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CHARACTERIZATIONS OF SOLUTION SETS OF A CLASS OF NONCONVEX SEMI-INFINITE PROGRAMMING PROBLEMS

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Dedicated to Professor Pham Huu Sach on the occasion of his 70th birthday

Abstract. In this paper we deal with characterizations of solution sets of a class nonconvex problems with an infinite number of constraints. Assuming the functions to be locally Lipschitz semiconvex functions, several types of characterizations of solution sets of the problems are given.

1. Introduction

The problem of establishing characterizations of solutions sets of optimization programs is a topic which has attracted attention of researchers for years. There were several results on characterizations of solutions sets of convex/nonconvex problems published [1], [6], [9], [10], [11], [13], [16], [18], and [20]. The motivation for this article came from [9] and [20], where several types of characterizations of solution sets of convex problems were established via Lagrange multipliers or minimizing sequences. Our aim is to extend these results to establish characterizations of solution sets of a class of nonconvex problems which have an infinite number of constraints. We are interested in the following problem:

$$\text{(P)}\quad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & f_t(x) \leq 0,\ t \in T, \\ & x \in C, \end{array}$$

where C is a closed convex subset of $\mathbb{R}^n$, $f, f_t : \mathbb{R}^n \to \mathbb{R}, t \in T$, are locally Lipschitz functions on an open subset containing C, T is a compact index set (possibly infinite). Problems in this form have been considered recently in several papers with various requirements on $f,\ f_t, t \in T$, and the setting space (see, e.g., [4], [5], [14], and [19]-[22]).

In this paper, characterizations of solution set of (P) will be given in several types. The technique used in this paper is based on theory of Nonsmooth Analysis introduced in the two well known books of Clarke [2], [3]. Most of our results in this paper are established by using a property of semiconvexity applied to involved functions.

We now describe the content of the paper. In the preliminaries, we recall basic concepts and results on semiconvex functions. In order to give characterizations

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of solution set of (P) via Lagrange multipliers, we firstly prove that the Lagrange function associated to (P) with a fixed multiplier corresponding to a given solution is constant on the solution set of the problem (P). Then, a theorem characterizing its solution set is given. In this part, we also give some corollaries on characterizations of solution set of (P) with relaxed assumptions applied to constraint functions. As a particular case, we recover results on characterizations of solution sets of a class of convex programs. For the second type, firstly we consider a point-to-set mapping defined on $\mathbb{R}^n$ by

$$\partial^c L(\cdot,\lambda)\cap(-N(C,\cdot)) : x \mapsto \partial^c L(\cdot,\lambda)(x)\cap(-N(C,x),$$

where $\partial^c L(\cdot,\lambda)(x)$ denotes Clarke-subdifferential at x of the Lagrange function L associated to (P) with a fixed Lagrange multiplier, and $N(C,x)$ is the normal cone to a nonempty closed convex subset C at $x \in C$. Then we prove that this mapping is constant on the solution set of (P). Based on this result, a characterization of solution set of (P) via subgradients is established. The last part of the paper is devoted to minimizing sequence characterizations of solution set of (P). Unlike characterizations mentioned above, where at least one solution of the problem under consideration is known, here a minimizing sequence of (P) is used instead. This is often the case when we do not know an exact solution but a minimizing sequence can be obtained by some numerical method.

The paper is organized as follows. Section 2 is devoted to preliminaries and the properties of semiconvex functions. Our main results are established in the third section (the last section of the paper). Several theorems concerning the types of characterizations of solution set of (P) are given. Besides, applications of the obtained results to some special cases are presented. Examples are given to illustrate certain results.

2. Preliminaries

Let us denote by $\mathbb{R}^{(T)}$ a following linear space:

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t\in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

For $\lambda \in \mathbb{R}^{(T)}$, its supporting set, $\mathrm{supp}\lambda := \{t \in T \mid \lambda_t \neq 0\}$, is a finite subset of T. The nonnegative cone of $\mathbb{R}^{(T)}$ is denoted by

$$\mathbb{R}^{(T)}_+ := \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0, t \in T\}.$$

It is easy to see that $\mathbb{R}^{(T)}_+$ is a convex cone of $\mathbb{R}^{(T)}$.

