb{K}^{m\times n}$ denotes the adjoint matrix of A. $\mathbb{K}^n (= \mathbb{K}^{n\times 1})$ is the n-dimentional vector space equipped with the vector norm $\|\cdot\|$, its dual space can be identified with $(\mathbb{K}^n)^* = (\mathbb{K}^{n\times 1})^* = \{u^* : \ u \in \mathbb{K}^n\}$, equipped with the dual norm. For $u^* \in (\mathbb{K}^n)^*$ we shall write $u^*(x) = u^*x, \forall x \in \mathbb{K}^n$. For a subset $M \subset \mathbb{K}^n$, we denote $M^\perp = \{u^* \in$

$(\mathbb{K}^n)^* : u^*x = 0$ for all $x \in M\}$. Let $\mathcal{F} : \mathbb{K}^n \rightrightarrows \mathbb{K}^m$ be a multi-valued operator. If the graph of $\mathcal{F}$, defined by

$$\operatorname{gr}\mathcal{F} = \big\{(x, y) \in \mathbb{K}^n \times \mathbb{K}^m : y \in \mathcal{F}(x)\big\}, \tag{2.1}$$

is a linear subspace of $\mathbb{K}^n \times \mathbb{K}^m$ then $\mathcal{F}$ is called a linear multi-valued operator. The domain and the nullspace of $\mathcal{F}$ are denoted, respectively, by $\operatorname{dom}\mathcal{F} = \big\{x \in \mathbb{K}^n : \mathcal{F}(x) \neq \emptyset\big\}$ and $\ker\mathcal{F} = \big\{x \in \operatorname{dom}\mathcal{F} : 0 \in \mathcal{F}(x)\big\}$. By definition, $\mathcal{F}(0)$ is a linear subspace and, for $x \in \operatorname{dom}\mathcal{F}$, we have the following equivalence

$$y \in \mathcal{F}(x) \iff \mathcal{F}(x) = y + \mathcal{F}(0). \tag{2.2}$$

Let $\mathcal{F} : \mathbb{K}^n \rightrightarrows \mathbb{K}^m$ be a multi-valued linear operator, then for given vector norms on $\mathbb{K}^n$ and $\mathbb{K}^m$, the norm of $\mathcal{F}$ is defined by

$$\|\mathcal{F}\| = \sup\Big\{\inf_{y\in\mathcal{F}(x)} \|y\| : x \in \operatorname{dom}\mathcal{F}, \|x\| = 1\Big\}. \tag{2.3}$$

For a linear multi-valued operator $\mathcal{F} : \mathbb{K}^n \rightrightarrows \mathbb{K}^m$, its adjoint operator $\mathcal{F}^* : (\mathbb{K}^m)^* \rightrightarrows (\mathbb{K}^n)^*$ and its inverse operator $\mathcal{F}^{-1} : \operatorname{Im}\mathcal{F} \rightrightarrows \mathbb{K}^n$ are defined, correspondingly, by

$$\mathcal{F}^*(v^*) = \big\{u^* \in (\mathbb{K}^m)^* : u^*x = v^*y \text{ for all } (x, y) \in \operatorname{gr}\mathcal{F}\big\}, \tag{2.4}$$

$$\mathcal{F}^{-1}(y) = \big\{x \in \mathbb{K}^n : y \in \mathcal{F}(x)\big\}. \tag{2.5}$$

Clearly, $\mathcal{F}^*$ and $\mathcal{F}^{-1}$ are also linear multi-valued operators and we have

$$(\mathcal{F}^*)^{-1} = (\mathcal{F}^{-1})^*, \quad \|\mathcal{F}\| = \|\mathcal{F}^*\|. \tag{2.6}$$

It can be proved that $\mathcal{F}$ is surjective (i.e. $\mathcal{F}(\mathbb{K}^n) = \mathbb{K}^m$) if and only if $\mathcal{F}^*$ is injective (i.e. $\mathcal{F}^{*-1}(0) = \{0\}$), or, equivalently, $\mathcal{F}^{*-1}$ is single-valued. Let $\mathcal{F} : \mathbb{K}^n \rightrightarrows \mathbb{K}^m, \mathcal{G} : \mathbb{K}^m \rightrightarrows \mathbb{K}^l$ are the linear multi-valued operators, then the operator $\mathcal{G}\mathcal{F} : \mathbb{K}^n \rightrightarrows \mathbb{K}^l$, defined by $(\mathcal{G}\mathcal{F})(x) = \mathcal{G}(\mathcal{F}(x))$ for all $x \in \operatorname{dom}\mathcal{F}$, is a linear multi-valued operator and if $\operatorname{Im}\mathcal{F} \subset \operatorname{dom}\mathcal{G}$ or $\operatorname{Im}\mathcal{G}^* \subset \operatorname{dom}\mathcal{F}^*$ then

$$(\mathcal{G}\mathcal{F})^* = \mathcal{F}^*\mathcal{G}^* \quad \text{and} \quad \|(\mathcal{G}\mathcal{F})^*\| = \|\mathcal{F}^*\mathcal{G}^*\| \le \|\mathcal{F}^*\|\,\|\mathcal{G}^*\| = \|\mathcal{F}\|\,\|\mathcal{G}\|. \tag{2.7}$$

