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STRONG CONVERGENCE THEOREMS FOR FINITE GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce an iterative sequence to approximate a common fixed point of finite generalized nonexpansive mappings in a Banach space. We first study two nonlinear operators: a W-mapping and a block mapping generated by finite mappings in a Banach space. Next, we prove strong convergence theorems by the hybrid methods for mathematical programming for these mappings. Using these results, we deal with the problem for finding a common element of finite sets in Banach spaces. This problem is connected with the problem of image recovery and the feasibility problem.

1. Introduction

Let H be a Hilbert space and let $\{C_i\}_{i=1}^r$ be a family of nonempty closed convex subsets of H such that $C_0 = \cap_{i=1}^r C_i$ is nonempty. Then the problem of image recovery is to find an element of C_0 by using an iterative sequence of the metric projections P_i of H onto C_i $(i = 1, 2, \dots, r)$, where

$$P_i(x) = \operatorname*{argmin}_{y \in C_i} \|x - y\|$$

for all $x \in H$. This problem is connected with the feasibility problem. In fact, if $\{g_i\}_{i=1}^r$ is a family of continuous convex functions from H into $\mathbb{R}$, then the convex feasibility problem is to find an element of the feasibility set

$$\bigcap_{i=1}^{r} \{x \in H : g_i(x) \leq 0\}.$$

We know that each P_i is a nonexpansive retraction of H onto C_i, that is,

$$\|P_i x - P_i y\| \leq \|x - y\|$$

for all $x, y \in H$ and $P_i^2 = P_i$. Further, it holds that $C_0 = \cap_{i=1}^r F(P_i)$, where $F(P_i)$ denotes the set of all fixed points of P_i $(i = 1, 2, \dots, r)$. Thus the problem of image recovery in a Hilbert space setting is extended to the problem of finding a common fixed point of a family of nonexpansive mappings.

In 1997, Takahashi [27] introduced the following mapping: Let C be a convex subset of a Banach space E. Let $T_1, T_2, \dots, T_r$ be finite mappings of C into itself and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 \leq \alpha_i \leq 1$ for each $i = 1, 2, \dots, r$. Then, a mapping W of C into itself is defined by

$$U_1 = \alpha_1 T_1 + (1 - \alpha_1) I,$$

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$$\begin{aligned} U_2 &= \alpha_2 T_2 U_1 + (1-\alpha_2)I, \\ &\vdots \\ U_{r-1} &= \alpha_{r-1} T_{r-1} U_{r-2} + (1-\alpha_{r-1})I, \\ W = U_r &= \alpha_r T_r U_{r-1} + (1-\alpha_r)I. \end{aligned}$$

Such a mapping W is called the W-mapping generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Using this mapping, Shimoji and Takahashi [26, 30] proved weak and strong convergence theorems for finding a common fixed point of $T_1, T_2, \dots, T_r$ under suitable conditions. Recently, the authors [8, 9] introduced the notion of generalized nonexpansive mappings and studied the asymptotic behavior of W-mappings generated by generalized nonexpansive mappings (see [10, 12]).

On the other hand, motivated by Aharoni and Censor [1], Kikkawa and Takahashi [16] introduced the following mapping: Let C be a convex subset of a Banach space E. Let $T_1, T_2, \dots, T_r$ be finite mappings of C into itself and let $\omega_1, \omega_2, \dots, \omega_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 \le \alpha_i \le 1$ and $0 \le \omega_i \le 1$ for each $i = 1, 2, \dots, r$, and $\sum_{i=1}^{r} \omega_i = 1$. Then, a mapping U of C into itself is defined by

$$U = \sum_{i=1}^{r} \omega_i (\alpha_i I + (1-\alpha_i) T_i).$$

Such a mapping U is called the block mapping generated by $T_1, T_2, \dots, T_r$, $\alpha_1, \alpha_2, \dots, \alpha_r$ and $\omega_1, \omega_2, \dots, \omega_r$. Using this mapping, they proved a weak convergence theorem for finding a common fixed point of $T_1, T_2, \dots, T_r$ under suitable conditions. Recently, the authors also studied the asymptotic behavior of block mappings generated by generalized nonexpansive mappings (see [11]).

The aim of this paper is to prove strong convergence theorems by the hybrid methods for finding a common fixed point of finite generalized nonexpansive mappings in a Banach space. We first study sunny generalized nonexpansive retracts. In particular, we show that the intersection of sunny generalized nonexpansive retracts also is a sunny generalized nonexpansive retract. Next, we study the W-mappings and the block mappings which are generated by finite generalized nonexpansive mappings in Banach space. Using these mappings, we prove strong convergence theorems by the hybrid methods introduced by Solodov and Svaiter [25] and Takahashi, Takeuchi and Kubota [31]. Moreover, using these results, we deal with the problem for finding a common element of finite sets in Banach spaces. This problem is connected with the problem of image recovery and the feasibility problem.

2. Preliminaries

Let E be a real Banach space with its dual E^*. We denote the strong convergence and weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x_0$, respectively. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to x^* in E^* by $x_n^* \overset{*}{\rightharpoonup} x^*$. A Banach space E is said to be strictly convex if $\|x\| = \|y\| = 1$ and $x \ne y$ imply $\|(x+y)/2\| < 1$. Also, E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \ge \varepsilon$$

imply $\|(x+y)/2\| \leq 1-\delta$. The following result was proved by Xu [32] (see also [33]).

Lemma 2.1 ([32]). *Let $s > 0$ and let E be a uniformly convex Banach space. Then, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|) \tag{2.1}$$

for all $x, y \in B_s := \{z \in E : \|z\| \leq s\}$ and λ with $0 \leq \lambda \leq 1$.

A Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in \{z \in E : \|z\| = 1\}(=: S(E))$. A Banach space E is said to be uniformly smooth if the limit (2.2) is attained uniformly for $x, y \in S(E)$.

The normalized duality mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for each $x \in E$. We also know the following properties (see [5, 28, 29] for details):

(1) $Jx \neq \emptyset$ for each $x \in E$.
(2) If E is reflexive, then J is surjective.
(3) If E is strictly convex, then J is one to one.
(4) If E is smooth, then J is single valued and norm to weak* continuous.
(5) If E is smooth, strictly convex and reflexive, then the duality mapping J_* from E^* into E is the inverse of J, that is, $J_* = J^{-1}$.
(6) If E is uniformly smooth, then the duality mapping J is uniformly norm to norm continuous on each bounded set of E.
(7) If E is uniformly convex, then E is reflexive and strictly convex.
(8) E is uniformly convex if and only if E^* is uniformly smooth.

Let E be a smooth Banach space and consider the following function studied in Alber [2] and Kamimura and Takahashi [15]:

$$V(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for each $x, y \in E$. It is obvious from the definition of V that

$$(\|x\| - \|y\|)^2 \leq V(x,y) \leq (\|x\| + \|y\|)^2 \tag{2.3}$$

for each $x, y \in E$. We also know that

$$V(x,y) = V(x,z) + V(z,y) + 2\langle x - z, Jz - Jy\rangle \tag{2.4}$$

for each $x, y, z \in E$ (see [15]). It is also easy to see that if E is additionally assumed to be strictly convex, then

$$V(x,y) = 0 \ \Leftrightarrow\ x = y.$$

See [20] for more details. The following lemma was well-known.

Lemma 2.2 ([15]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} V(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty closed convex subset of a smooth Banach space E and let T be a mapping from C into itself. We denote by $F(T)$ the set of all fixed points of T. A point p in C is said to be a generalized asymptotic fixed point [14] of T if C contains a sequence $\{x_n\}$ such that $Jx_n \overset{*}{\rightharpoonup} Jp$ and $\|Jx_n - JTx_n\| \to 0$. The set of all generalized asymptotic fixed points of T is denoted by $\check{F}(T)$. A mapping T is called generalized nonexpansive [8, 9] if $F(T) \neq \emptyset$ and $V(Tx, p) \leq V(x, p)$ for each $x \in C$ and $p \in F(T)$.

Let D be a nonempty subset of E. A mapping $R : E \to D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx$$

for each $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a retraction if $Rx = x$ for each $x \in D$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto D is uniquely decided (see [8, 9]). Then, such a sunny generalized nonexpansive retraction of E onto D is denoted by R_D. A nonempty subset D of E is said to be a sunny generalized nonexpansive retract (resp. a generalized nonexpansive retract) of E if there exists a sunny generalized nonexpansive retraction (resp. a generalized nonexpansive retraction) of E onto D (see [8, 9] for more details). The set of fixed points of a sunny generalized nonexpansive retraction of E onto D is, of course, D.

We know the following results for sunny generalized nonexpansive retractions in Banach spaces.

Lemma 2.3 ([8, 9])**.** *Let D be a nonempty subset of a smooth and strictly convex Banach space E. Let R be a retraction of E onto D. Then R is sunny and generalized nonexpansive if and only if*

$$\langle x - Rx, JRx - Jy \rangle \geq 0$$

for each $x \in E$ and $y \in D$.

Lemma 2.4 ([9, 10])**.** *Let D be a nonempty subset of a reflexive, strictly convex, and smooth Banach space E. If R is the sunny generalized nonexpansive retraction of E onto D, then*

$$V(x, Rx) + V(Rx, u) \leq V(x, u)$$

for each $x \in E$ and $u \in D$.

Lemma 2.5 ([14])**.** *Let D be a nonempty subset of a reflexive, strictly convex, and smooth Banach space E. If R is the sunny generalized nonexpansive retraction of E onto D, then $\check{F}(R) = F(R)$.*

3. Sunny generalized nonexpansive retracts

In this section, we deal with some properties for sunny generalized nonexpansive retracts in a Banach space. We first recall the following results in [14, 17].