For $z = (z_t)_t \subset Z$, Z being a linear space, we understand that

$$\langle \lambda, z\rangle := \sum_{t\in T} \lambda_t z_t = \sum_{\mathrm{t\in supp}\lambda} \lambda_t z_t$$

and

$$\sum_{t\in T} \lambda_t f_t := \sum_{\mathrm{t\in supp}\lambda} \lambda_t f_t$$

where $f_t : \mathbb{R}^n \to \mathbb{R}, t \in T$. If $\{\mathrm{Y_t} \mid \mathrm{t} \in \mathrm{supp}\lambda\}$ is a class of non-empty sets of some linear space, we define

$$\sum_{t\in T} \lambda_t Y_t := \sum_{\mathrm{t}\in\mathrm{supp}\lambda} \lambda_t Y_t.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function at $x \in \mathbb{R}^n$. The *generalized directional derivative* of f at x in the direction $d \in \mathbb{R}^n$ is defined by (see [2])

$$f^c(x; d) := \limsup_{\substack{h\to 0\\ t\downarrow 0}} \frac{f(x + h + td) - f(x + h)}{t}$$

and the *Clarke's subdifferential* of f at x, denoted by $\partial^c f(x)$, is

$$\partial^c f(x) := \{u \in \mathbb{R}^n \mid u(d) \leq f^c(x; d), \forall d \in \mathbb{R}^n\}.$$

For $d \in \mathbb{R}^n$, the *directional derivative* of f at x in the direction d, denoted by $f'(x; d)$, is defined by the following limit (if it exists)

$$\lim_{t\downarrow 0} \frac{f(x + td) - f(x)}{t}.$$

The function f is said to be *quasidifferentiable* or *regular* (in the sense of Clarke) at x if $f'(x; d)$ exists and equals to $f^c(x; d)$ for every $d \in \mathbb{R}^n$ (see [2], [3]).

Let D be a closed convex subset of $\mathbb{R}^n$. The normal cone to D at x is

$$N(D, x) = \{u \in \mathbb{R}^n \mid u(y - x) \leq 0, \forall y \in D\}.$$

Definition 2.1 ([17, Definition 2]). Let Ω be a nonempty subset of $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *semiconvex at* $x \in \Omega$ if f is locally Lipschitz at x, regular at x, and

$$\big(x + d \in \Omega, d \in \mathbb{R}^n, f'(x; d) \geq 0\big) \Longrightarrow f(x + d) \geq f(x).$$

The function f is said to be *semiconvex on* Ω if f is semiconvex at every $x \in \Omega$.

From the definition above, we can easily verify that

$$(f \text{ is semiconvex at } x, \exists u \in \partial^c f(x) : u(z - x) \geq 0) \Rightarrow (f(z) \geq f(x)). \tag{2.1}$$

Lemma 2.2. *Suppose that f is semiconvex on a convex set $C \subset \mathbb{R}^n$. Then for $x \in C$, $d \in \mathbb{R}^n$ with $x + d \in C$,*

$$f(x + d) \leq f(x) \Longrightarrow f'(x; d) \leq 0.$$

The lemma above was presented in [17, Theorem 8] with C is a convex subset of $\mathbb{R}^n$ and in [20, Lemma 4.1] with C is a convex subset of a Banach space. We note that the notion of semiconvexity presented above was used in several papers such as [8], [12], [20], [21] (an extension of this notion called ϵ-semiconvexity is proposed in [15]). When a semiconvex function is differentiable, it is called "pseudoconvex" (see [7], [16], [23]).

It is well known that if $f : \mathbb{R}^n \to \mathbb{R}$ is a pseudoconvex function then the lower level set of f, $\{x \in C \mid f(x) \leq \alpha, \alpha \in \mathbb{R}\}$, is a convex subset of $\mathbb{R}^n$, where C is a convex subset of $\mathbb{R}^n$. When f is semiconvex on $\mathbb{R}^n$ and C is also convex, it is easy to verify that f is quasiconvex on C. Consequently, the level set above is a convex subset of $\mathbb{R}^n$ (for details, see [7], p. 119). For Problem (P), if $f_t, t \in T$, are semiconvex on C then for every $t \in T$, $\{x \in C \mid f_t(x) \leq 0\}$ is convex. Hence, the feasible set of (P) is convex (it is also closed).

3. Characterizations of solution sets

Let us denote by A the feasible set of (P). The solution set of (P) is

$$S := \{z \in A \mid f(z) \leq f(x), \forall x \in A\}.$$

To derive characterizations of solution set S of (P), we suppose that $S \neq \emptyset$. In this paper, z is a given solution of (P). We also assume that under some condition there exists $\lambda \in \mathbb{R}_+^{(T)}$ (Lagrange multiplier) such that the following optimality condition holds (for details, see [21]).

$$0 \in \partial^c f(z) + \sum_{t \in T} \lambda_t \partial^c f_t(z) + N(C, z), \lambda_t f_t(z) = 0, \ \forall t \in T. \tag{3.1}$$

In this case, we denote the supporting set of the λ by T_+. Frequently, the Lagrange function $L : \mathbb{R}^n \times \mathbb{R}_+^{(T)} \to \mathbb{R}$ of (P) is defined by

$$L(x, \lambda) = f(x) + \sum_{t \in T} \lambda_t f_t(x).$$

3.1. **Lagrange multiplier characterizations.** To obtain characterizations of solution sets of (P) via Lagrange multipliers, we need the following lemma.