If $\mathcal{F}$ is the linear single-valued operator defined by $\mathcal{F}(x) = \mathcal{F}_G(x) = Gx$, where $G \in \mathbb{K}^{m\times n}$ and $x \in \mathbb{K}^n$, then, clearly, the norm of $\mathcal{F}_G$ defined by (2.3) is just the operator norm of matrix G:

$$\|\mathcal{F}_G\| = \|G\|.$$

In the sequence, when dealing with this operator in the context of the theory of multi-valued linear operators, we shall use the notation $\mathcal{F}_G(x) = G(x)$. It is easily seen that the adjoint operator $(\mathcal{F}_G)^* : (\mathbb{K}^m)^* \longrightarrow (\mathbb{K}^n)^*$ is also linear single-valued operator which is given by $(\mathcal{F}_G)^*(v^*) = v^*G$, $\forall v^* \in (\mathbb{K}^m)^*$. For the sake of simplicity, we shall identify $(\mathcal{F}_G)^*$ with G^*, that reads

$$(\mathcal{F}_G)^*(v^*) = G^*(v^*) = v^*G, \forall v^* \in (\mathbb{K}^m)^*. \tag{2.8}$$

Remark that the notation G^*v is understood, as usual, the product of matrix $G^* \in \mathbb{K}^{n\times m}$ and column vector $v \in \mathbb{K}^m$ and we have $(G^*v)^* = G^*(v^*)$. Finally, let $P \in \mathbb{K}^{m\times l}, Q \in \mathbb{K}^{l\times n}$ and $\mathcal{F}_{PQ} : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ is the linear single-valued operator defined by $\mathcal{F}_{PQ}(x) = (PQ)x$. Then the adjoint operator $(\mathcal{F}_{PQ})^* : (\mathbb{K}^m)^* \longrightarrow (\mathbb{K}^n)^*$ is also linear single-valued operator and we have, by (2.8), $\forall v^* \in (\mathbb{K}^m)^*$,

$$(PQ)^*(v^*) = (\mathcal{F}_{PQ})^*(v^*) = v^*(PQ) = Q^*(P^*(v^*)) = (Q^*P^*)(v^*) = (\mathcal{F}^*_Q\mathcal{F}^*_P)(v^*),$$

and, by (2.7),

$$\|(PQ)^*\| \le \|Q^*\| \, \|P^*\|.$$

3. Structured radius of surjectivity

Assume that the matrix $W \in \mathbb{K}^{n\times m}$ is *surjective*, i.e. $W\mathbb{K}^m = \mathbb{K}^n$, and is subjected to affine perturbations of the form:

$$W \rightsquigarrow \widetilde{W} = W + D\Delta E. \tag{3.1}$$

Here $D \in \mathbb{K}^{n\times l}, E \in \mathbb{K}^{q\times m}$ are given matrices defining the structure of perturbations, $\Delta \in \mathbb{K}^{l\times q}$ is unknown disturbance matrix.

Definition 3.1. Let $W \in \mathbb{K}^{n\times m}$ be surjective. Given a norm $\|\cdot\|$ on $\mathbb{K}^{l\times q}$, the structured radius of surjectivity of W with respect to affine perturbations of the form (3.1) is defined by

$$r(W; D, E) = \inf\big\{\|\Delta\| : \Delta \in \mathbb{K}^{l\times q} \text{ s.t } \widetilde{W} = W + D\Delta E \text{ non-surjective}\big\}. \tag{3.2}$$

If $W + D\Delta E$ is surjective for all $\Delta \in \mathbb{K}^{l\times q}$ then we set $r(W; D, E) = +\infty$.

Define the multi-valued operator $EW^{-1}D : \mathbb{K}^l \rightrightarrows \mathbb{K}^q$ by setting

$$(EW^{-1}D)(u) = E(W^{-1}(Du)), \quad \forall u \in \mathbb{K}^l,$$

where $W^{-1} : \mathbb{K}^n \rightrightarrows \mathbb{K}^{n+m}$ is the (multi-valued) inverse operator of W.

Theorem 3.2. *Assume that the surjective matrix W is subjected to structured perturbations of the form* (3.1)*. Then the structured radius of surjectivity of W is given by the formula*

$$r(W; D, E) = \frac{1}{\|EW^{-1}D\|}. \tag{3.3}$$

Proof. Since the operator W is surjective W^{*-1} is single-valued. Assume that

$$\widetilde{W} = W + D\Delta E$$

is non-surjective, for some $\Delta \in \mathbb{K}^{l\times q}$. This implies that there exists $y_0^* \in (\mathbb{K}^n)^*, y_0^* \neq 0$ such that $(W + D\Delta E)^*(y_0^*) = W^*(y_0^*) + (E^*\Delta^*D^*)(y_0^*) = 0$. Since $W_{\lambda_0}^{*-1}$ is single-valued, we have

$$y_0^* = -(W^{*-1}E^*\Delta^*)(D^*(y_0^*)) \tag{3.4}$$

and, hence, $D^*(y_0^*) \neq 0$. By applying D^* to the left of the both sides of (3.4), we obtain

$$D^*(y_0^*) = -(D^*W^{*-1}E^*\Delta^*)(D^*(y_0^*)).$$

Therefore,

$$0 < \|D^*(y_0^*)\| \le \|D^*W^{*-1}E^*\| \; \|\Delta^*(D^*(y_0^*))\| \le \|D^*W^{*-1}E^*\| \; \|\Delta^*\| \; \|D^*(y_0^*)\|.$$

Since $\operatorname{Im} W^{-1} \subset \operatorname{dom} E = \mathbb{K}^m$ and $\operatorname{Im}(EW^{-1})^* \subset \operatorname{dom} D^* = (\mathbb{K}^n)^*$, we have, by using (2.7), $(EW^{-1})^* = W^{*-1}E^*$ and

$$(EW^{-1}D)^* = D^*(EW^{-1})^* = D^*W^{*-1}E^*.$$

By (2.6), we get

$$\|\Delta^*\| = \|\Delta\| \geq \frac{1}{\|D^*W^{*-1}E^*\|} = \frac{1}{\|EW^{-1}D\|}.$$

