Journal of Nonlinear and Convex Analysis
Volume 12, Number 1, 2011, 69–80

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

A NEW CHARACTERIZATION ON OPTIMALITY AND DUALITY FOR NONDIFFERENTIABLE MINIMAX FRACTIONAL PROGRAMMING PROBLEMS

HANG-CHIN LAI AND JEN-CHWAN LIU

Abstract. In this paper, we employ the elementary method and technique to prove the necessary and sufficient optimality conditions for nondifferentiable minimax fractional programming problem involving convexity. By the process, we deduce the formulation of a one parametric dual problem in which we prove three duality theorems: weak duality, strong duality, and strict converse duality theorem.

1. Introduction

In an optimization problem, one may consider the objective function which is a finite system of ratio positive functions with convex/concave functions as the form:

$$\min_{x\in X}\max_{1\leq i\leq r}\frac{f_i(x)}{g_i(x)}$$

where X is a feasible solution set in $\Re^n$. Sometimes we would like to transform such problem into a nonfractional problem with a parameter as the following form:

$$\min_{x\in X}\max_{1\leq i\leq r}[f_i(x)-\lambda g_i(x)]$$

by suitable constraints. Because the new objective function is produced by the sum of two convex functions, it still preserves the convex properties. However, when we can treat the finite system of minimum programming problem such as $\min_{x\in X}\max_{1\leq i\leq r}\frac{f_i(x)}{g_i(x)}$ by changing $\{1,2,\cdots,r\}$ to be a compact space Y in $\Re^m$, we wonder whether there have the same optimal solution between $\min_{x\in X}\max_{y\in Y}\frac{f(x,y)}{g(x,y)}$ and $\min_{x\in X}\max_{y\in Y}[f(x,y)-\lambda g(x,y)]$.

Furthermore, due to the different viewpoints for the various types of objective functions in differentiable or nondifferentiable fractional programming problems, many authors intend to establish the Kuhn Tucker type conditions and employ the conditions to search optimal solutions. For these tasks, one can consult [1, 10-18]. For instance, Lai et al. [8] have obtained the Kuhn Tucker type necessary optimality conditions for nondifferentiable fractional programming problems, yet they needed some complicated assumptions (constraint qualifications) to derive the necessary optimality conditions. Furthermore, from the necessary conditions, we also establish the sufficient optimality conditions with extra assumptions.

2010 *Mathematics Subject Classification.* Primary 26A51, 49A50, 90C25.

Key words and phrases. Minimax fractional programming, weak duality theorem, strict converse duality theorem.

Research supported by National Science Council, Taiwan.

In this paper, we will improve the complexity method of [8], which has been studied by Lai et. al as the following form:

$$(P_0) \qquad \min_{x\in X} \sup_{y\in Y} \frac{\phi(x,y)+(x^TAx)^{1/2}}{\psi(x,y)-(x^TBx)^{1/2}}$$
$$\text{subject to} \quad h_j(x)\le 0, \quad j=1,2,\ldots,p, \quad x\in\Re^n.$$

It is obvious that if x^TAx or x^TBx is equal zero at some point $x_0\in\Re^n$, the problem (P_0) is a nondifferentiable minimax fractional programming problem. For this purpose, we consider a more general nondifferentiable minimax fractional programming problem as follows

$$(P) \qquad \min_{x\in X} \max_{y\in Y} \frac{f(x,y)}{g(x,y)} (= \min_{x\in X} F(x))$$
$$\text{subject to} \quad X=\{\, x\in K \mid h_j(x)\le 0, \quad j=1,2,\ldots,p \,\}$$

where K is a compact convex subset of $\Re^n$ and Y is a compact subset of $\Re^m$. For each $y\in Y$, $f(\cdot,y):\Re^n\times\Re^m\to\Re$ and $-g(\cdot,y):\Re^n\times\Re^m\to\Re$ and $h_j(\cdot):\Re^n\to\Re$ for $j=1,2,\ldots,p$ are real-valued convex subdifferentiable functions. In particular, if $f(x,y)=\phi(x,y)+(x^TAx)^{1/2}$, $g(x,y)=\psi(x,y)-(x^TBx)^{1/2}$ and $K=\Re^n$, where A and B are positive semidefinite $n\times n$ matrices, both functions $\phi(\cdot,\cdot):\Re^n\times\Re^m\to\Re$ and $\psi(\cdot,\cdot):\Re^n\times\Re^m\to\Re$ are C^1 functions on $\Re^n\times\Re^m$, and $h_j(\cdot):\Re^n\to\Re$ for $j=1,2,\ldots,p$ are C^1 map on $\Re^n$, then the problem (P) becomes a special nondifferentiable minimax fractional programming problem (P_0).