Lemma 3.1. *For Problem (P), let z be a given solution such that the condition* (3.1) *holds. If the functions f and $f_t, t \in T$, are regular at z and the function $L(\cdot, \lambda)$ is semiconvex at z, then $L(\cdot, \lambda)$ is constant on S. Moreover, for all $y \in S$, $f_t(y) = 0$ for all $t \in T_+$.*

Proof. Suppose that $z \in S$ and there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that (3.1) holds. Then there exist $u \in \partial^c f(z), v_t \in \partial^c f_t(z), t \in T, w \in N(C, z)$ and $\lambda_t f_t(z) = 0$ for all $t \in T$ such that

$$u + \sum_{t \in T} \lambda_t v_t = -w.$$

Since C is a closed convex subset of X, $w(y - z) \leq 0$ for all $y \in C$. Hence,

$$(u + \sum_{t \in T} \lambda_t v_t)(y - z) \geq 0, \forall y \in C. \tag{3.2}$$

On the other hand, since $L(\cdot, \lambda)$ is regular at z, we have

$$(f + \sum_{t \in T_+} \lambda_t f_t)^c(z; y - z) = (f + \sum_{t \in T_+} \lambda_t f_t)'(z; y - z). \tag{3.3}$$

Moreover, by the regularity of $f, f_t, t \in T$ at z, from (3.2) and (3.3) we deduce

$$(f + \sum_{t \in T_+} \lambda_t f_t)'(z; y - z) \geq 0.$$

Based on the semiconvexity property of $L(\cdot,\lambda)$ at z, we get

$$f(y)+\sum_{t\in T_+}\lambda_t f_t(y)\geq f(z)+\sum_{t\in T_+}\lambda_t f_t(z),\forall y\in C.$$

Since $\lambda_t f_t(z)=0$ for all $t\in T$,

$$f(y)+\sum_{t\in T_+}\lambda_t f_t(y)\geq f(z),\forall y\in C.$$

When $y\in S$, we get $y\in A$ (i.e., $f_t(y)\leq 0$ for all $t\in T$) and $f(y)=f(z)$. Hence,

$$f(z)=f(y)\geq f(y)+\sum_{t\in T_+}\lambda_t f_t(y)\geq f(z).$$

It follows that $\sum_{t\in T_+}\lambda_t f_t(y)=0$, i.e., $f_t(y)=0$ for all $t\in T_+$. We see that $L(\cdot,\lambda)$ is constant on S. □

Theorem 3.2. *For Problem (P), let z be a given solution such that the condition* (3.1) *holds. Suppose that f is semiconvex on C and $L(\cdot,\lambda)$ is semiconvex at z. Suppose further that the functions $f_t, t\in T$, are regular at z and A is convex. Then $S=S_1=\bar{S}_1$ where*

$$\begin{aligned} S_1 &= \{x\in C \mid \exists u\in\partial^c f(z)\cap\partial^c f(x), u(z-x)=0, f_t(x)=0\ \forall t\in T_+ \\ &\quad \textit{and } f_t(x)\leq 0\ \forall t\in T\setminus T_+\}, \\ \bar{S}_1 &= \{x\in C \mid \exists u\in\partial^c f(x), u(z-x)=0, f_t(x)=0\ \forall t\in T_+ \\ &\quad \textit{and } f_t(x)\leq 0\ \forall t\in T\setminus T_+\}. \end{aligned}$$

Proof. It is obvious that $S_1\subset\bar{S}_1$. So, it needs to prove that $S\subset S_1$ and $\bar{S}_1\subset S$. Firstly we prove that $\bar{S}_1\subset S$. Let $x\in\bar{S}_1$. Then there exists $u\in\partial^c f(x)$, such that $u(z-x)=0$, $f_t(x)=0$ for all $t\in T_+$, and $f_t(x)\leq 0$ for all $t\in T\setminus T_+$. From $u\in\partial^c f(x)$, $u(z-x)=0$, and $z\in C$, since f is semiconvex on C, $f(z)\geq f(x)$. Furthermore, since $x,z\in A$ and z is a solution of (P), $x\in S$.