Since the above inequality holds for any disturbance matrix $\Delta \in \mathbb{K}^{l\times q}$ such that $D\Delta E$ destroys surjectivity, we obtain by definition,

$$r(W; D, E) \geq \frac{1}{\|EW^{-1}D\|}. \tag{3.5}$$

To prove the converse inequality, for any small $\epsilon > 0$ such that $\|EW^{-1}D\| - \epsilon > 0$. Since $D^*W^{*-1}E^*$ is single-valued it follows that its norm is the operator norm and hence there exists $v^*_\epsilon \in (\mathbb{K}^q)^* : \|v^*_\epsilon\| = 1, v^*_\epsilon \in \operatorname{dom}(D^*W^{*-1}E^*)$ such that

$$0 < \|EW^{-1}D\| - \epsilon = \|D^*W^{*-1}E^*\| - \epsilon \leq \|(D^*W^{*-1}E^*)(v^*_\epsilon)\| \leq \|D^*W^{*-1}E^*\|. \tag{3.6}$$

Letting $u^*_\epsilon = -W^{*-1}(E^*(v^*_\epsilon)) \neq 0$, we have

$$W^*(u^*_\epsilon) = -E^*(v^*_\epsilon) \text{ and } D^*(u^*_\epsilon) = -(D^*W^{*-1}E^*)(v^*_\epsilon) \neq 0.$$

By Hahn-Banach Theorem, there exists $h \in \mathbb{K}^l$ such that $\|h\| = 1, (D^*(u^*_\epsilon))h = \|D^*(u^*_\epsilon)\|$. Thus, we can define a rank-one perturbation $\Delta_\epsilon \in \mathbb{K}^{l\times q}$ by setting

$$\Delta_\epsilon = \frac{1}{\|D^*(u^*_\epsilon)\|} h v^*_\epsilon.$$

Then, it is obvious that

$$\|\Delta_\epsilon\| = \|D^*(u^*_\epsilon)\|^{-1} = \|(D^*W^{*-1}E^*)(v^*_\epsilon)\|^{-1} \leq \frac{1}{\|EW^{-1}D\| - \epsilon},$$

and, using (2.8), $(\Delta^*_\epsilon D^*)(u^*_\epsilon) = \Delta^*_\epsilon(D^*(u^*_\epsilon)) = D^*(u^*_\epsilon)\Delta_\epsilon = v^*_\epsilon$. Hence, $(E^*\Delta^*_\epsilon D^*)(u^*_\epsilon) = E^*(v^*_\epsilon)$ and, therefore,

$$W^*(u^*_\epsilon) + (E^*\Delta^*_\epsilon D^*)(u^*_\epsilon) = 0,$$

with $u^*_\epsilon \neq 0$, which implies that the perturbed matrix $\widetilde{W} = W + D\Delta_\epsilon E$ is non-surjective. Thus, by definition,

$$r(W; D, E) \leq \|\Delta_\epsilon\| \leq \frac{1}{\|EW^{-1}D\| - \epsilon}. \tag{3.7}$$

Letting $\epsilon \to 0$ we obtain the converse inequality. The proof is complete. □

We note that Theorem 3.2 has been proved for the case when the norms of matrices under consideration are operator norms induced by arbitrary vector norms in corresponding vector spaces. In the case when the vector spaces under consideration are equipped with Euclidean norms (for example, $\|x\| = \sqrt{x^*x}, x \in \mathbb{K}^n$) we can derive from Theorem 3.2 a more computable formula for r. To this end, we need the following technical result (see [19]).

Lemma 3.3. *Assume $G \in \mathbb{K}^{n\times p}$ has full row rank: $\operatorname{rank} G = n$ and $\mathbb{K}^n, \mathbb{K}^p$ are equipped with Euclidean norms. Then, for the linear operator $\mathcal{F}_G(z) = Gz$, we have*

$$d(0, \mathcal{F}_G^{-1}(y)) = \|G^\dagger y\|, \quad \|\mathcal{F}_G^{-1}\| = \|G^\dagger\|, \tag{3.8}$$

where $G^\dagger$ denotes the Moore-Penrose pseudoinverse of $G : G^\dagger = G^*(GG^*)^{-1}$.

Corollary 3.4. *Assume that the matrix $W \in \mathbb{K}^{n\times m}$ is surjective and subjected to perturbations of the form (3.1) where $E \in \mathbb{K}^{q\times(n+m)}$ has full column rank. Then the structured radius of surjectivity of W is given by*

$$r(W; D, E) = \frac{1}{\|(W(E^*E)^{-1/2})^\dagger D\|}. \tag{3.9}$$

Proof. Since E has full column rank, $E^*E \in \mathbb{K}^{(n+m)\times(n+m)}$ is a positive definite matrix. Therefore, we can decompose it as $E^*E = (E^*E)^{1/2}(E^*E)^{1/2}$ with an invertible $(E^*E)^{1/2} \in \mathbb{K}^{(n+m)\times(n+m)}$ (see, e.g. [10]). Since $\|Ev\| = \sqrt{(Ev)^*Ev} = \sqrt{v^*E^*Ev} = \|(E^*E)^{1/2}v\|$, we have, by definition, for all $u \in \mathbb{K}^l$,