Theorem 3.1 ([17])**.** *Let E be a smooth, reflexive, and strictly convex Banach space and let C be a nonempty subset of E. Then, the following conditions are equivalent.*

1. *C is a sunny generalized nonexpansive retract of E;*
2. *C is a generalized nonexpansive retract of E;*
3. *JC is closed and convex.*

In this case, C is closed.

Lemma 3.2 ([14]). *Let E be a reflexive, strictly convex, and smooth Banach space and let T be a generalized nonexpansive mapping from E into itself such that $F(T)$ is nonempty. Then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Using Theorem 3.1, we can prove the following theorem.

Theorem 3.3. *Let E be a reflexive, strictly convex, and smooth Banach space and let $\mathscr{C}$ be a family of sunny generalized nonexpansive retracts of E such that $\cap_{C\in\mathscr{C}}C$ is nonempty. Then $\cap_{C\in\mathscr{C}}C$ is a sunny generalized nonexpansive retract of E.*

Proof. It is obvious that $J\cap_{C\in\mathscr{C}}C = \cap_{C\in\mathscr{C}}JC$. In fact, we have that

$$\begin{aligned} x \in J\cap_{C\in\mathscr{C}} C &\Leftrightarrow J^{-1}x \in \cap_{C\in\mathscr{C}}C \\ &\Leftrightarrow J^{-1}x \in C, \quad \forall C \in \mathscr{C} \\ &\Leftrightarrow x \in JC, \quad \forall C \in \mathscr{C} \\ &\Leftrightarrow x \in \cap_{C\in\mathscr{C}}JC. \end{aligned}$$

From Theorem 3.1, JC is closed and convex for each $C \in \mathscr{C}$ and hence $\cap_{C\in\mathscr{C}}JC$ is closed and convex. So, we have that $J\cap_{C\in\mathscr{C}}C$ is closed and convex. Therefore, from Theorem 3.1, we have that $\cap_{C\in\mathscr{C}}C$ is a sunny generalized nonexpansive retract of E. □

As a direct consequence of Lemma 3.2 and Theorem 3.3, we obtain the following result.

Theorem 3.4. *Let E be a reflexive, strictly convex, and smooth Banach space and let $\mathcal{T}$ be a family of generalized nonexpansive mappings T from E into itself such that $F(\mathcal{T})$ is nonempty, where $F(\mathcal{T})$ is the set of all common fixed points of $\mathcal{T}$. Then $F(\mathcal{T})$ is a sunny generalized nonexpansive retract of E.*

Proof. By Lemma 3.2, $F(T)$ is a sunny generalized nonexpansive retract of E. Further, since $F(\mathcal{T}) = \cap_{T\in\mathcal{T}}F(T)$ is nonempty, by Theorem 3.3 we have the desired result. □

4. Strong convergence theorem

In this section, we first introduce a new condition for a family of generalized nonexpansive mappings in a Banach space: Let E be a Banach space and let C be a nonempty closed convex subset of E. Let $\{T_n\}$ and $\mathcal{T}$ be families of generalized nonexpansive mappings of C into itself such that $\emptyset \neq F(\mathcal{T}) \subset \cap_{n=1}^{\infty}F(T_n)$, where $F(T_n)$ is the set of all fixed pints of T_n and $F(\mathcal{T})$ is the set of all common fixed pints of $\mathcal{T}$. Motivated by Nakajo, Shimoji and Takahashi [21], we consider the following condition concerning $\{T_n\}$ and $\mathcal{T}$:

(∗∗) For each bounded sequence $\{z_n\} \subset C$ with $\lim_{n\to\infty}\|z_n - T_nz_n\| = 0$ and for each $T \in \mathcal{T}$, there exists a sequence $\{u_n^T\}$ of C such that

$$\lim_{n\to\infty}\|u_n^T - Tu_n^T\| = \lim_{n\to\infty}\|z_n - u_n^T\| = 0.$$

Using condition $(**)$, we prove a strong convergence theorem for a family of generalized nonexpansive mappings in a Banach space which unifies two hybrid methods introduced by Solodov and Svaiter [25] and Takahashi, Takeuchi and Kubota [31]. This theorem is proved in more general Banach spaces.

Theorem 4.1. *Let E be a uniformly smooth and uniformly convex Banach space, let $\{T_n\}$ and $\mathcal{T}$ be families of generalized nonexpansive mappings from E into itself such that $\emptyset \neq F(\mathcal{T}) \subset \cap_{i=1}^{\infty} F(T_i)$ and $\check{F}(T) = F(T)$ for each $T \in \mathcal{T}$, and let $\{Y_n\}$ be a sequence of sunny generalized nonexpansive retracts of E such that $F(\mathcal{T}) \subset Y_n$ for each $n \in \mathbb{N}$. Suppose that $\{T_n\}$ satisfy the condition $(**)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, and*

$$\begin{cases} y_n = T_n x_n, \\ X_n = \{z \in E : V(y_n, z) \leq V(x_n, z)\}, \\ x_{n+1} = R_{X_n \cap Y_n} x, \ n = 1, 2, 3, \ldots. \end{cases}$$

Suppose that the sequence $\{x_n\}$ satisfy $x_n = R_{Y_n} x$ for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $R_{F(\mathcal{T})} x$, where $R_{F(\mathcal{T})}$ is a sunny generalized nonexpansive retraction of E onto $F(\mathcal{T})$.

Proof. We first show that JX_n is closed and convex for each $n \in \mathbb{N}$. Let $\{z_m^*\} \subset JX_n$ with $\lim_{m\to\infty} z_m^* = z_0^* \in E^*$. Then define $z_m := J^{-1} z_m^* \in X_n$. Since E is uniformly convex, E^* is uniformly smooth and hence the duality mapping J^{-1} on E^* is norm to norm continuous. Therefore, we have

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty} J^{-1} z_m^* = J^{-1} z_0^*.$$

From the definition of X_n and the uniformly smoothness of E, we know that X_n is closed. Hence $J^{-1} z_0^* \in X_n$. Therefore we obtain $z_0^* \in JX_n$. This implies that JX_n is closed. We show that JX_n is convex. Let $u^*, v^* \in JX_n$, and let $\lambda \in (0,1)$. Then there exist $u, v \in X_n$ such that $u^* = Ju$ and $v^* = Jv$. Put $z = J^{-1}(\lambda u^* + (1-\lambda) v^*)$. We have from the definition of X_n that

$$\begin{aligned} & V(x_n, z) - V(y_n, z) \\ = \ & \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, Jz\rangle \\ = \ & \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, \lambda u^* + (1-\lambda) v^*\rangle \\ = \ & \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, \lambda Ju + (1-\lambda) Jv\rangle \\ = \ & \lambda\Big(\|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, Ju\rangle\Big) \\ & +(1-\lambda)\Big(\|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, Jv\rangle\Big) \\ = \ & \lambda\Big(V(x_n, u) - V(y_n, u)\Big) + (1-\lambda)\Big(V(x_n, v) - V(y_n, v)\Big) \\ \geq \ & \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0 \end{aligned}$$

and hence $z \in X_n$. Therefore we obtain $\lambda u^* + (1-\lambda) v^* = Jz \in JX_n$. This implies that JX_n is convex for each $n \in \mathbb{N}$.

We next show that $F(\mathcal{T}) \subset X_n$ for each $n \in \mathbb{N}$. Let $p \in F(\mathcal{T}) \subset \cap_{i=1}^{\infty} F(T_i)$ and let $n \in \mathbb{N}$. Then from

$$V(y_n, p) = V(T_n x_n, p) \leq V(x_n, p),$$

we have $p \in X_n$ and hence $F(\mathcal{T}) \subset X_n$. Therefore, from Theorem 3.1, X_n is a sunny generalized nonexpansive retract for each $n \in \mathbb{N}$. Further, since $F(\mathcal{T}) \subset Y_n$, we obtain $F(\mathcal{T}) \subset X_n \cap Y_n$ for each $n \in \mathbb{N}$. Since Y_n is a sunny generalized nonexpansive retract for each $n \in \mathbb{N}$, by Theorem 3.3 we have that $X_n \cap Y_n$ is a sunny generalized nonexpansive retract for each $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Let $p \in F(\mathcal{T}) \subset X_n \cap Y_n$. Using $x_{n+1} = R_{X_n \cap Y_n} x$ and Lemma 2.4, we have

$$V(x, x_{n+1}) \leq V(x, p) - V(x_{n+1}, p) \leq V(x, p) \tag{4.1}$$

for each $n \in \mathbb{N}$. It is obvious that (4.1) holds for $n = 0$. In fact, we have $V(x, x_1) = V(x, x) = 0$. Hence we obtain

$$V(x, x_n) \leq V(x, p) - V(x_n, p) \leq V(x, p)$$

for each $n \in \mathbb{N}$. Therefore, $\{V(x, x_n)\}$ is bounded. Moreover, from (2.3) we have that $\{x_n\}$ is bounded. From $x_n = R_{Y_n} x$, $x_{n+1} = R_{X_n \cap Y_n} x \in Y_n$ and Lemma 2.4, we have

$$V(x, x_n) \leq V(x, x_{n+1}) - V(x_n, x_{n+1}) \leq V(x, x_{n+1})$$

for each $n \in \mathbb{N}$. Therefore $\{V(x, x_n)\}$ is nondecreasing. So, there exists the limit of $\{V(x, x_n)\}$. Since

$$V(x_n, x_{n+1}) \leq V(x, x_{n+1}) - V(x, x_n)$$

for each $n \in \mathbb{N}$, we have that $\lim_{n\to\infty} V(x_n, x_{n+1}) = 0$. From $x_{n+1} = R_{X_n \cap Y_n} x \in X_n$ and the definition of X_n, we also have

$$V(y_n, x_{n+1}) \leq V(x_n, x_{n+1})$$

for each $n \in \mathbb{N}$. Tending $n \to \infty$, we have $\lim_{n\to\infty} V(y_n, x_{n+1}) = 0$. Using Lemma 2.2, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