We now prove that $S\subset S_1$. Let $x\in S$. By Lemma 3.1, we get $f_t(x)=0$ for all $t\in T_+$ and $f_t(x)\leq 0$ for all $t\in T\setminus T_+$. Since z satisfies condition (3.1) with $\lambda\in\mathbb{R}_+^{(T)}$, there exist $u\in\partial^c f(z), v_t\in\partial^c f_t(z)$, $t\in T$, $w\in N(C,z)$ and $\lambda_t f_t(z)=0$ for all $t\in T$ such that

$$u+\sum_{t\in T}\lambda_t v_t=-w.$$

Since C is a closed convex subset of X, $w(x-z)\leq 0$ for all $x\in C$. Hence, for $x\in S\subset C$, we get

$$(u+\sum_{t\in T}\lambda_t v_t)(x-z)\geq 0,$$

i.e.,

$$u(x-z)\geq-\sum_{t\in T}\lambda_t v_t(x-z)=-\sum_{t\in T_+}\lambda_t v_t(x-z). \tag{3.4}$$

Since $\lambda_t f_t(z)=0$ for all $t\in T$,

$$(\lambda_t f_t)'(z;x-z)=\lim_{\theta\downarrow 0}\frac{\lambda_t f_t(z+\theta(x-z))-\lambda_t f_t(z)}{\theta}=\lim_{\theta\downarrow 0}\frac{\lambda_t f_t(z+\theta(x-z))}{\theta}. \tag{3.5}$$

Since A is a convex subset of X, we get $z+\theta(x-z) \in A$ when $x, z \in A$ and $\theta \in (0,1)$. Hence, $\lambda_t f_t(z+\theta(x-z)) \leq 0$ for all $t \in T$ when θ is small enough. From this and (3.5), we obtain $(\lambda_t f_t)^{'}(z; x-z) \leq 0$, $t \in T$. It is obvious that $\lambda_t f_t, t \in T$, are regular at z, i.e., $(\lambda_t f_t)^{'}(z; x-z) = (\lambda_t f_t)^c(z; x-z)$. It follows that $\lambda_t v_t(x-z) \leq 0$ for all $t \in T$ if $v_t \in \partial^c f_t(z)$. These and (3.4) imply that $u(x-z) \geq 0$. On the other hand, since $f(x) = f(z)$ and f is semiconvex at z, by Lemma 2.2, $f^{'}(z; x-z) \leq 0$. Hence,

$$u(x-z) \leq f^c(z; x-z) = f^{'}(z; x-z) \leq 0,$$

where $u \in \partial f^c(z)$. Thus, $u(x-z) = 0$.

To complete the proof, we need to prove that $u \in \partial f(z) \cap \partial f(x)$. Since $u \in \partial^c f(z)$, it remains to show that $u \in \partial^c f(x)$. By the regularity of f at z, also at x, we have $f'(x; d) = f^c(x; d)$ and $f'(z; d) = f^c(z; d)$ for all $d \in \mathbb{R}^n$. We claim that there does not exist any $d_0 \in \mathbb{R}^n$ such that $f'(x; d_0) < f'(z; d_0)$. Indeed, suppose to contrary that there exists $d_0 \in \mathbb{R}^n$ such that $f'(x; d_0) < f'(z; d_0)$, i.e.,

$$\lim_{t_1 \downarrow 0} \frac{f(x+t_1 d_0) - f(x)}{t_1} - \lim_{t_2 \downarrow 0} \frac{f(z+t_2 d_0) - f(z)}{t_2} < 0.$$

Then,

$$\lim_{t \downarrow 0} \frac{f(x+td_0) - f(x)}{t} - \frac{f(z+td_0) - f(z)}{t} < 0,$$

Since $f(x) = f(z)$, we get

$$\lim_{t \downarrow 0} \frac{f(x+td_0) - f(z+td_0)}{t} < 0.$$

Thus, there exists $t_0 \in (0,1)$ and $\alpha > 0$ small enough such that

$$f(x+td_0) - f(z+td_0) < -\alpha < 0, \forall t \in (0, t_0). \tag{3.6}$$

It is easy to see that $h(t) := f(x+td_0) - f(z+td_0)$ is continuous at $t = 0$. In the inequality above, by letting $t \to 0$, we get $f(x) - f(z) < 0$, a contradiction. So, if $u(d) \leq f'(z; d) = f^c(z; d)$ for all $d \in \mathbb{R}^n$ then

$$u(d) \leq f'(x; d) = f^c(x; d), \forall d \in \mathbb{R}^n.$$

This shows that $u \in \partial^c f(z)$ implies $u \in \partial^c f(x)$. We obtain $u \in \partial^c f(z) \cap \partial^c f(x)$. Hence, $x \in S_1$. The proof is complete. $\square$

As we discussed in the last part of Section 2, if $f_t, t \in T$, are semiconvex on C, then the feasible set of (P) is convex. We obtain a corollary from Theorem 3.2 with the proof omitted.