(3.10)
$$\begin{aligned} d(0,(EW^{-1}D)(u)) &= \inf_{v\in(W^{-1}D)(u)} \|Ev\| \\ &= \inf_{v\in(W^{-1}D)(u)} \|(E^*E)^{1/2}v\| = d(0,((E^*E)^{1/2}W^{-1}D)(u)). \end{aligned}$$

Since matrix $W(E^*E)^{-1/2} \in \mathbb{K}^{n\times(n+m)}$ has full row rank and, obviously, $(E^*E)^{1/2}W^{-1} = (W(E^*E)^{-1/2})^{-1}$ when they are considered as multi-valued linear operators, we can apply Lemma 3.3 to deduce, for all $u \in \mathbb{K}^l$,

$$d(0,(EW^{-1}D)(u)) = d(0,((E^*E)^{1/2}W^{-1}D)(u))) = \|(W(E^*E)^{-1/2})^\dagger Du\|,$$

which implies that

$$\|EW^{-1}D\| = \|(W(E^*E)^{-1/2})^\dagger D\|. \tag{3.11}$$

The result now follows from Theorem 3.2. □

Now, we consider the more general situation, assuming that the matrix $W \in \mathbb{K}^{n\times m}$ is subjected to structured multi-perturbations of the form

$$W \rightsquigarrow \widetilde{W} = W + \sum_{i=1}^{N} D_i\Delta_i E_i, \tag{3.12}$$

where $D_i \in \mathbb{K}^{n\times l_i}, E_i \in \mathbb{K}^{q_i\times m}, i \in \underline{N} = \{1,\dots,N\}$ are given structure matrices and $\Delta_i \in \mathbb{K}^{l_i\times q_i}, i \in \underline{N}$ are unknown perturbations. The size of each perturbation $\Delta = (\Delta_1,\cdots,\Delta_N) \in \mathcal{D}_{\mathbb{K}} = \Pi_{i=1}^{N}\mathbb{K}^{l_i\times q_i}$ is measured by

$$\|\Delta\| = \sum_{i=1}^{N} \|\Delta_i\|, \tag{3.13}$$

where the norms $\|\Delta_i\|$ are operator norms on $\mathbb{K}^{l_i\times q_i}$ induced by given vector norms on the spaces $\mathbb{K}^{l_i}, \mathbb{K}^{q_i}, i \in \underline{N}$, respectively.

Definition 3.5. The structured radius of surjectivity of W w.r.t multi-perturbations of the form (3.12) is defined by

(3.14)
$$r(W; D_i, E_i, i \in \underline{N}) = \inf\Big\{\|\Delta\| : \Delta \in \mathcal{D}_{\mathbb{K}}, W + \sum_{i=1}^{N} D_i\Delta_i E_i \text{ non-surjective }\Big\},$$

where $\|\Delta\|$ is given by (3.13).

Let $Q \in \mathbb{K}^{n\times m}$ and $P \in \mathbb{K}^{l\times m}$, we shall write

$$P \preceq Q \ \text{ iff } \ \|Px\| \leq \|Qx\|, \quad \text{for all } \ x \in \mathbb{K}^m. \tag{3.15}$$

It is easy to see that if $P \preceq Q$ then $\ker Q \subset \ker P$ and $\operatorname{Im} P^* = (\ker P)^\perp \subset (\ker Q)^\perp = \operatorname{Im} Q^*$. As above, for each $i \in \underline{N}$ we define the multi-valued operator $E_i W^{-1} D_i : \mathbb{K}^{l_i} \rightrightarrows \mathbb{K}^{q_i}$ by setting

$$(E_i W^{-1} D_i)(u_i) = E_i W^{-1}(D_i u_i),\ u_i \in \mathbb{K}^{l_i}.$$

We need the following result which is a simple consequence of the Hahn-Banach Theorem.

Lemma 3.6. *Assume U is a non-trivial subspace of $\mathbb{K}^k$, and $0 \neq \hat{v}_0^* \notin U^\perp$. Then, there exists $0 \neq v_0^* \in \hat{v}_0^* + U^\perp, 0 \neq x_0 \in U$ such that $|v_0^* x_0| = \|v_0^*\|\|x_0\|$.*

The following theorem gives the formula of structured radius of surjectivity of W with respect to multi-perturbations of the form (3.12).

Theorem 3.7. *Let $W \in \mathbb{K}^{n\times m}$ be surjective and $H \in \mathbb{K}^{k\times m}$ be a given matrix. Assume that W is subjected to multi-perturbations of the form (3.12) with $E_i \preceq H$ for all $i \in \underline{N}$. Then*

$$\frac{1}{\max_{i\in\underline{N}} \|HW^{-1}D_i\|} \leq r(W; D_i, E_i, i \in \underline{N}) \leq \frac{1}{\max_{i\in\underline{N}} \|E_i W^{-1} D_i\|}. \tag{3.16}$$

Proof. Assume that the perturbed matrix $\widetilde{W}$ defined as in (3.12) is non-surjective. Then, $y_0^* \in (\mathbb{K}^n)^*, y_0 \neq 0$ such that

$$W^*(y_0^*) + \sum_{i=1}^{N} (E_i^* \Delta_i^* D_i^*)(y_0^*) = 0. \tag{3.17}$$

Letting $u_0^* = W^*(y_0^*)$ we have $y_0^* = W^{*-1}(u_0^*),\ u_0^* \in \operatorname{dom}(W^{*-1})$ and