In this paper, without loss of generality, we may assume that

$$f(x,y)\ge 0 \quad \text{and} \quad g(x,y)>0 \quad \text{for all} \quad (x,y)\in X\times Y.$$

We will treat problem (P) by parametric process to establish necessary and sufficient optimality conditions for the nondifferentiable minimax fractional programming problems (P). In Section 2, we will introduce some known notations and main lemmas. Then we establish the necessary and sufficient optimality conditions for (P) by parametric method in Section 3. Finally, we not only formulate a one-parametric dual problem but also prove the duality theorems in Section 4.

2. Notations and preliminary results

Throughout the paper, $\Re^n$ is the n-dimensional Euclidean space and $\Re^n_+$ is its nonnegative orthant. For a continuous function $\Phi:\Re^n\to\Re$, the subdifferential of Φ at x is defined and denoted by the set

$$\partial\Phi(x)=\left\{\, \xi\in\Re^n \mid \Phi(u)-\Phi(x)\ge \xi^T(u-x) \quad \text{for any} \quad u\in\Re^n \,\right\}.$$

If $\partial\Phi(x)\ne\emptyset$, we say that the Φ is subdifferentiable at x and any element ξ in $\partial\Phi(x)$ is called the subgradient of Φ at x. We note that a convex function is subdifferentiable. In this paper, we will prove that the necessary optimality condition is also the sufficient optimality condition. We would suppose that functions $f,-g$ and h are convex throughout.

In [7], Kanniappan and Sastry consider the following convex programming problem:

$$(SP) \qquad \text{minimize } \Phi(x) \quad \text{subject to} \quad x \in X \subset \Re^n$$

where X is the feasible set in problem (P). They derived the necessary and sufficient optimality conditions for (SP). We state it as a result in the following lemma:

Lemma 2.1. *Suppose that problem (SP) satisfies a regularity condition, that is, there exists $x' \in X$ such that $h_j(x') < 0$ for all $j = 1, 2, \ldots, p$. Let Φ and h_j, $j = 1, 2, \ldots, p$, be nonsmooth convex functions on $\Re^n$. Then a point $x_0 \in \Re^n$ is an optimal solution of (SP) if and only if there exists a multiplier $\mu^* \in \Re^p_+$ such that*

$$0 \in \partial\,\Phi(x_0) + \sum_{j=1}^{p} \mu_j^*\, \partial\, h_j(x_0) + N(x_0\,/\,K) \quad \textit{and} \quad \sum_{j=1}^{p} \mu_j^*\, h_j(x_0) = 0,$$

where $N(x_0\,/\,K)$, the normal cone with respect to K at x_0, is defined by

$$N(x_0\,/\,K) = \{\,\eta \in \Re^n \mid \eta^T(x - x_0) \le 0 \quad \textit{for all} \quad x \in K\,\}.$$

From the assumption in problem (P), since Y is compact, there exists a positive number λ depending on x $(x \in X)$ such that

$$\max_{y \in Y} \frac{f(x,y)}{g(x,y)} = \lambda$$

and

$$\frac{f(x,y)}{g(x,y)} \le \lambda \quad \text{for all} \quad y \in Y \quad \text{or} \quad f(x,y) - \lambda g(x,y) \le 0 \quad \text{for all} \quad (x,y) \in X \times Y.$$

Consequently, we can reduce the problem (P) to an equivalent nonfractional parametric problem:

$$(P_\lambda) \qquad v(\lambda) = \min_{x \in X} \max_{y \in Y} \,(f(x,y) - \lambda g(x,y)) \quad (\le 0)$$

where $\lambda \in \Re_+ = [0, \infty)$ is a parameter. For convenience, we define for each $x \in X$,

$$Y(x) = \left\{ y \in Y \mid \frac{f(x,y)}{g(x,y)} = \max_{z \in Y} \frac{f(x,z)}{g(x,z)} \right\} \qquad \text{and}$$

$$Y_\lambda(x) = \left\{ y \in Y \mid f(x,y) - \lambda g(x,y) = \max_{z \in Y} \,(f(x,z) - \lambda g(x,z)) \right\}.$$

We can prove that the problem (P) is equivalent to the problem (P_{λ^*}) for the optimal value λ^* of (P). We note that the functions in the following lemma only need the continuity but don't need convexity. This lemma is our main technique to derive the necessary and sufficient optimality conditions for problem (P).