Corollary 3.3. *For Problem (P), let z be a given solution such that the condition* (3.1) *holds. If the functions $f, f_t, t \in T$, are semiconvex on C and if the function $L(\cdot, \lambda)$ is semiconvex at z, then $S = S_1 = \bar{S}_1$ where*

$$\begin{aligned} S_1 &= \{x \in C \mid \exists u \in \partial^c f(z) \cap \partial^c f(x), u(z-x) = 0, f_t(x) = 0 \ \forall t \in T_+ \\ &\qquad \textit{and } f_t(x) \leq 0 \ \forall t \in T \setminus T_+\}, \\ \bar{S}_1 &= \{x \in C \mid \exists u \in \partial^c f(x), u(z-x) = 0, f_t(x) = 0 \ \forall t \in T_+ \\ &\qquad \textit{and } f_t(x) \leq 0 \ \forall t \in T \setminus T_+\}. \end{aligned}$$

Example 3.4.

$$\begin{array}{lll}(\mathrm{P}_1) & \text{Minimize} & \sin(y-x)\\ & \text{subject to} & \sin(tx-y)\le 0, t\in[0,1]\\ & & (x,y)\in\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1, y\ge 0\}.\end{array}$$

Set

$$f(x,y)=\sin(y-x), f_t(x,y)=\sin(tx-y), C=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1, y\ge 0\}.$$

It is easy to check that f and $f_t, t\in T$, are semiconvex functions on C. For $(x,y)\in C$, a simple computation shows that $y-x, tx-y\in[-\sqrt{2},\sqrt{2}]\subset[-\frac{\pi}{2},\frac{\pi}{2}]$. Hence,

$$\sin(tx-y)\le 0, t\in[0,1]\Leftrightarrow tx-y\le 0, t\in[0,1].$$

The feasible set of the problem is

$$A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1, y\ge 0, y\ge x\}.$$

For every $(x,y)\in A$, we have $\sin(y-x)\ge 0$. The point $z=(0,0)$ is a solution of (P). We can easily check the validity of the condition (3.1). So, the solution set of (P_1) can be determined as follows. Choose $u_1=-\cos(y-x), u_2=\cos(y-x)$. Then $(u_1,u_2)\in\partial^c f(x,y)$. For $(x,y)\in C$, we have

$$(u_1,u_2)\big((0,0)-(x,y)\big)=0\Leftrightarrow(x-y)\cos(y-x)=0\Leftrightarrow y=x.$$

Hence,

$$\begin{aligned} S &=\{(x,y)\in C \mid x-y=0, tx-y\le 0, t\in[0,1]\},\\ &=\{(x,y)\in C \mid x-y=0, x-y\le 0, y\ge 0\},\\ &=\{(x,y)\in\mathbb{R}^2 \mid x=y, 0\le x\le \tfrac{\sqrt{2}}{2}\}.\end{aligned}$$

Since a convex function is a semiconvex function [15], in case f is semiconvex on C and $f_t, t\in T$, are convex on $\mathbb{R}^n$, it is easy to see that the conclusion of Theorem 3.3 is valid. In particular, if f and $f_t, t\in T$, are convex functions, the Clarke subdifferentials coincide with the ones in the sense of convex analysis. We obtain the following corollary with noting that ∂f denotes the convex subdifferential of f.

Corollary 3.5. *For Problem (P), let z be a given solution such that the condition* (3.1) *holds. If f and $f_t, t\in T$, are convex functions then $S=S'_1=\bar{S}'_1$ where*

$$\begin{aligned} S'_1 &=\{x\in C \mid \exists u\in\partial f(z)\cap\partial f(x), u(z-x)=0, f_t(x)=0\ \forall t\in T_+\\ &\qquad \textit{and } f_t(x)\le 0\ \forall t\in T\setminus T_+\},\\ \bar{S}'_1 &=\{x\in C \mid \exists u\in\partial f(x), u(z-x)=0, f_t(x)=0\ \forall t\in T_+\\ &\qquad \textit{and } f_t(x)\le 0\ \forall t\in T\setminus T_+\}.\end{aligned}$$

The formulas above were presented in [20] where convex infinite programs were considered with T is an arbitrary index set.

We now give some more formulas of the solution set of (P). It is worth mentioning that

$$[z\in A, \sum_{t\in T_+}\lambda_t f_t(z)=0]\Leftrightarrow[x\in C, f_t(z)=0\ \forall t\in T_+, f_t(z)\le 0\ \forall t\in T\setminus T_+]$$

and, by Lemma 3.1, $L(x,\lambda) = f(x)$ for all $x \in S$. We can deduce a following corollary with the proof omitted.