$$u_0^* = -\sum_{i=1}^{N} (E_i^* \Delta_i^* D_i^* W^{*-1})(u_0^*). \tag{3.18}$$

Since $E_i \preceq H$, $\operatorname{Im} E_i^* \subset \operatorname{Im} H^*$ for all $i \in \underline{N}$. Therefore $u_0^* \in \operatorname{Im} H^*$. Assume $u_0^* = H^* \hat{v}_0^*$. Since $y_0^* \neq 0$, $u_0^* \neq 0$ and $\hat{v}_0^* \notin \ker H^* = (\operatorname{Im} H)^\perp$. By Lemma 3.6, there exists $0 \neq v_0^* \in \hat{v}_0^* + (\operatorname{Im} H)^\perp$ and $0 \neq x_0 \in \operatorname{Im} H$ such that $|v_0^* x_0| = \|v_0^*\|\|x_0\|$. It implies that $u_0^* = H^*(v_0^*)$ and $x_0 = Hz_0$ with some $z_0 \in \mathbb{K}^{n+m}$. We have

$$H^*(v_0^*) = -\sum_{i=1}^{N} (E_i^* \Delta_i^* D_i^* W^{*-1})(H^*(v_0^*)), \tag{3.19}$$

which implies that

$$H^*(v_0^*) z_0 = -\sum_{i=1}^{N} (E_i^* \Delta_i^* D_i^* W^{*-1} H^*)(v_0^*) z_0. \tag{3.20}$$

Therefore, by (2.7) and taking into account the assumption that $E_i \preceq H$ for all $i \in \underline{N}$, we can deduce

$$\|v_0^*\|\|x_0\| = |v_0^* x_0| = |v_0^* H z_0| = |H^*(v_0^*)z_0| \leq \sum_{i=1}^{N} |(E_i^* \Delta_i^* D_i^* W^{*-1} H^*)(v_0^*)z_0|$$
$$= \sum_{i=1}^{N} |(\Delta_i^* D_i^* W^{*-1} H^*)(v_0^*) E_i z_0| \leq \sum_{i=1}^{N} \|(\Delta_i^* D_i^* W^{*-1} H^*)(v_0^*)\| \, \|E_i z_0\|$$
$$\leq \|v_0^*\| \, \|H z_0\| \sum_{i=1}^{N} \|\Delta_i^*\| \, \|D_i^* W^{*-1} H^*\| \leq \|v_0^*\| \, \|x_0\| \, \|\Delta\| \max_{i \in \underline{N}} \|H W^{-1} D_i\|.$$

This implies that

$$\|\Delta\| \geq \frac{1}{\max_{i \in \underline{N}} \|H W^{-1} D_i\|}, \tag{3.21}$$

which yields the first inequality in (3.16). To prove the second inequality, choose $k \in \underline{N}$ and $\epsilon > 0$ such that $\|E_k W^{-1} D_k\| - \epsilon = \max_{i \in \underline{N}} \|E_i W^{-1} D_i\| - \epsilon > 0$. Then, as was shown in the proof of Theorem 3.2, there exists a disturbance matrix $\Delta_{k\epsilon} \in \mathbb{K}^{l_k \times q_k}$ such that $W + D_k \Delta_{k\epsilon} E_k$ is non-surjective and

$$\|\Delta_{k\epsilon}\| \leq \frac{1}{\|E_k W^{-1} D_k\| - \epsilon}.$$

Therefore, for the perturbation $\widetilde{\Delta} = (\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_N)$, with $\widetilde{\Delta}_j = 0, \forall j \neq k,\ \widetilde{\Delta}_k = \Delta_{k\epsilon}$, we see that the perturbed matrix $\widetilde{W} = W + \sum_{i=1}^{N} E_i \widetilde{\Delta}_i D_i$ is non-surjective and

$$\|\widetilde{\Delta}\| = \|\Delta_{k\epsilon}\| \leq \frac{1}{\max_{i \in \underline{N}} \|E_i W^{-1} D_i\| - \epsilon}.$$

This implies that the second inequality in (3.16), completing the proof. □

It is obvious that for any $P \in \mathbb{K}^{n \times m},\ \ P \preceq \|P\| \, I_m$, where I_m is the identity matrix in $\mathbb{K}^{m \times m}$. Using this observation we get the following consequence of Theorems 3.7.

Corollary 3.8. *The radius of surjectivity of matrix W w.r.t complex multi-perturbations of the form* (3.12) *satisfies the inequalities*

$$\frac{1}{\alpha \max_{i \in \underline{N}} \|W^{-1} D_i\|} \leq r(W; D_i, E_i, i \in \underline{N}) \leq \frac{1}{\max_{i \in \underline{N}} \|E_i W^{-1} D_i\|}, \tag{3.22}$$

where $\alpha = [\max_{i \in \underline{N}} \|E_i\|]$.

Now, we give a particular case where the bounds established in the Theorem 3.7 yield a formula for calculating the structured radius of surjectivity.

Corollary 3.9. *If $E_i = \alpha_i E_1, \alpha_i \in \mathbb{K}$, for all $i \in \underline{N}$, then the distance to non-surjectivity of a surjective matrix W w.r.t. complex multi-perturbations of the form (3.12) is given by the formula*

$$r(W; D_i, E_i, i \in \underline{N}) = \frac{1}{\max_{i \in \underline{N}} \|E_i W^{-1} D_i\|}. \tag{3.23}$$

Example 3.10. Consider the surjective matrix $W = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Assume that W is subjected to structured perturbation of the form