From

$$\|T_n x_n - x_n\| = \|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

we have $\lim_{n\to\infty} \|T_n x_n - x_n\| = 0$. From the assumption $(**)$, for each $T \in \mathcal{T}$, there exists sequence $\{u_n^T\}$ of E to itself such that

$$\lim_{n\to\infty} \|u_n^T - T u_n^T\| = \lim_{n\to\infty} \|x_n - u_n^T\| = 0.$$

Since E is uniformly smooth, the duality mapping J is uniformly norm to norm continuous on each bounded subset of E. Therefore, we obtain that

$$\lim_{n\to\infty} \|J u_n^T - J T u_n^T\| = \lim_{n\to\infty} \|J x_n - J u_n^T\| = 0.$$

Let $\{J x_{n_k}\}$ be a subsequence of $\{J x_n\}$ such that $J x_{n_k} \rightharpoonup p^*$ for some $p^* \in E^*$. Then, for any $T \in \mathcal{T}$ there exists $\{u_{n_k}^T\} \subset E$ such that $J u_{n_k}^T \rightharpoonup p^*$ and $\|J u_{n_k}^T - J T u_{n_k}^T\| \to 0$. So, $J^{-1} p^* \in \check{F}(T)$. Putting $p = J^{-1} p^*$, we have $p \in \check{F}(T) = F(T)$. This implies that $p \in F(\mathcal{T})$.

Finally, we show that $x_n \to R_{F(\mathcal{T})} x$. Let $\{J x_{n_k}\}$ be a subsequence of $\{J x_n\}$ such that $J x_{n_k} \rightharpoonup Jp \in JF(\mathcal{T})$ and let $z = R_{F(\mathcal{T})} x$. From Lemma 2.4, $x_{n+1} = R_{X_n \cap Y_n} x$, and $z \in F(\mathcal{T}) \subset X_n \cap Y_n$, we have $V(x, x_{n+1}) \leq V(x, z)$ for each $n \in \mathbb{N}$. Since $x = x_1$, it is clear that $0 = V(x, x) \leq V(x, z)$. Therefore, we have

$$V(x, x_n) \leq V(x, z) \tag{4.2}$$

for each $n \in \mathbb{N}$. On the other hand, since the norm $\|\cdot\|$ is weakly lower semicontinuous, we have

$$\begin{aligned} V(x,p) &= \|x\|^2 - 2\langle x, Jp\rangle + \|Jp\|^2 \\ &\leq \liminf_{k\to\infty}\big(\|x\|^2 - 2\langle x, Jx_{n_k}\rangle + \|Jx_{n_k}\|^2\big) \\ &= \liminf_{k\to\infty} V(x, x_{n_k}) \\ &\leq \limsup_{k\to\infty} V(x, x_{n_k}) \leq V(x,z). \end{aligned} \tag{4.3}$$

From (4.3) and Lemma 2.4, we get

$$V(x, z) + V(z, p) \leq V(x, p) \leq V(x, z).$$

This implies $V(z, p) = 0$. So, we have $z = p$ and hence $Jz = Jp = p^*$. From $z = R_{F(\mathcal{T})}x$, we obtain that $Jx_n \rightharpoonup Jz = JR_{F(\mathcal{T})}x$. By (2.4), we have

$$V(z, x_n) = V(z, x) + V(x, x_n) + 2\langle z - x, Jx - Jx_n\rangle \tag{4.4}$$

for each $n \in \mathbb{N}$. By (4.2) and (4.4), we obtain

$$\begin{aligned} \limsup_{n\to\infty} V(z, x_n) &= \limsup_{n\to\infty}\Big\{V(z,x) + V(x,x_n) + 2\langle z - x, Jx - Jx_n\rangle\Big\} \\ &\leq \limsup_{n\to\infty}\Big\{V(z,x) + V(x,z) + 2\langle z - x, Jx - Jx_n\rangle\Big\} \\ &= V(z,x) + V(x,z) + 2\langle z - x, Jx - Jz\rangle \\ &= V(z,z) = 0. \end{aligned}$$

Therefore, we have $\limsup_{n\to\infty} V(z, x_n) = 0$. This implies $\lim_{n\to\infty} V(z, x_n) = 0$. From Lemma 2.2, we obtain $\lim_{n\to\infty}\|z - x_n\| = 0$. Therefore, we obtain that $\{x_n\}$ converges strongly to $R_{F(\mathcal{T})}x$. This completes the proof. □

5. Families of generalized nonexpansive mappings with the new conditions

In this section, we give two examples of a family of generalized nonexpansive mappings which satisfies the condition(∗∗).

Let C be a nonempty convex subset of a Banach space E, let $S_1, S_2, \ldots, S_r$ be mappings from C into itself and let $\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,r}$ be real numbers such that $0 \leq \alpha_{n,i} \leq 1$ for each $i = 1, 2, \ldots, r$ and $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, Takahashi [27] introduced a mapping W of C into itself as follows:

$$\begin{aligned} U_{n,1} &= \alpha_{n,1}S_1 + (1 - \alpha_{n,1})I, \\ U_{n,2} &= \alpha_{n,2}S_2U_{n,1} + (1 - \alpha_{n,2})I, \\ &\vdots \\ U_{n,r-1} &= \alpha_{n,r-1}S_{r-1}U_{n,r-2} + (1 - \alpha_{n,r-1})I, \\ W_n = U_{n,r} &= \alpha_{n,r}S_rU_{n,r-1} + (1 - \alpha_{n,r})I. \end{aligned}$$

Recall that such a mapping W_n is called a W-mapping generated by $S_1, S_2, \dots, S_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ (see also [26, 30]).

We recall the following result for a W-mapping generated by finite generalized nonexpansive mappings in a Banach space.

Lemma 5.1 ([10]). *Let C be a nonempty closed convex subset of a smooth and uniformly convex Banach space E, let $\{S_i\}_{i=1}^r$ be a finite family of generalized nonexpansive mappings from C into itself such that $\cap_{i=1}^r F(S_i)$ is nonempty, and let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq r\}$ be a set in $(0, 1]$ such that $\alpha_{n,i} \neq 1$ for each $i = 1, 2, \dots, r-1$ and $n \in \mathbb{N}$. Let $\{W_n\}$ be a sequence of W-mappings of C into itself generated by $S_1, S_2, \dots, S_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$. Then, $F(W_n) = \cap_{i=1}^r F(S_i)$ for each $n \in \mathbb{N}$.*

We obtain the following result for a W-mapping generated by finite generalized nonexpansive mappings in a Banach space.

Lemma 5.2. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E, let $\{S_i\}_{i=1}^r$ be a finite family of generalized nonexpansive mappings from C into itself such that $\cap_{i=1}^r F(S_i)$ is nonempty, and let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq r\}$ be a set in $(0, 1]$ such that $\liminf_{n\to\infty} \alpha_{n,i}(1 - \alpha_{n,i}) > 0$ and $\alpha_{n,i} \neq 1$ for each $i = 1, 2, \dots, r-1$ and $n \in \mathbb{N}$. Let $\{W_n\}$ be a sequence of W-mappings of C into itself generated by $S_1, S_2, \dots, S_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$. Then $\{T_n\}$ with $T_n = W_n$ $(\forall n \in \mathbb{N})$ and $\mathcal{T} = \{S_1, S_2, \dots, S_r\}$ satisfy the condition $(**)$ with $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = \cap_{i=1}^r F(S_i)$.*

Proof. Let $p \in \cap_{i=1}^r F(S_i)$ and let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_n - W_n z_n\| = 0$. Then, from the definition of W_n, we have that