Corollary 3.6. *For Problem (P), suppose that $f, f_t, t \in T$, and $L(\cdot,\lambda)$ are semiconvex functions on C. Then $S = S_2 = \bar{S}_2$, where*

$$\begin{array}{ll} S_2 & = \{x \in A \mid \sum_{t\in T_+} \lambda_t f_t(x) = 0,\ \exists u \in \partial^c L(\cdot,\lambda)(x),\ u(z-x) = 0\} \\ \bar{S}_2 & = \{x \in A \mid \sum_{t\in T_+} \lambda_t f_t(x) = 0,\ \exists u \in \partial^c L(\cdot,\lambda)(z) \cap \partial^c L(\cdot,\lambda)(x),\ u(z-x) = 0\}. \end{array}$$

3.2. **Subgradient characterizations.** Let $z \in S$ be such that the condition (3.1) is satisfied with $\lambda \in \mathbb{R}^{(T)}_+$. We obtain

$$\big(\partial^c f(z) + \sum_{t\in T} \lambda_t \partial^c f_t(z)\big) \cap \big(-N(C,z)\big) \neq \emptyset, \lambda_t f_t(z) = 0,\ \forall t \in T.$$

Note that

$$L(z,\lambda) = f(z) + \sum_{t\in T} \lambda_t f_t(z) = f(z) + \sum_{t\in T_+} \lambda_t f_t(z) = \big(f + \sum_{t\in T_+} \lambda_t f_t\big)(z).$$

Hence,

$$\partial^c L(\cdot,\lambda)(z) = \partial^c\big(f + \sum_{t\in T_+} \lambda_t f_t\big)(z) \subset \partial^c f(z) + \sum_{t\in T_+} \lambda_t \partial^c f_t(z) = \partial^c f(z) + \sum_{t\in T} \lambda_t \partial^c f_t(z).$$

Since the sum $\sum_{t\in T} \lambda_t f_t(z)$ is a finite sum, if the functions $f, f_t, t \in T$ are regular then, by applying Corollary 3 of [2], the following equality holds:

$$\partial^c\big(f + \sum_{t\in T} \lambda_t f_t\big)(z) = \partial^c f(z) + \sum_{t\in T} \lambda_t \partial^c f_t(z).$$

In this case we get $\partial^c L(\cdot,\lambda)(z) \cap \big(-N(C,z)\big) \neq \emptyset$.

In order to give characterizations of solution set of (P) via subgradients, we need a following lemma.

Lemma 3.7. *For Problem (P), let z be a given solution such that the condition* (3.1) *holds. Suppose that $f, f_t, t \in T$, are regular over C and $L(\cdot,\lambda)$ is semiconvex on C. For each $y \in S$, it holds*

$$\partial^c L(\cdot,\lambda)(z) \cap \big(-N(C,z)\big) = \partial^c L(\cdot,\lambda)(y) \cap \big(-N(C,y)\big).$$

Proof. Suppose that z is a solution of (P) such that the condition (3.1) holds with some $\lambda \in \mathbb{R}^{(T)}_+$. Firstly, for each $y \in S$, we prove that

$$\partial^c L(\cdot,\lambda)(y) \cap \big(-N(C,y)\big) \subset \partial^c L(\cdot,\lambda)(z) \cap \big(-N(C,z)\big).$$

Let $y \in S$ and let $u \in \partial^c L(\cdot,\lambda)(y) \cap \big(-N(C,y)\big)$. Then

$$\left\{ \begin{array}{l} u \in \partial^c L(\cdot,\lambda)(y), \\ u \in -N(C,y). \end{array} \right. \tag{3.7}$$

Since $u \in \partial^c L(\cdot,\lambda)(y)$ and $L(\cdot,\lambda)$ is regular at y,

$$L^{'}(\cdot,\lambda)(y;d) = L^c(\cdot,\lambda)(y;d) \geq u(d), \forall d \in X.$$

Since $u \in -N(C,y)$, $u(x-y) \geq 0$ for all $x \in C$. Hence, $L(x,\lambda) \geq L(y,\lambda)$ for all $x \in C$ by the semiconvexity of $L(\cdot,\lambda)$ on C. Thus, the point y minimizes $L(\cdot,\lambda)$

over C. Since $y, z \in S$, by Lemma 3.1, we get $L(y,\lambda) = L(z,\lambda)$. Hence, z also minimizes $L(\cdot,\lambda)$ over C. Using the same arguments as in the second part of the proof of Theorem 3.2, we can see that $u \in \partial^c L(\cdot,\lambda)(z)$.