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \delta_1 & 1+\delta_1 & 2+\delta_2 \\ 1+\delta_1 & \delta_1 & \delta_2 \end{bmatrix}, \tag{3.24}$$

where $\delta_i \in \mathbb{C}, i \in \overline{1,2}$ are disturbance parameters. The above perturbed model can be represented in the form $W \rightsquigarrow W + D\Delta E$ with $D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We have, for each $v \in \mathbb{C}$,

$$EW^{-1}D(v) = EW^{-1}\begin{pmatrix} v \\ v \end{pmatrix}$$

$$= \left\{ E \begin{pmatrix} p \\ q \\ r \end{pmatrix} : q + 2r = p = v \right\} = \left\{ \begin{pmatrix} v+q \\ v/2 - q/2 \end{pmatrix} : q \in \mathbb{C} \right\}.$$

Thus, for each $v \in \mathbb{C}$, the problem of computing $d(0, EW^{-1}D(v))$ is reduced to the calculation of the distance from the origin to the straight line in $\mathbb{C}^2$ whose equation can be rewritten in the form $x_1 + 2x_2 = 2v$ with $x_1 = v + q$ and $x_2 = v/2 - q/2$. Let $\mathbb{C}^2$ be endowed with the vector norms $\|\cdot\|_\infty$, then we can deduce,

$$2|v| \le |x_1| + 2|x_2| \le 3\max\{|x_1|, |x_2|\} = 3\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_\infty.$$

This implies that

$$\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_\infty \ge \frac{2|v|}{3},$$

which yields the equality if $x_1 = x_2 = \frac{2v}{3}$. Therefore, $\|EW^{-1}D\| = \frac{2}{3}$. By applying Theorem 3.2, we obtain $r(W; D, E) = \frac{3}{2}$.

4. Complex stabilizability radius

In this section, the results of the previous section will be used to obtain the formula for stabilizability radius of a linear system. Consider the system (A, B):

$$\begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0, t \ge 0, \end{cases} \tag{4.1}$$

where $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}$. Recall that the system (4.1) is said to be *stabilizable* if there exists a linear feedback control $u = Kx$ with $K \in \mathbb{C}^{m\times n}$ such that the closed-loop system $\dot{x} = (A+BK)x$ is asymptotically stable, or equivalently $\sigma(A+BK) := \{s \in \mathbb{C} : \det(Is - A - BK) = 0\} \subset \mathbb{C}^- := \{s \in \mathbb{C} : \text{Re } s < 0\}$. Assume that system (4.1) is subjected to structured perturbations of the form

$$[A, B] \rightsquigarrow [\widetilde{A}, \widetilde{B}] = [A, B] + D\Delta E, \tag{4.2}$$

where $D \in \mathbb{C}^{n\times l}, E \in \mathbb{C}^{q\times(n+m)}$ are given structuring matrices and $\Delta \in \mathbb{K}^{l\times q}$ is unknown disturbance matrix. Then, for a stabilizable system (A, B), we can define the notion of *complex stabilizability radius* as follows

$$\Lambda_{\mathbb{C}}^{D,E}(A,B) = \inf\{\|\Delta\| : \Delta \in \mathbb{C}^{l\times q},\ [A,B] + D\Delta E \ \text{ is not stabilizable}\}. \tag{4.3}$$

By using the formulas for the radius of surjectivity given by Theorem 3.2 and Theorem 3.7, we derive now formulas of structured stabilizability radius for arbitrary operator norms. Define, for each $\lambda \in \mathbb{C}$, the linear single-valued operator $W_\lambda : \mathbb{C}^{n+m} \to \mathbb{C}^n$ by setting $W_\lambda z = [A - \lambda I, B]z,\ \forall z \in \mathbb{C}^{n+m}$, and the multi-valued operator $EW_\lambda^{-1}D : \mathbb{C}^l \rightrightarrows \mathbb{C}^q$ by setting

$$(EW_\lambda^{-1}D)(u) = E(W_\lambda^{-1}(Du)), \quad \forall u \in \mathbb{C}^l,$$

where $W_\lambda^{-1} : \mathbb{C}^n \rightrightarrows \mathbb{C}^{n+m}$ is the (multi-valued) inverse operator of W_λ. By Hautus Theorem (see e.g. [8]), the system (A, B) is stabilizable iff W_λ is surjective for all $\lambda \in \bar{\mathbb{C}}_+$, where $\bar{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\}$. Therefore, by using Theorem 3.2, we get

Theorem 4.1. *Assume that system* (4.1) *is stabilizable and subjected to structured perturbations of the form* (4.2). *Then the complex stabilizability radius of* (4.1) *is given by the formula*

$$\Lambda_{\mathbb{C}}^{D,E}(A,B) = \frac{1}{\sup_{\lambda\in\bar{\mathbb{C}}_+} \|EW_\lambda^{-1}D\|}. \tag{4.4}$$

The formula (4.4) looks very similar to the well-known formula of complex stability radius of a asymptotically stable linear system $\dot{x} = Ax, t \geq 0$ where the matrix A is subjected to structured perturbations $A \rightsquigarrow A + D\Delta E$ (see, e.g. [9]). We note that the study of stability radius has atracted much attention over last two decades and algorithms for computing the complex stability radius has been developed by several authors (see, e.g. [6] and [7]). The formula (4.4), involves, however, calculation of the norm of the linear multi-valued operator $EW_\lambda^{-1}D$ which does not have an explicit representation. We now derive from this result some more computable formulas, particularly in the case, where the matrix norm is the spectral norm (i.e. the operator norm induced by Euclidean vector norms of the form $\|x\| = \sqrt{x^*x}$).

First, we note that if vector spaces $\mathbb{C}^n, \mathbb{C}^m$ are equipped with Euclidean norms then for any rectangular matrix $W \in \mathbb{C}^{n\times m}$ its operator norm is the spectral norm and we have $\|W\| = \sigma_{\min}[W] = \|W^\dagger\|^{-1}$, where $W^\dagger = W^*(WW^*)^{-1}$ is the Moore-Penrose pseudoinverse of W. Therefore, from Theorem 4.1, we get the following formula for unstructured stabilizability radius of a pair (A, B) which is obviously similar to the Eising result (see [5]).