$$\begin{aligned}
&V(W_n z_n, p) = V(U_{n,r} z_n, p) \\
&\quad \leq \alpha_{n,r} V(S_r U_{n,r-1} z_n, p) + (1 - \alpha_{n,r}) V(z_n, p) \\
&\quad \leq \alpha_{n,r} V(U_{n,r-1} z_n, p) + (1 - \alpha_{n,r}) V(z_n, p) \\
&\quad \leq \alpha_{n,r}\alpha_{n,r-1} V(S_{r-1} U_{n,r-2} z_n, p) + \alpha_{n,r}(1 - \alpha_{n,r-1}) V(z_n, p) \\
&\qquad\quad + (1 - \alpha_{n,r}) V(z_n, p) \\
&\quad \leq \alpha_{n,r}\alpha_{n,r-1} V(U_{n,r-2} z_n, p) + (1 - \alpha_{n,r}\alpha_{n,r-1}) V(z_n, p) \\
&\quad \leq \alpha_{n,r}\alpha_{n,r-1}\alpha_{n,r-2} V(S_{r-2} U_{n,r-3} z_n, p) \\
&\qquad + \alpha_{n,r}\alpha_{n,r-1}(1 - \alpha_{n,r-2}) V(z_n, p) + (1 - \alpha_{n,r}\alpha_{n,r-1}) V(z_n, p) \\
&\quad \leq \alpha_{n,r}\alpha_{n,r-1}\alpha_{n,r-2} V(U_{n,r-3} z_n, p) + (1 - \alpha_{n,r}\alpha_{n,r-1}\alpha_{n,r-2}) V(z_n, p) \\
&\quad \vdots \\
&\quad \leq \alpha_{n,r}\alpha_{n,r-1} \cdots \alpha_{n,2} V(U_{n,1} z_n, p) + (1 - \alpha_{n,r}\alpha_{n,r-1} \cdots \alpha_{n,2}) V(z_n, p) \\
&\quad \leq \alpha_{n,r}\alpha_{n,r-1} \cdots \alpha_{n,2}\alpha_{n,1} V(S_1 z_n, p) \\
&\qquad + \alpha_{n,r}\alpha_{n,r-1} \cdots \alpha_{n,2}(1 - \alpha_{n,1}) V(z_n, p) + (1 - \alpha_{n,r}\alpha_{n,r-1} \cdots \alpha_{n,2}) V(z_n, p) \\
&\quad = \alpha_{n,r}\alpha_{n,r-1} \cdots \alpha_{n,2}\alpha_{n,1} V(S_1 z_n, p) + (1 - \alpha_{n,r}\alpha_{n,r-1} \cdots \alpha_{n,2}\alpha_{n,1}) V(z_n, p) \\
&\quad \leq \alpha_{n,r}\alpha_{n,r-1} \cdots \alpha_{n,2}\alpha_{n,1} V(z_n, p) + (1 - \alpha_{n,r}\alpha_{n,r-1} \cdots \alpha_{n,2}\alpha_{n,1}) V(z_n, p) \\
&\quad = V(z_n, p)
\end{aligned}$$

for each $n \in \mathbb{N}$. Since $\{z_n\}$ is bounded, from (2.3), $\{W_n z_n\}$ is bounded. Further, we obtain that, for each $i = 1, 2, \dots, r$,

$$\begin{aligned} V(W_n z_n, p) \leq & \alpha_{n,r}\alpha_{n,r-1}\cdots\alpha_{n,i} V(U_{n,i-1}z_n, p) \\ & + (1 - \alpha_{n,r}\alpha_{n,r-1}\cdots\alpha_{n,i}) V(z_n, p) \leq V(z_n, p) \end{aligned} \tag{5.1}$$

where $U_{n,0} = I$ for each $n \in \mathbb{N}$. So, we have

$$V(z_n, p) - V(U_{n,i-1}z_n, p) \leq \frac{V(z_n, p) - V(W_n z_n, p)}{\alpha_{n,r}\alpha_{n,r-1}\cdots\alpha_{n,i}}. \tag{5.2}$$

From (2.4), we obtain

$$\begin{aligned} & V(z_n, p) - V(W_n z_n, p) \\ & = V(z_n, W_n z_n) + 2\langle z_n - W_n z_n, JW_n z_n - Jp\rangle \\ & \leq \|z_n\|^2 - 2\langle z_n, JW_n z_n\rangle + \|W_n z_n\|^2 + 2\|z_n - W_n z_n\|\|JW_n z_n - Jp\| \\ & = \langle z_n, Jz_n - JW_n z_n\rangle - \langle z_n - W_n z_n, JW_n z_n\rangle \\ & \qquad + 2\|z_n - W_n z_n\|\|JW_n z_n - Jp\| \\ & \leq \|z_n\|\|Jz_n - JW_n z_n\| + \|z_n - W_n z_n\|\|W_n z_n\| \\ & \qquad + 2\|z_n - W_n z_n\|\|JW_n z_n - Jp\| \\ & \leq \|z_n\|\|Jz_n - JW_n z_n\| + \|z_n - W_n z_n\|\big(3\|W_n z_n\| + 2\|Jp\|\big). \end{aligned} \tag{5.3}$$

Since E is uniformly smooth, the duality mapping J is uniformly norm to norm continuous on each bounded set of E. Hence we obtain

$$\lim_{n\to\infty} \|Jz_n - JW_n z_n\| = 0.$$

From (5.3) and the boundedness of $\{z_n\}$ and $\{W_n z_n\}$, we have

$$\lim_{n\to\infty} \Big(V(z_n, p) - V(W_n z_n, p)\Big) = 0. \tag{5.4}$$

From (5.2), (5.4) and $\liminf_{n\to\infty} \alpha_{n,i}(1 - \alpha_{n,i}) > 0$, we also have

$$\lim_{n\to\infty} \Big(V(z_n, p) - V(U_{n,i-1}z_n, p)\Big) = 0 \tag{5.5}$$

for each $i = 1, 2, \dots, r$, where $U_{n,0} = I$ for each $n \in \mathbb{N}$.

Since $\{z_n\}$ is bounded, by (5.1) $\{U_{n,i-1}z_n\}$ is bounded and hence $\{S_i U_{n,i-1}z_n\}$ is also bounded for each $i = 1, 2, \dots, r$, where $U_{n,0} = I$. Put $s_{r-1} = \sup_{n\in\mathbb{N}}\{\|z_n\|, \|S_r U_{n,r-1}z_n\|\}$. By Lemma 2.1, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ satisfying (2.1), where $B_{s_{r-1}} = \{x \in E \ : \|x\| \leq s_{r-1}\}$. We have

$$\begin{aligned} & V(W_n z_n, p) \\ & \quad = \quad \|\alpha_{n,r} S_r U_{n,r-1}z_n + (1 - \alpha_{n,r})z_n\|^2 \\ & \qquad\qquad -2\langle \alpha_{n,r} S_r U_{n,r-1}z_n + (1 - \alpha_{n,r})z_n, Jp\rangle + \|p\|^2 \\ & \quad \leq \quad \alpha_{n,r}\|S_r U_{n,r-1}z_n\|^2 + (1 - \alpha_{n,r})\|z_n\|^2 \\ & \qquad\qquad -\alpha_{n,r}(1 - \alpha_{n,r}) g(\|S_r U_{n,r-1}z_n - z_n\|) \\ & \qquad\qquad\quad -2\alpha_{n,r}\langle S_r U_{n,r-1}z_n, Jp\rangle - 2(1 - \alpha_{n,r})\langle z_n, Jp\rangle + \|p\|^2 \end{aligned}$$

$$\begin{aligned}
&= \alpha_{n,r}\Big(\|S_rU_{n,r-1}z_n\|^2 - 2\langle S_rU_{n,r-1}z_n, Jp\rangle + \|p\|^2\Big) \\
&\quad +(1-\alpha_{n,r})\Big(\|z_n\|^2 - 2\langle z_n, Jp\rangle + \|p\|^2\Big) \\
&\quad -\alpha_{n,r}(1-\alpha_{n,r})g(\|S_rU_{n,r-1}z_n - z_n\|) \\
&= \alpha_{n,r}V(S_rU_{n,r-1}z_n, p) + (1-\alpha_{n,r})V(z_n, p) \\
&\quad -\alpha_{n,r}(1-\alpha_{n,r})g(\|S_rU_{n,r-1}z_n - z_n\|) \\
&\leq \alpha_{n,r}V(U_{n,r-1}z_n, p) + (1-\alpha_{n,r})V(z_n, p) \\
&\quad -\alpha_{n,r}(1-\alpha_{n,r})g(\|S_rU_{n,r-1}z_n - z_n\|) \\
&\leq \alpha_{n,r}V(z_n, p) + (1-\alpha_{n,r})V(z_n, p) - \alpha_{n,r}(1-\alpha_{n,r})g(\|S_rU_{n,r-1}z_n - z_n\|) \\
&= V(z_n, p) - \alpha_{n,r}(1-\alpha_{n,r})g(\|S_rU_{n,r-1}z_n - z_n\|)
\end{aligned}$$

and hence

$$\alpha_{n,r}(1-\alpha_{n,r})g(\|S_rU_{n,r-1}z_n - z_n\|) \leq V(z_n, p) - V(W_nz_n, p).$$

From (5.4) and $\liminf_{n\to\infty}\alpha_{n,r}(1-\alpha_{n,r}) > 0$, we have

$$\lim_{n\to\infty} g(\|S_rU_{n,r-1}z_n - z_n\|) = 0.$$

Then the properties of g yield that

$$\lim_{n\to\infty} \|S_rU_{n,r-1}z_n - z_n\| = 0.$$

Next, put $s_{r-2} = \sup_{n\in\mathbb{N}}\{\|U_{n,r-2}z_n\|, \|S_{r-1}U_{n,r-2}z_n\|\}$. By Lemma 2.1, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ satisfying (2.1), where $B_{s_{r-2}} = \{x \in E : \|x\| \leq s_{r-2}\}$. Therefore we have

$$\begin{aligned}
&V(U_{n,r-1}z_n, p) \\
&= \|\alpha_{n,r-1}S_{r-1}U_{n,r-2}z_n + (1-\alpha_{n,r-1})z_n\|^2 \\
&\quad -2\langle \alpha_{n,r-1}S_{r-1}U_{n,r-2}z_n + (1-\alpha_{n,r-1})z_n, Jp\rangle + \|p\|^2 \\
&\leq \alpha_{n,r-1}\|S_{r-1}U_{n,r-2}z_n\|^2 + (1-\alpha_{n,r-1})\|z_n\|^2 \\
&\quad -\alpha_{n,r-1}(1-\alpha_{n,r-1})g(\|S_{r-1}U_{n,r-2}z_n - z_n\|) \\
&\quad -2\alpha_{n,r-1}\langle S_{r-1}U_{n,r-2}z_n, Jp\rangle - 2(1-\alpha_{n,r-1})\langle z_n, Jp\rangle + \|p\|^2 \\
&\leq \alpha_{n,r-1}V(S_{r-1}U_{n,r-2}z_n, p) + (1-\alpha_{n,r-1})V(z_n, p) \\
&\quad -\alpha_{n,r-1}(1-\alpha_{n,r-1})g(\|S_{r-1}U_{n,r-2}z_n - z_n\|) \\
&\leq V(z_n, p) - \alpha_{n,r-1}(1-\alpha_{n,r-1})g(\|S_{r-1}U_{n,r-2}z_n - z_n\|)
\end{aligned}$$

and hence

$$\alpha_{n,r-1}(1-\alpha_{n,r-1})g(\|S_{r-1}U_{n,r-2}z_n - z_n\|) \leq V(z_n, p) - V(U_{n,r-1}z_n, p).$$