We now prove that $u \in -N(C,z)$, i.e., $u(x-z) \geq 0$ for all $x \in C$. Since $u \in -N(C,y)$ and C is convex, $u(x-y) \geq 0$ for all $x \in C$. On the other hand, we have $u(x-z) = u(x-y) + u(y-z)$. We claim that $u(y-z) = 0$. Indeed, from (3.7), we get $u(z-y) \geq 0$. The conclusion $u(y-z) = 0$ is fulfilled if the inequality $u(z-y) > 0$ does not hold. Suppose to contrary that $u(z-y) > 0$. Then, by the regularity of $L(\cdot,\lambda)$ at y, we have

$$\lim_{\theta\downarrow 0} \frac{L(y+\theta(z-y),\lambda) - L(y,\lambda)}{\theta} = L(\cdot,\lambda)'(y; z-y) = L(\cdot,\lambda)^c(y; z-y) \geq u(z-y) > 0.$$

Since $L(y,\lambda) = L(z,\lambda)$,

$$\lim_{\theta\downarrow 0} \frac{L(y+\theta(z-y),\lambda) - L(z,\lambda)}{\theta} > 0.$$

Hence, there exist $\theta_0 \in (0,1)$ and $\alpha > 0$ small enough such that if $\theta \in (0,\theta_0)$ then

$$L(y+\theta(z-y),\lambda) - L(z,\lambda) > \alpha > 0. \tag{3.8}$$

It is easy to see that the function $G(\theta) := L(y+\theta(z-y),\lambda)$ is continuous at $\theta = 0$. By letting $\theta \to 0$ in (3.8), we obtain $L(y,\lambda) > L(z,\lambda)$, a contradiction. Hence, $u(y-z) = 0$ and we get $u(x-z) \geq 0$ for all $x \in C$, i.e., $u \in -N(C,z)$. Thus, $u \in \partial^c L(\cdot,\lambda)(z) \cap (-N(C,z))$. So,

$$\partial^c L(\cdot,\lambda)(y) \cap \big(-N(C,y)\big) \subset \partial^c L(\cdot,\lambda)(z) \cap \big(-N(C,z)\big).$$

Using similar arguments we can show that

$$\partial^c L(\cdot,\lambda)(z) \cap \big(-N(C,z)\big) \subset \partial^c L(\cdot,\lambda)(y) \cap \big(-N(C,y)\big).$$

□

Theorem 3.8. *For Problem (P), let z be a given solution such that the condition* (3.1) *holds. Suppose that $f, f_t, t \in T$, are regular over C and $L(\cdot,\lambda)$ is semiconvex on C. Then $S = S_3$, where*

$$S_3 := \{x \in C \mid \partial^c L(\cdot,\lambda)(x) \cap \big(-N(C,x)\big) = \partial^c L(\cdot,\lambda)(z) \cap \big(-N(C,z)\big),$$
$$f_t(x) = 0\ \forall t \in T_+ \text{ and } f_t(x) \leq 0\ \forall t \in T \setminus T_+\}.$$

Proof. Let $x \in S$. Then $x \in A$, i.e., $f_t(x) \leq 0$ for all $t \in T$. On the other hand, by Lemma 3.1, $f_t(x) = 0$ for all $t \in T_+$. Hence, we obtain $x \in C$, $f_t(x) = 0$ for all $t \in T_+$ and $f_t(x) \leq 0$ for all $t \in T \setminus T_+$. In addition, if $x \in S$, then, by applying Lemma 3.7, we get $\partial^c L(\cdot,\lambda)(x) \cap \big(-N(C,x)\big) = \partial^c L(\cdot,\lambda)(z) \cap \big(-N(C,z)\big)$. So, $S \subset S_3$.

Let $x \in S_3$. Then $x \in C$ and $f_t(x) = 0$ for all $t \in T_+$ and $f_t(x) \leq 0$ for all t $\in T \setminus T_+$, and there exists $u \in \partial^c L(\cdot,\lambda)(x) \cap \big(-N(C,x)\big)$. Hence, $u \in \partial^c L(\cdot,\lambda)(x)$ and $u \in -N(C,x)$, i.e., $u(y-x) \geq 0$ for all $y \in C$. This implies that

$$L^{'}(\cdot,\lambda)(x; y-x) \geq 0, \forall y \in C.$$

Since $L(\cdot,\lambda)$ is semiconvex on C, it follows that $L(z,\lambda) \geq L(x,\lambda)$, i.e.,

$$f(z) + \sum_{t\in T_+} \lambda_t f_t(z) \geq f(x) + \sum_{t\in T_+} \lambda_t f_t(x).$$

Note that we have $f_t(z) = 0$ for $t \in T_+$ by (3.1) and $f_t(x) = 0$ for all $t \in T_+$ as $x \in S_3$. These and the inequality above imply that $f(z) \geq f(x)$, i.e., $x \in S$. Hence, $S_3 \subset S$. □

3.3. **Minimizing sequence characterizations.** In this subsection characterizations of solution set of (P) will be given by using minimizing sequences. This result is an extension of the one applied to convex programs presented recently in [20]. To start with it, we suppose that $\inf\{f(x) : x \in A\} = \alpha$ is finite. Recall that a sequence $\{a_n\} \subset A$, A is the feasible set of (P), is called a minimizing sequence of (P) if $\lim_{n\to+\infty} f(a_n) = \alpha$.