Corollary 4.2. *Let stabilizable system (A, B) be subjected to unstructured complex perturbations of the form*

$$[A,B] \rightsquigarrow [A,B] + \Delta,\ \Delta = [\Delta_1, \Delta_2] \in \mathbb{C}^{n\times m},$$

and the norm of disturbance matrices Δ in (4.3) is spectral norm, then the system's complex stabilizability radius is given by

$$\Lambda_{\mathbb{C}}(A,B) = \frac{1}{\sup_{\lambda\in\bar{\mathbb{C}}_+} \|W_\lambda^\dagger\|}.$$

By applying Corollary 3.4 we get

Corollary 4.3. *Let the matrix E has full column rank. Then the complex stabilizability radius of (A,B) w.r.t. matrix spectral norm and structured perturbations of the form (4.2) is given by*

$$\Lambda_{\mathbb{C}}^{D,E}(A,B) = \frac{1}{\sup_{\lambda\in\bar{\mathbb{C}}_+}\|(W_\lambda(E^*E)^{-1/2})^\dagger D\|}. \tag{4.5}$$

We now assume that the matrix pair $(A,B) \in \mathbb{C}^{n\times n}\times\mathbb{C}^{n\times m}$ is subjected to structured multi-perturbations of the form

$$[A,B] \rightsquigarrow [\widetilde{A},\widetilde{B}] = [A,B] + \sum_{i=1}^{N} D_i\Delta_i E_i, \tag{4.6}$$

where $D_i \in \mathbb{C}^{n\times l_i}, E_i \in \mathbb{C}^{q_i\times(n+m)}, i \in \underline{N} = \{1,\dots,N\}$ are given structure matrices and $\Delta_i \in \mathbb{C}^{l_i\times q_i}, i \in \underline{N}$ are unknown perturbations. The size of each perturbation $\Delta = (\Delta_1,\dots,\Delta_N) \in \mathcal{D}_{\mathbb{C}} = \Pi_{i=1}^{N}\mathbb{C}^{l_i\times q_i}$ is measureded by :

$$\|\Delta\| = \sum_{i=1}^{N}\|\Delta_i\|, \tag{4.7}$$

where the norms $\|\Delta_i\|$ are operator norms on $\mathbb{C}^{l_i\times q_i}$ induced by given vector norms on the spaces $\mathbb{C}^{l_i}, \mathbb{C}^{q_i}, i \in \underline{N}$, respectively. Then the complex stabilizability radius of (A,B) under multi-perturbations of the form (4.6) is defined by

$$\Lambda_{\mathbb{C}}^{mp}(A,B) = \inf\big\{\|\Delta\| : \Delta\in\mathcal{D}_{\mathbb{C}}, [A,B] + \sum_{i=1}^{N} D_i\Delta_i E_i \text{ unstabilizable }\big\}. \tag{4.8}$$

By using Theorem 3.7, we get

Theorem 4.4. *Let $H \in \mathbb{C}^{k\times(n+m)}$ be a given matrix. Assume the matrix pair (A,B) is stabilizable and subjected to multi-perturbations of the form (3.12) with $E_i \preceq H$ for all $i \in \underline{N}$. Then, the complex stabilizability radius of (A,B) satisfies the inequality*

$$\frac{1}{\max_{i\in\underline{N}}\sup_{\lambda\in\bar{\mathbb{C}}_+}\|HW_\lambda^{-1}D_i\|} \le \Lambda_{\mathbb{C}}^{mp}(A,B) \le \frac{1}{\max_{i\in\underline{N}}\sup_{\lambda\in\bar{\mathbb{C}}_+}\|E_iW_\lambda^{-1}D_i\|}, \tag{4.9}$$

where W_λ denotes $[A-\lambda I, B]$.

We illustrate the above result by an example.

Example 4.5. Consider the linear control system (A,B) described by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{4.10}$$

where $A = \begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}, B = \begin{bmatrix}0\\1\end{bmatrix}$. By Hautus characterization, the system is stabilizable. Assume that, the control matrix $[A,B]$ is subjected to structured perturbation of the form

$$\begin{bmatrix}-1 & 1 & 0\\ 0 & -2 & 1\end{bmatrix} \rightsquigarrow \begin{bmatrix}-1+\delta_1 & 1+\delta_1 & \delta_2\\ 0+\delta_1 & -2+\delta_1 & 1+\delta_2\end{bmatrix}, \tag{4.11}$$

where $\delta_i \in \mathbb{C}, i \in \overline{1,2}$ are disturbance parameters. The above perturbed model can be represented in the form $[A,B] \rightsquigarrow [A,B] + D\Delta E$ with $D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
We have

$$\begin{aligned} E[A-\lambda I, B]^{-1}D(v) &= E[A-\lambda I, B]^{-1}\begin{pmatrix} v \\ v \end{pmatrix} \\ &= \left\{ E\begin{pmatrix} p \\ q \\ r \end{pmatrix} : -(\lambda+1)p + q = -(\lambda+2)q + r = v \right\} \\ &= \left\{ \begin{pmatrix} v + (\lambda+2)p \\ (\lambda+3)v + (\lambda+1)(\lambda+2)p \end{pmatrix} : q \in \mathbb{C} \right\}. \end{aligned}$$