From (5.5) and $\liminf_{n\to\infty}\alpha_{n,r-1}(1-\alpha_{n,r-1}) > 0$, we obtain

$$\lim_{n\to\infty} g(\|S_{r-1}U_{n,r-2}z_n - z_n\|) = 0.$$

Then the properties of g yield that

$$\lim_{n\to\infty} \|S_{r-1}U_{n,r-2}z_n - z_n\| = 0.$$

By such a method, we have

$$\lim_{n\to\infty} \|S_i U_{n,i-1} z_n - z_n\| = 0.$$

for each $i = r-2, r-3, \dots, 1$, where $U_{n,0} = I$ for each $n \in \mathbb{N}$. So, from

$$\|z_n - U_{n,i} z_n\| = \|z_n - \alpha_{n,i} S_i U_{n,i-1} z_n - (1-\alpha_{n,i}) z_n\| = \alpha_{n,i} \|z_n - S_i U_{n,i-1} z_n\|,$$

we have

$$\lim_{n\to\infty} \|z_n - U_{n,i} z_n\| = 0 \tag{5.6}$$

for each $i = 1, 2, \dots, r$. Note that (5.6) hold for $i = 0$. In fact, we have that $\|z_n - U_{n,0} z_n\| = \|z_n - z_n\| = 0$. Then from

$$\|S_i U_{n,i-1} z_n - U_{n,i-1} z_n\| \le \|S_i U_{n,i-1} z_n - z_n\| + \|z_n - U_{n,i-1} z_n\|,$$

we also obtain

$$\lim_{n\to\infty} \|S_i U_{n,i-1} z_n - U_{n,i-1} z_n\| = 0.$$

Put $u_n^{S_i} := U_{n,i-1} z_n$ for each $i = 1, 2, \dots, r$ and $n \in \mathbb{N}$. Then, we have that $\{T_n\}$ and $\mathcal{T}$ satisfy the condition $(**)$. Further, from Lemma 5.1, we have $\cap_{i=1}^{\infty} F(T_n) = F(\mathcal{T}) = \cap_{i=1}^{r} F(S_i)$. This completes the proof. □

Let C be a nonempty closed convex subset of a Banach space E and let $S_1, S_2, \dots, S_r$ be mappings from C into itself. Then, motivated by Aharoni and Censor [1], Kikkawa and Takahashi [16] introduced a mappings U_n of C into itself as follows:

$$U_n = \sum_{i=1}^{r} \omega_n(i)(\alpha_{n,i} I + (1-\alpha_{n,i}) S_i) \tag{5.7}$$

for all $n \in \mathbb{N}$, where $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \le i \le r\} \subset [0,1]$, $\{\omega_n(i) : n, i \in \mathbb{N}, 1 \le i \le r\} \subset [0,1]$ and $\sum_{i=1}^{r} \omega_n(i) = 1$ for each $n \in \mathbb{N}$. Recall that such a mapping U_n is called a block mapping defined by $S_1, S_2, \dots, S_r$, $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ and $\omega_n(1), \omega_n(2), \dots, \omega_n(r)$.

We recall the following two results for a block mapping generated by finite generalized nonexpansive mappings in a Banach space.

Lemma 5.3 ([11]). *Let C be a nonempty closed convex subset of a smooth Banach space E, let $\{S_i\}_{i=1}^{r}$ be a finite family of generalized nonexpansive mappings from C into itself such that $\cap_{i=1}^{r} F(S_i) \ne \emptyset$ and let $\{U_n\}$ be a sequence of block mappings defined by* (5.7)*, where $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \le i \le r\} \subset [0,1]$, $\{\omega_n(i) : n, i \in \mathbb{N}, 1 \le i \le r\} \subset [0,1]$ and $\sum_{i=1}^{r} \omega_n(i) = 1$ for all $n \in \mathbb{N}$. Then $V(U_n x, z) \le V(x, z)$ for each $x \in C$, $z \in \cap_{i=1}^{r} F(S_i)$ and $n \in \mathbb{N}$.*

Theorem 5.4 ([11]). *Let C be a nonempty closed convex subset of a smooth and strictly convex Banach space E, let $\{S_i\}_{i=1}^{r}$ be a finite family of generalized nonexpansive mappings from C into itself such that $\cap_{i=1}^{r} F(S_i) \ne \emptyset$ and let $\{U_n\}$ be a sequence of block mappings defined by* (5.7)*, where $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \le i \le r\} \subset [0,1)$, $\{\omega_n(i) : n, i \in \mathbb{N}, 1 \le i \le r\} \subset (0,1]$ and $\sum_{i=1}^{r} \omega_n(i) = 1$ for all $n \in \mathbb{N}$. Then $F(U_n) = \cap_{i=1}^{r} F(S_i)$ for each $n \in \mathbb{N}$.*

Now, we obtain the following result for a block mapping generated by finite generalized nonexpansive mappings in a smooth Banach space.

Lemma 5.5. *Let C be a nonempty closed convex subset of a smooth and uniformly convex Banach space E, let $\{S_i\}_{i=1}^r$ be a finite family of generalized nonexpansive mappings from C into itself such that $\cap_{i=1}^r F(S_i) \neq \emptyset$. Let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq r\} \subset [0,1)$ and $\{\omega_n(i) : n, i \in \mathbb{N}, 1 \leq i \leq r\} \subset (0,1]$ be sets such that $\liminf_n \alpha_{n,i}(1-\alpha_{n,i}) > 0$, $\liminf_n \omega_n(i) > 0$ for each $i = 1, 2, \ldots, r$ and $\sum_{i=1}^r \omega_n(i) = 1$ for all $n \in \mathbb{N}$, let $\{U_n\}$ be a sequence of block mappings generated by $S_1, S_2, \ldots, S_r$, $\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,r}$ and $\omega_n(1), \omega_n(2), \ldots, \omega_n(r)$. Then $\{T_n\}$ with $T_n = U_n$ ($\forall n \in \mathbb{N}$) and $\mathcal{T} = \{S_1, S_2, \ldots, S_r\}$ satisfy the condition $(**)$ with $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = \cap_{i=1}^r F(S_i)$.*

Proof. Let $p \in \cap_{i=1}^r F(S_i)$ and let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_n - U_n z_n\| = 0$. Since S_i is generalized nonexpansive and $\{z_n\}$ is bounded, then $\{S_i z_n\}$ is bounded for each $i = 1, 2, \ldots, r$. Take $s > 0$ such that $\{z_n\}, \{S_i z_n\} \subset B_s$ ($i = 1, 2, \ldots, r$), where $B_s = \{x \in E : \|x\| \leq s\}$. Then, Lemma 2.1 ensures the existence of a strictly increasing, continuous and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and

$$\|tz_n + (1-t)S_i z_n\|^2 \leq t\|z_n\|^2 + (1-t)\|S_i z_n\|^2 - t(1-t)g(\|z_n - S_i z_n\|)$$

for each $t \in [0,1], n \in \mathbb{N}$, and $i = 1, 2, \ldots, r$. Then we have

$$\begin{aligned}
&V(U_n z_n, p)\\
&= V\left(\sum_{i=1}^r \omega_n(i)\Big(\alpha_{n,i} z_n + (1-\alpha_{n,i})S_i z_n\Big), p\right)\\
&\leq \sum_{i=1}^r \omega_n(i) V\big(\alpha_{n,i} z_n + (1-\alpha_{n,i})S_i z_n, p\big)\\
&= \sum_{i=1}^r \omega_n(i)\Big(\|\alpha_{n,i} z_n + (1-\alpha_{n,i})S_i z_n\|^2\\
&\qquad -2\langle \alpha_{n,i} z_n + (1-\alpha_{n,i})S_i z_n, Jp\rangle + \|p\|^2\Big)\\
&\leq \sum_{i=1}^r \omega_n(i)\Big(\alpha_{n,i}\|z_n\|^2 + (1-\alpha_{n,i})\|S_i z_n\|^2 - \alpha_{n,i}(1-\alpha_{n,i})g(\|z_n - S_i z_n\|)\\
&\qquad -2\alpha_{n,i}\langle z_n, Jp\rangle - 2(1-\alpha_{n,i})\langle S_i z_n, Jp\rangle + \|p\|^2\Big)\\
&= \sum_{i=1}^r \omega_n(i)\Big(\alpha_{n,i} V(z_n, p) + (1-\alpha_{n,i})V(S_i z_n, p)\\
&\qquad -\alpha_{n,i}(1-\alpha_{n,i})g(\|z_n - S_i z_n\|)\Big)\\
&\leq \sum_{i=1}^r \omega_n(i)\Big(\alpha_{n,i} V(z_n, p) + (1-\alpha_{n,i})V(z_n, p) - \alpha_{n,i}(1-\alpha_{n,i})g(\|z_n - S_i z_n\|)\Big)\\
&= \sum_{i=1}^r \omega_n(i)\Big(V(z_n, p) - \alpha_{n,i}(1-\alpha_{n,i})g(\|z_n - S_i z_n\|)\Big)
\end{aligned}$$

$$= \quad V(z_n, p) - \sum_{i=1}^{r} \omega_n(i)\alpha_{n,i}(1-\alpha_{n,i})g(\|z_n - S_i z_n\|)$$

for each $n \in \mathbb{N}$ and hence

$$\sum_{i=1}^{r} \omega_n(i)\alpha_{n,i}(1-\alpha_{n,i})g(\|z_n - S_i z_n\|) \leq V(z_n, p) - V(U_n z_n, p). \tag{5.8}$$