Theorem 3.9. *Suppose that $\{a_n\} \subset A$ is a minimizing sequence of* (P). *If f is a semiconvex function on C and A is a closed convex subset of X then the solution set of* (P) *is*

$$S_4 = \{x \in A \mid \exists u \in \partial^c f(x), u(a_n - x) \geq 0, \forall n \in N\}.$$

Proof. Let $x \in S_4$. Then $x \in A$ and there exists $u \in \partial^c f(x)$ such that $u(a_n - x) \geq 0$ for all $n \in \mathbb{N}$. Since f is semiconvex at x, by (2.1), we obtain

$$f(a_n) - f(x) \geq 0, \forall n \in \mathbb{N}.$$

Letting $n \to +\infty$, we get $f(x) \leq \alpha$. Since $x \in A$, x is a solution of (P).

Conversely, suppose that x is a solution of (P). Then,

$$0 \in \partial^c(f + \delta_A)(x) \subset \partial^c f(z) + \partial^c \delta_A(x). \tag{3.9}$$

Thus there exists $u \in \partial^c f(x)$ such that $-u \in \partial^c \delta_A(x)$. Since A is a closed convex subset of $\mathbb{R}^n$, $\partial^c \delta_A(x) = N_A(x)$. It follows that $u(y - x) \geq 0$ for all $y \in A$. Hence, $u(a_n - x) \geq 0$, for all $n \in N$. The proof is complete. □

As $f_t, t \in T$, are semiconvex functions on C, the feasible set A is convex (discussion in the last part of Section 2). We get a following corollary with the proof omitted.

Corollary 3.10. *Suppose that $\{a_n\} \subset A$ is a minimizing sequence of* (P). *If f and $f_t, t \in T$, are semiconvex functions on C then $S = S_4$ where*

$$S_4 = \{x \in A \mid \exists u \in \partial^c f(x), u(a_n - x) \geq 0, \forall n \in N\}.$$

Example 3.11.

$$(\mathrm{P}_2) \quad \begin{array}{ll} \text{Minimize} & e^{x-y} \\ \text{subject to} & f_t(x,y) \leq 0, t \in [1,2] \\ & (x,y) \in C. \end{array}$$

where $f_t(x,y) = \begin{cases} tx^3 - y \leq 0, t \in [0,1], \\ ty - 2x \leq 0, t \in (1,2], \end{cases}$ and $C := \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

Set $f(x, y) = e^{x-y}$, $T = [0, 2]$ and

$$f_t(x, y) = \begin{cases} tx^3 - y, t \in [0, 1], \\ ty - 2x, t \in (1, 2]. \end{cases}$$

We can check that the feasible set of (P_2) is

$$\begin{aligned} A &= \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1, x \geq y \geq x^3\} \\ &= \{(x, y) \mid 0 \leq x \leq 1, x \geq y \geq x^3\} \end{aligned} \tag{3.10}$$

and $e^{x-y} \geq 1$ for any $(x, y) \in A$.

It is easy to check f is semiconvex on C and f_t are not semiconvex for all $t \in T$. So, Corollary 3.10 can not be applied. However, since the feasible set of (P_2) is closed and convex, the solution set can be determined by Theorem 3.9. Note that

$$\partial^c f(x, y) = \{(u, v) \mid ud_1 + vd_2 \leq (d_1 - d_2)e^{x-y}\}, \forall (d_1, d_2) \in \mathbb{R}^2.$$

Choose $(u, v) = (e^{x-y}, -e^{x-y})$. Then $(u, v) \in \partial^c f(x, y)$. Now, by using a minimizing sequence $(a_n) = (1/n, 1/n)_n$, we have

$$S = \{(x, y) \in A \mid \exists (u, v) \in \partial^c f(x, y), (u, v)\big((1/n, 1/n) - (x, y)\big) \geq 0\}, \forall n \in \mathbb{N}.$$

A simple computation gives

$$S = \{(x, y) \in \mathbb{R}^2 \mid y = x, 0 \leq x \leq 1\}.$$

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