Thus, for each $v \in \mathbb{C}$, the problem of computing $d(0, E[A-\lambda I, B]^{-1}D(v))$ is reduced to the calculation of the distance from the origin to the straight line in $\mathbb{C}^2$ whose equation can be rewritten in the form $x_2 - (\lambda+1)x_1 = 2v$ with $x_1 = v + (\lambda+2)p, x_2 = (\lambda+3)v + (\lambda+1)(\lambda+2)p$. Note that if $\lambda = -2$ then this line is reduced to the point $\begin{pmatrix} v \\ v \end{pmatrix}$. Assume $\lambda \neq -2$ and let $\mathbb{C}^2$ be endowed with the vector norms $\|\cdot\|_\infty$, then we can deduce,

$$2|v| \leq |x_2| + |\lambda+1||x_1| \leq (1 + |\lambda+1|)\max\{|x_1|, |x_2|\} = (1 + |\lambda+1|)\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_\infty.$$

This implies that

$$\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\|_\infty \geq \frac{2|v|}{1 + |\lambda+1|},$$

which yields the equality if $x_2 = \dfrac{2v}{1+|\lambda+1|}$ and $x_1 = e^{i\varphi}x_2$, where φ is chosen such that $-(\lambda+1)e^{i\varphi} = |\lambda+1|$. Therefore,

$$\|E[A-\lambda I, B]^{-1}D\| = \sup_{|v|=1} d\big(0, E[A-\lambda I, B]^{-1}D(v)\big) = \begin{cases} \dfrac{2}{|\lambda+1|+1} & \text{if } \lambda \neq -2, \\ 1 & \text{if } \lambda = -2, \end{cases}$$

By applying Theorem 4.1, we obtain the complex stabilizability radius $\Lambda_{\mathbb{C}}^{D,E}(A,B) = 1$.

5. Conclusion

In this paper we developed a unifying approach to the problem of calculating the radius of surjectivity of a surjective rectangular matrix, which is based on the theory of linear multi-valued operators. Our result generalized, in particular, the classical Eckart-Young Theorem to structured perturbations. We applied the obtained results to establish some formulas for the stabilizability radius of linear control systems under structured perturbations and multi-perturbations of system matrices. Our approach can be developed further for calculating the distance from ill-posedness of conic systems of the form $Ax = b, x \in K \subset \mathbb{K}^m$, where K is a closed convex cone, as well as for stability radius of convex processes $\dot{x} \in \mathcal{F}(x), t \geq 0$. These problems are the topics of our further study.

References

[1] R. Cross, *Multi-valued Linear Operators*, Marcel Dekker, New York, 1998.
[2] A. L. Donchev, A. S. Lewis and R. T. Rockafellar, *The radius of metric regularity*, Transactions of Amer. Mathematical Society **355** (2003), 493–517
[3] A. L. Dontchev and R. T. Rockafellar, *Implicit Functions and Solution Mappings*, Monographs in Mathematics, Springer-Verlag, Berlin, 2009.
[4] C. Eckart and G. Young, *The approximation of one matrix by another of lower rank*, Psychometrika **1** (1936), 211–218.
[5] R. Eising, *Between controllable and uncontrollable*, Systems & Control Letters **5** (1984), 263–264.
[6] M. Gu, *New methods for estimating the distance to uncontrollability*, SIAM J. Matrix Anal. Appl. **21** (2000), 989–1003.
[7] E. Mengi, *On the estimation of the distance to uncontrollability for higher order systems*, SIAM J. Matrix Anal. Appl. **30** (2006), 154–172.
[8] M. L. J. Hautus, *Controllability and observability conditions of linear autonomous systems*, Nederl. Acad. Wetensch. Proc. Ser. A72 **31** (1969), 443–448.
[9] D. Hinrichsen and A. J. Pritchard, *Stability radius for structured perturbations and the algebraic Riccati equation*, Systems Control Lett. **8** (1986), 105–113.
[10] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, London, 1985.
[11] M. Karow and D. Kressner, *On the structured distance to uncontrollability*, Systems & Control Letters **58** (2009), 128–132.
[12] A. Lewis, *Ill-conditioned convex processes and linear inequalities*, Mathematics on Operations Research **24** (1999), 829–834.
[13] A. Lewis, *Ill-conditioned inclusions*, Set-Valued Analysis **9** (2001), 375–381.
[14] A. Lewis, R. Henrion and A. Seerger, *Distance to uncontrollability for convex processes*, SIAM J. Control Optim. **45** (2006), 26–50.
[15] J. Pena, *A characterization of the distance to infeasibility under block-structured perturbations*, Linear Algebra and its Applications **370** (2003), 193–216.
[16] J. Pena, *On the block-structured distance to non-surjectivity of sublinear mappings*, Mathematical Programming, Ser. A **103** (2005), 561–573.
[17] J. Renegar, *Some perturbation theory for linear programming*, Math. Programming **65** (1994), 73–91.
[18] J. Renegar, *Linear programming, complexity theory and elementary functional analysis*, Mathematical Programming **70** (1995), 279–351.
[19] N. K. Son and D. D. Thuan, *Structured distance to uncontrollability under multi-perturbations: an approach using multi-valued linear operators*, Systems & Control Letters **59** (2010), 476–483.

Manuscript received January 10, 2011
revised March 17, 2011

Nguyen Khoa Son
Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet Rd., 10307 Hanoi, Vietnam
E-mail address: `nkson@vast.ac.vn`

Do Duc Thuan
Department of Applied Mathematics and Informatics, Hanoi University of Technology, 1 Dai Co Viet Str., Hanoi, Vietnam
E-mail address: `ducthuank7@gmail.com`