From (2.4), we obtain

$$\begin{aligned}
&V(z_n, p) - V(U_n z_n, p) \\
&= V(z_n, U_n z_n) + 2\langle z_n - U_n z_n, JU_n z_n - Jp\rangle \\
&\leq \|z_n\|^2 - 2\langle z_n, JU_n z_n\rangle + \|U_n z_n\|^2 + 2\|z_n - U_n z_n\|\|JU_n z_n - Jp\| \\
&= \langle z_n, Jz_n - JU_n z_n\rangle - \langle z_n - U_n z_n, JU_n z_n\rangle \\
&\qquad + 2\|z_n - U_n z_n\|\|JU_n z_n - Jp\| \\
&\leq \|z_n\|\|Jz_n - JU_n z_n\| + \|z_n - U_n z_n\|\|U_n z_n\| \\
&\qquad + 2\|z_n - U_n z_n\|\|JU_n z_n - Jp\| \\
&\leq \|z_n\|\|Jz_n - JU_n z_n\| + \|z_n - U_n z_n\|\Big(3\|U_n z_n\| + 2\|Jp\|\Big)
\end{aligned} \tag{5.9}$$

Since E is uniformly smooth, the duality mapping J is uniformly norm to norm continuous on each bounded set of E. Hence we obtain

$$\lim_{n\to\infty} \|Jz_n - JU_n z_n\| = 0.$$

From (5.9) and the boundedness of $\{z_n\}$ and $\{U_n z_n\}$, we have

$$\lim_{n\to\infty} \Big(V(z_n, p) - V(U_n z_n, p)\Big) = 0.$$

Combining this with (5.8), we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{r} \omega_n(i)\alpha_{n,i}(1-\alpha_{n,i})g(\|z_n - S_i z_n\|) = 0.$$

Since $\liminf_n \omega_n(i) > 0$ and $\liminf \alpha_{n,i}(1-\alpha_{n,i}) > 0$ for each $i = 1, 2, \ldots, r$, we have

$$\lim_{n\to\infty} g(\|z_n - S_i z_n\|) = 0$$

for each $i = 1, 2, \ldots, r$. Then the properties of g yield that

$$\lim_{n\to\infty} \|z_n - S_i z_n\| = 0$$

for each $i = 1, 2, \ldots, r$. Put $u_n^{S_i} := z_n$ for each $i = 1, 2, \ldots, r$. Then, we have that $\{T_n\}$ and $\mathcal{T}$ satisfy the condition $(**)$. Further, from Lemma 5.4, we have $\cap_{i=1}^{\infty} F(T_n) = F(\mathcal{T}) = \cap_{i=1}^{r} F(S_i)$. This completes the proof. □

6. Hybrid methods in mathematical programing

In this section, using Theorem 4.1, we prove a strong convergence theorem for a family of generalized nonexpansive mappings in a Banach space by the hybrid method in mathematical programing introduced by Solodov and Svaiter [25]. This theorem extend Nakajo-Takahashi's result ([22]) for a nonexpansive mapping in a Hilbert space to a more general Banach space and a famiily of mappings.

Theorem 6.1. *Let E be a uniformly smooth and uniformly convex Banach space and let $\{T_n\}$ and $\mathcal{T}$ be families of generalized nonexpansive mappings from E into itself which satisfy $\emptyset \neq F(\mathcal{T}) \subset \cap_{i=1}^{\infty} F(T_i)$, $\check{F}(T) = F(T)$ for each $T \in \mathcal{T}$ and the condition $(**)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, and*

$$\begin{cases} y_n = T_n x_n, \\ C_n = \{z \in E : V(y_n, z) \leq V(x_n, z)\}, \\ D_n = \{z \in E : \langle x - x_n, Jx_n - Jz \rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap D_n} x, \ n = 1, 2, 3, \ldots. \end{cases}$$

Then $\{x_n\}$ converges strongly to $R_{F(\mathcal{T})}x$, where $R_{F(\mathcal{T})}$ is a sunny generalized nonexpansive retraction of E onto $F(\mathcal{T})$.

Proof. We first show that JD_n is closed and convex for each $n \in \mathbb{N}$. Let $\{z_m^*\} \subset JD_n$ with $\lim_{m\to\infty} z_m^* = z_0^* \in E^*$. Then define $z_m := J^{-1} z_m^* \in D_n$. Since E is uniformly convex, E^* is uniformly smooth and hence the duality mapping J^{-1} on E^* is norm to norm continuous. Therefore, we have

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty} J^{-1} z_m^* = J^{-1} z_0^*.$$

From the definition of D_n and the uniformly smoothness of E, it is obvious that D_n is closed. So, we have $J^{-1} z_0^* \in D_n$. Therefore we obtain $z_0^* \in JD_n$. This implies that JD_n is closed for each $n \in \mathbb{N}$. We show that JD_n is convex. Let $u^*, v^* \in JD_n$, and let $\lambda \in (0, 1)$. Then there exist $u, v \in D_n$ such that $u^* = Ju$ and $v^* = Jv$. Put $z = J^{-1}(\lambda u^* + (1 - \lambda)v^*)$. We obtain from the definition of D_n

$$\begin{aligned} &\langle x - x_n, Jx_n - Jz \rangle \\ &= \langle x - x_n, Jx_n - \lambda Ju - (1 - \lambda)Jv \rangle \\ &= \lambda \langle x - x_n, Jx_n - Ju \rangle + (1 - \lambda)\langle x - x_n, Jx_n - Jv \rangle \\ &\geq \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0 \end{aligned}$$

and hence $z \in D_n$. So, we have $Jz = \lambda u^* + (1 - \lambda)v^* \in JD_n$. This implies that JD_n is convex for each $n \in \mathbb{N}$.

We next show that $F(\mathcal{T}) \subset D_n$ for each $n \in \mathbb{N}$. It is clear that $F(\mathcal{T}) \subset D_1 = E$. Suppose that $F(\mathcal{T}) \subset D_k$ for some $k \in \mathbb{N}$. As in the proof of Theorem 4.1, we have that $F(\mathcal{T}) \subset C_n$, JC_n is closed and convex for each $n \in \mathbb{N}$. Therefore $C_k \cap D_k$ is nonempty. So, from Theorem 3.1 we have that $C_k \cap D_k$ is a sunny generalized nonexpansive retract of E. Then there exists an element $x_{k+1} \in C_k \cap D_k$ such that $x_{k+1} = R_{C_k \cap D_k} x$, where $R_{C_k \cap D_k}$ is a sunny generalized nonexpansive retraction of E onto $C_k \cap D_k$. From Lemma 2.3, there holds

$$\langle x - x_{k+1}, Jx_{k+1} - Jz \rangle \geq 0$$

for each $z \in C_k \cap D_k$. Since $F(\mathcal{T}) \subset C_k \cap D_k$, we have $\langle x - x_{k+1}, Jx_{k+1} - Jp\rangle \geq 0$ for each $p \in F(\mathcal{T})$ and hence $F(\mathcal{T}) \subset D_{k+1}$. By induction, we have that $F(\mathcal{T}) \subset D_n$ for each $n \in \mathbb{N}$. Therefore, from Theorem 3.1 D_n is a sunny generalized nonexpansive retract of E and $F(\mathcal{T}) \subset D_n$.

Put $X_n = C_n$ and $Y_n = D_n$. It is obvious from Lemma 2.3 and the definition of D_n that $x_n = R_{Y_n}x$ for each $n \in \mathbb{N}$. Then X_n and Y_n satisfy the assumption of Theorem 4.1. Therefore, by Theorem 4.1, the sequence $\{x_n\}$ converges strongly to $R_{F(\mathcal{T})}x$. □

Using Theorem 6.1 and Lemma 5.2, we obtain the following result for a W-mapping generated by a finite family of generalized nonexpansive mappings.

Theorem 6.2. *Let E be a uniformly smooth and uniformly convex Banach space, let $\{S_i\}_{i=1}^r$ be a finite family of generalized nonexpansive mappings from E into itself such that $\cap_{i=1}^r F(S_i)$ is nonempty and $F(S_i) = \check{F}(S_i)$ for each $i = 1, 2, \ldots, r$, and let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq r\}$ be a set in $(0, 1]$ such that $\liminf_{n\to\infty} \alpha_{n,i}(1 - \alpha_{n,i}) > 0$ and $\alpha_{n,i} \neq 1$ for each $i = 1, 2, \ldots, r - 1$ and $n \in \mathbb{N}$. Let $\{W_n\}$ be a sequence of W-mappings of E into itself generated by $S_1, S_2, \ldots, S_r$ and $\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,r}$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, and*

$$(6.1) \qquad \begin{cases} y_n = W_n x_n, \\ C_n = \{z \in E : V(y_n, z) \leq V(x_n, z)\}, \\ D_n = \{z \in E : \langle x - x_n, Jx_n - Jz\rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap D_n} x, \ n = 1, 2, 3, \ldots. \end{cases}$$

Then $\{x_n\}$ converges strongly to Rx, where R is a sunny generalized nonexpansive retraction of E onto $\cap_{i=1}^r F(S_i)$.

Proof. Put $T_n = W_n$ for each $n \in \mathbb{N}$ and $\mathcal{T} = \{S_1, S_2, \ldots, S_r\}$. From Lemma 5.2, we have that $\{T_n\}$ and $\mathcal{T}$ satisfy the condition $(**)$ with $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = \cap_{i=1}^r F(S_i)$. Therefore, using Theorem 6.1, we obtain the desired result. □

Using Theorem 6.1 and Lemma 5.5, we also obtain the following result for a block mapping generated by a finite family of generalized nonexpansive mappings.

Theorem 6.3. *Let E be a uniformly smooth and uniformly convex Banach space, let $\{S_i\}_{i=1}^r$ be a finite family of generalized nonexpansive mappings from E into itself such that $F(S_i) = \check{F}(S_i)$ for each $i = 1, 2, \ldots, r$ and $\cap_{i=1}^r F(S_i) \neq \emptyset$. Let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq r\} \subset [0, 1)$ and $\{\omega_n(i) : n, i \in \mathbb{N}, 1 \leq i \leq r\} \subset (0, 1]$ be sets such that $\liminf_n \alpha_{n,i}(1 - \alpha_{n,i}) > 0$, $\liminf_n \omega_n(i) > 0$ for each $i = 1, 2, \ldots, r$ and $\sum_{i=1}^r \omega_n(i) = 1$ for all $n \in \mathbb{N}$, let $\{U_n\}$ be a sequence of block mappings generated by $S_1, S_2, \ldots, S_r$, $\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,r}$ and $\omega_n(1), \omega_n(2), \ldots, \omega_n(r)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, and*

$$(6.2) \qquad \begin{cases} y_n = U_n x_n, \\ C_n = \{z \in E : V(y_n, z) \leq V(x_n, z)\}, \\ D_n = \{z \in E : \langle x - x_n, Jx_n - Jz\rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap D_n} x, \ n = 1, 2, 3, \ldots. \end{cases}$$

Then $\{x_n\}$ converges strongly to Rx, where R is a sunny generalized nonexpansive retraction of E onto $\cap_{i=1}^r F(S_i)$.

Proof. Put $T_n = U_n$ for each $n \in \mathbb{N}$ and $\mathcal{T} = \{S_1, S_2, \dots, S_r\}$. As in the proof of Theorem 6.2, by Theorem 6.1 and Lemma 5.5 we obtain the desired result. □

As a direct consequence of Theorem 6.3, we have the following result which is connected with [14].

Theorem 6.4. *Let E be a uniformly smooth and uniformly convex Banach space, let T be a generalized nonexpansive mapping from E into itself such that $F(T) = \check{F}(T)$ and let $\{\alpha_n\}$ be a sequence in $[0, 1)$ such that* $\liminf_n \alpha_n(1 - \alpha_n) > 0$*. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in E : V(y_n, z) \leq V(x_n, z)\}, \\ D_n = \{z \in E : \langle x - x_n, Jx_n - Jz\rangle \geq 0\}, \\ x_{n+1} = R_{C_n \cap D_n} x, \ n = 1, 2, 3, \dots. \end{cases}$$

Then $\{x_n\}$ converges strongly to Rx, where R is a sunny generalized nonexpansive retraction of E onto $F(T)$.

Proof. In the case of $r = 1$, we know that $U_n = \alpha_n x_n + (1 - \alpha_n)Tx_n$ for each $n \in \mathbb{N}$. Therefore, by Theorem 6.3 we obtain the desired result. □

7. Shrinking projection methods

In this section, using Theorem 4.1, we prove a strong convergence theorem for a family of generalized nonexpansive mappings in a Banach space by the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [31]. We extend Takahashi-Takeuchi-Kubota's result ([31]) to a more general Banach space and a more general family of mappings.

Theorem 7.1. *Let E be a uniformly smooth and uniformly convex Banach space and let $\{T_n\}$ and $\mathcal{T}$ be families of generalized nonexpansive mappings from E into itself which satisfy $\emptyset \neq F(\mathcal{T}) \subset \cap_{i=1}^{\infty} F(T_i)$, $\check{F}(T) = F(T)$ for each $T \in \mathcal{T}$, and the condition $(**)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, $C_1 = E$, and*

$$\begin{cases} y_n = T_n x_n, \\ C_{n+1} = \{z \in C_n : V(y_n, z) \leq V(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x, \ n = 1, 2, 3, \dots. \end{cases}$$

Then $\{x_n\}$ converges strongly to $R_{F(\mathcal{T})}x$, where $R_{F(\mathcal{T})}$ is a sunny generalized nonexpansive retraction of E onto $F(\mathcal{T})$.

Proof. We first show that JC_n is closed and convex for each $n \in \mathbb{N}$. Since E is reflexive, strictly convex and smooth, it is clear that $JC_1 = JE = E^*$ is closed and convex. Suppose that JC_k is closed and convex for some $k \in \mathbb{N}$. Let $\{z_m^*\} \subset JC_{k+1}$ with $\lim_{m\to\infty} z_m^* = z^* \in E^*$ and define $z_m := J^{-1}z_m^*$. Then, $\{z_m\} \subset C_{k+1} \subset C_k$. Therefore, we have $\{z_m^*\} \subset JC_k$. Since JC_k is closed, then we obtain $z^* \in JC_k$ and hence $J^{-1}z^* \in C_k$. We also have from $z_m \in C_{k+1} \subset C_k$ that

$$\begin{aligned} & V(y_k, J^{-1}z^*) - V(x_k, J^{-1}z^*) \\ = \ & \|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, z^*\rangle \\ = \ & \lim_{m\to\infty} \left\{ \|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, z_m^*\rangle \right\} \end{aligned}$$

$$\begin{aligned} &= \lim_{m\to\infty}\big\{V(y_k,z_m)-V(x_k,z_m)\big\} \\ &\le \lim_{m\to\infty} 0 = 0 \end{aligned}$$

and hence $J^{-1}z^* \in C_{k+1}$. So, we have $z^* \in JC_{k+1}$. This implies that JC_{k+1} is closed. Let $u^*, v^* \in JC_{k+1}$ and let $\lambda \in (0,1)$. Then there exists $u, v \in C_{k+1} \subset C_k$ such that $u^* = Ju$ and $v^* = Jv$. Put $z^* := \lambda Ju + (1-\lambda)Jv$. From the convexity of JC_k, we obtain $z^* \in JC_k$ and hence $J^{-1}z^* \in C_k$. We also have from $u, v \in C_{k+1} \subset C_k$ that

$$\begin{aligned} &V(y_k, J^{-1}z^*) - V(x_k, J^{-1}z^*) \\ &\quad= \|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, z^*\rangle \\ &\quad= \|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, \lambda Ju + (1-\lambda)Jv\rangle \\ &\quad= \lambda\big\{\|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, Ju\rangle\big\} \\ &\qquad\quad +(1-\lambda)\big\{\|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, Jv\rangle\big\} \\ &\quad= \lambda\big\{V(y_k,u) - V(x_k,u)\big\} + (1-\lambda)\big\{V(y_k,v) - V(x_k,v)\big\} \\ &\quad\le \lambda\cdot 0 + (1-\lambda)\cdot 0 = 0 \end{aligned}$$

and hence $J^{-1}z^* \in C_{k+1}$. So, we have $z^* \in JC_{k+1}$. This implies that JC_{k+1} is convex. So, we have that JC_{k+1} is closed and convex. By induction, JC_n is closed and convex for each $n \in \mathbb{N}$.

We next show that $F(\mathcal{T}) \subset C_n$ for each $n \in \mathbb{N}$. It is clear that $F(\mathcal{T}) \subset E = C_1$. Suppose that $F(\mathcal{T}) \subset C_k$ for some $k \in \mathbb{N}$. Let $p \in F(\mathcal{T}) \subset \cap_{i=1}^{\infty} F(T_i)$. From the definition of T_k, we have

$$V(y_k, p) = V(T_k x_k, p) \le V(x_k, p)$$

and hence $p \in C_{k+1}$. This means $F(\mathcal{T}) \subset C_{k+1}$. So, we have $F(\mathcal{T}) \subset C_n$ for each $n \in \mathbb{N}$. From Theorem 3.1, we have that C_n is a sunny generalized nonexpansive retract of E for each $n \in \mathbb{N}$.

Put $X_n = \{z \in E : V(y_n, z) \le V(x_n, z)\}$ and $Y_n = C_n$. Then X_n and Y_n satisfies the assumption of Theorem 4.1 and $C_{n+1} = X_n \cap Y_n$. Therefore, by Theorem 4.1, the sequence $\{x_n\}$ converges strongly to $R_{F(\mathcal{T})}x$. □

Using Theorem 7.1 and Lemma 5.2, we obtain the following result for a W-mapping generated by a finite family of generalized nonexpansive mappings.

Theorem 7.2. *Let E be a uniformly smooth and uniformly convex Banach space, let $\{S_i\}_{i=1}^r$ be a finite family of generalized nonexpansive mappings from E into itself such that $\cap_{i=1}^r F(S_i)$ is nonempty and $F(S_i) = \check{F}(S_i)$ for each $i = 1, 2, \ldots, r$, and let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \le i \le r\}$ be a set in $(0,1]$ such that $\liminf_{n\to\infty} \alpha_{n,i}(1-\alpha_{n,i}) > 0$ and $\alpha_{n,i} \ne 1$ for each $i = 1, 2, \ldots, r-1$ and $n \in \mathbb{N}$. Let $\{W_n\}$ be a sequence of W-mappings of E into itself generated by $S_1, S_2, \ldots, S_r$ and $\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,r}$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$, $C_1 = E$, and*

$$\begin{cases} y_n = W_n x_n, \\ C_{n+1} = \{z \in C_n : V(y_n, z) \le V(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x, \ n = 1, 2, 3, \ldots. \end{cases} \tag{7.1}$$

Then $\{x_n\}$ *converges strongly to* Rx*, where* R *is a sunny generalized nonexpansive retraction of* E *onto* $\cap_{i=1}^r F(S_i)$*.*

Proof. Put $T_n = W_n$ for each $n \in \mathbb{N}$ and $\mathcal{T} = \{S_1, S_2, \dots, S_r\}$. From Lemma 5.2, we have that $\{T_n\}$ and $\mathcal{T}$ satisfy the condition $(**)$ with $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = \cap_{i=1}^r F(S_i)$. Therefore, using Theorem 7.1, we obtain the desired result. □

Using Theorem 7.1 and Lemma 5.5, we also obtain the following result for a block mapping generated by a finite family of generalized nonexpansive mappings.

Theorem 7.3. *Let* E *be a uniformly smooth and uniformly convex Banach space, let* $\{S_i\}_{i=1}^r$ *be a finite family of generalized nonexpansive mappings from* E *into itself such that* $F(S_i) = \check{F}(S_i)$ *for each* $i = 1, 2, \dots, r$ *and* $\cap_{i=1}^r F(S_i) \neq \emptyset$*. Let* $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq r\} \subset [0, 1)$ *and* $\{\omega_n(i) : n, i \in \mathbb{N}, 1 \leq i \leq r\} \subset (0, 1]$ *be sets such that* $\liminf_n \alpha_{n,i}(1 - \alpha_{n,i}) > 0$, $\liminf_n \omega_n(i) > 0$ *for each* $i = 1, 2, \dots, r$ *and* $\sum_{i=1}^r \omega_n(i) = 1$ *for all* $n \in \mathbb{N}$*, let* $\{U_n\}$ *be a sequence of block mappings generated by* $S_1, S_2, \dots, S_r$*,* $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ *and* $\omega_n(1), \omega_n(2), \dots, \omega_n(r)$*. Let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in E$*,* $C_1 = E$*, and*

$$\left\{ \begin{array}{l} y_n = U_n x_n, \\ C_{n+1} = \{z \in C_n : V(y_n, z) \leq V(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x, \ n = 1, 2, 3, \dots . \end{array} \right. \tag{7.2}$$

Then $\{x_n\}$ *converges strongly to* Rx*, where* R *is a sunny generalized nonexpansive retraction of* E *onto* $\cap_{i=1}^r F(S_i)$*.*

Proof. Put $T_n = U_n$ for each $n \in \mathbb{N}$ and $\mathcal{T} = \{S_1, S_2, \dots, S_r\}$. As in the proof of Theorem 7.2, by Theorem 7.1 and Lemma 5.5 we obtain the desired result. □

As a direct consequence of Theorem 7.3, we have the following result.

Theorem 7.4. *Let* E *be a uniformly smooth and uniformly convex Banach space, let* T *be a generalized nonexpansive mapping from* E *into itself such that* $F(T) = \check{F}(T)$ *and let* $\{\alpha_n\}$ *be a sequence in* $[0, 1)$ *such that* $\liminf_n \alpha_n(1 - \alpha_n) > 0$*. Let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in E$*, and*

$$\left\{ \begin{array}{l} y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_{n+1} = \{z \in C_n : V(y_n, z) \leq V(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x, \ n = 1, 2, 3, \dots . \end{array} \right.$$

Then $\{x_n\}$ *converges strongly to* Rx*, where* R *is a sunny generalized nonexpansive retraction of* E *onto* $F(T)$*.*

Proof. In the case of $r = 1$, we know that $U_n = \alpha_n x_n + (1 - \alpha_n) T x_n$ for each $n \in \mathbb{N}$. Therefore, by Theorem 7.3 we obtain the desired result. □

8. Deduced results

In this section, we consider the problem of image recovery in a Banach space by using the sunny generalized nonexpansive retraction which is a generalization of the metric projection in a Hilbert space. These results for sunny generalized nonexpansive retractions are new.

Using Theorems 6.2 and 6.3, and Lemma 2.5, we first obtain the following two results which are connected with the feasibility problem and the problem of image recovery.

Corollary 8.1. *Let E be a uniformly smooth and uniformly convex Banach space, let $\{D_i\}_{i=1}^r$ be a finite family of sunny generalized nonexpansive retracts of E such that $\cap_{i=1}^r D_i$ is nonempty, and let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq r\}$ be a set in $(0,1]$ such that $\liminf_{n\to\infty} \alpha_{n,i}(1-\alpha_{n,i}) > 0$ and $\alpha_{n,i} \neq 1$ for each $i = 1, 2, \dots, r-1$ and $n \in \mathbb{N}$. Let $\{W_n\}$ be a sequence of W-mappings of E into itself generated by $R_1, R_2, \dots, R_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$, where each R_i is the sunny generalized nonexpansive retraction of E onto D_i. Let $\{x_n\}$ be a sequence generated by (6.1). Then $\{x_n\}$ converges strongly to Rx, where R is a sunny generalized nonexpansive retraction of E onto $\cap_{i=1}^r D_i$.*

Proof. From Lemma 2.5, we know that since each R_i is a sunny generalized nonexpansive retraction, $\check{F}(R_i) = F(R_i)$ for each i. So, we have the desired result by Theorem 6.2. □

Corollary 8.2. *Let E be a uniformly smooth and uniformly convex Banach space, let $\{D_i\}_{i=1}^r$ be a finite family of sunny generalized nonexpansive retracts of E such that $\cap_{i=1}^r D_i \neq \emptyset$. Let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq r\} \subset [0,1)$ and $\{\omega_n(i) : n, i \in \mathbb{N}, 1 \leq i \leq r\} \subset (0,1]$ be sets such that $\liminf_n \alpha_{n,i}(1-\alpha_{n,i}) > 0$, $\liminf_n \omega_n(i) > 0$ for each $i = 1, 2, \dots, r$ and $\sum_{i=1}^r \omega_n(i) = 1$ for all $n \in \mathbb{N}$, let $\{U_n\}$ be a sequence of block mappings generated by $R_1, R_2, \dots, R_r$, $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ and $\omega_n(1), \omega_n(2), \dots, \omega_n(r)$, where each R_i is the sunny generalized nonexpansive retraction of E onto D_i. Let $\{x_n\}$ be a sequence generated by (6.2). Then $\{x_n\}$ converges strongly to Rx, where R is a sunny generalized nonexpansive retraction of E onto $\cap_{i=1}^r D_i$.*

Proof. In the same way as Corollary 8.1, we have the desired result by Theorem 6.3 and Lemma 2.5. □

Next, using Theorems 7.2 and 7.3, and Lemma 2.5, we obtain the following two results which are connected with the feasibility problem and the problem of image recovery.

Corollary 8.3. *Let E be a uniformly smooth and uniformly convex Banach space, let $\{D_i\}_{i=1}^r$ be a finite family of sunny generalized nonexpansive retracts of E such that $\cap_{i=1}^r D_i$ is nonempty, and let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq r\}$ be a set in $(0,1]$ such that $\liminf_{n\to\infty} \alpha_{n,i}(1-\alpha_{n,i}) > 0$ and $\alpha_{n,i} \neq 1$ for each $i = 1, 2, \dots, r-1$ and $n \in \mathbb{N}$. Let $\{W_n\}$ be a sequence of W-mappings of E into itself generated by $R_1, R_2, \dots, R_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$, where each R_i is the sunny generalized nonexpansive retraction of E onto D_i. Let $\{x_n\}$ be a sequence generated by (7.1). Then $\{x_n\}$ converges strongly to Rx, where R is a sunny generalized nonexpansive retraction of E onto $\cap_{i=1}^r D_i$.*

Proof. From Lemma 2.5, we know that since each R_i is a sunny generalized nonexpansive retraction, $\check{F}(R_i) = F(R_i)$ for each i. So, we have the desired result by Theorem 7.2. □

Corollary 8.4. *Let E be a uniformly smooth and uniformly convex Banach space, let $\{D_i\}_{i=1}^r$ be a finite family of sunny generalized nonexpansive retracts of E such* *that* $\cap_{i=1}^r D_i \neq \emptyset$. *Let* $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq r\} \subset [0, 1)$ *and* $\{\omega_n(i) : n, i \in \mathbb{N}, 1 \leq i \leq r\} \subset (0, 1]$ *be sets such that* $\liminf_n \alpha_{n,i}(1-\alpha_{n,i}) > 0$, $\liminf_n \omega_n(i) > 0$ *for each* $i = 1, 2, \ldots, r$ *and* $\sum_{i=1}^r \omega_n(i) = 1$ *for all* $n \in \mathbb{N}$, *let* $\{U_n\}$ *be a sequence of block mappings generated by* $S_1, S_2, \ldots, S_r$, $\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,r}$ *and* $\omega_n(1), \omega_n(2), \ldots, \omega_n(r)$, *where each* R_i *is the sunny generalized nonexpansive retraction of* E *onto* D_i. *Let* $\{x_n\}$ *be a sequence generated by* (7.2). *Then* $\{x_n\}$ *converges strongly to* Rx, *where* R *is a sunny generalized nonexpansive retraction of* E *onto* $\cap_{i=1}^r D_i$.

Proof. In the same way as Corollary 8.3, we have the desired result by Theorem 7.3 and Lemma 2.5. □

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