Lemma 2.2.

(a) *Problem (P) has an optimal solution x_0 with optimal value λ^* if and only if $v(\lambda^*) = 0$ and x_0 is an optimal solution of (P_{λ^*}).*

(b) *If x_0 is an optimal solution of (P) with optimal value λ^*, then $Y(x_0) = Y_{\lambda^*}(x_0)$.*

Proof. (a) If x_0 is an optimal solution of (P) with optimal value $\lambda^* = \lambda^*(x_0)$, that is,

$$\lambda^* = \min_{x\in X}\max_{z\in Y}\frac{f(x,z)}{g(x,z)} = \max_{z\in Y}\frac{f(x_0,z)}{g(x_0,z)} \geq \frac{f(x_0,z)}{g(x_0,z)} \quad \text{for all} \quad z. \tag{2.1}$$

It follows that

$$f(x_0,z) - \lambda^* g(x_0,z) \leq 0 \quad \text{for all} \quad z \in Y. \tag{2.2}$$

Thus, we have

$$\min_{x\in X}\max_{z\in Y}\,(f(x,z) - \lambda^* g(x,z)) \leq \max_{z\in Y}\,(f(x_0,z) - \lambda^* g(x_0,z)) \leq 0.$$

Since $\max_{z\in Y}\,(f(\cdot,z) - \lambda^* g(\cdot,z))$ is a continuous function on the compact set X for any $z \in Y \subset \Re^m$, there exists $x_1 \in X$ such that

$$\min_{x\in X}\max_{z\in Y}\,(f(x,z) - \lambda^* g(x,z)) = \max_{z\in Y}\,(f(x_1,z) - \lambda^* g(x_1,z)) \leq 0. \tag{2.3}$$

The expression (2.3) is exactly equal to 0. If not, it would be

$$\min_{x\in X}\max_{z\in Y}\,(f(x,z) - \lambda^* g(x,z)) < 0, \tag{2.4}$$

that is,

$$f(x_1,z) - \lambda^* g(x_1,z) < 0 \quad \text{for all} \quad z \in Y.$$

It follows that

$$\frac{f(x_1,z)}{g(x_1,z)} < \lambda^* \quad \text{for all} \quad z \in Y.$$

Thus

$$\max_{z\in Y}\frac{f(x_1,z)}{g(x_1,z)} < \lambda^*. \tag{2.5}$$

On the other hand, the expression (2.1) yields the inequality

$$\max_{z\in Y}\frac{f(x_1,z)}{g(x_1,z)} \geq \lambda^*,$$

which contradicts the inequality (2.5). Hence, (2.4) is not true, and (2.3) becomes

$$\min_{x\in X}\max_{z\in Y}\,(\,f(x,z) - \lambda^* g(x,z)\,) = 0. \tag{2.6}$$

Similarly, the inequality (2.2) will be

$$\max_{z\in Y}\,(\,f(x_0,z) - \lambda^* g(x_0,z)\,) = 0. \tag{2.7}$$

Then (2.7) yields

$$\min_{x\in X}\max_{z\in Y}\,(\,f(x,z) - \lambda^* g(x,z)\,) = \max_{z\in Y}\,(\,f(x_0,y) - \lambda^* g(x_0,z)\,) = 0.$$

Therefore, x_0 is an optimal solution of (P_{λ^*}) and $v(\lambda^*) = 0$.
Conversely, if x_0 is an optimal solution of (P_{λ^*}) with optimal value $v(\lambda^*) = 0$, then

$$\min_{x\in X}\max_{z\in Y}\,(\,f(x,z) - \lambda^* g(x,z)\,) = \max_{z\in Y}\,(\,f(x_0,y) - \lambda^* g(x_0,z)\,) = 0. \tag{2.8}$$

It follows that

$$\max_{z\in Y}\,(f(x,z) - \lambda^* g(x,z)) \geq 0 \quad \text{for all} \quad x \in X.$$

Since Y is a compact subset of $\Re^m$, by the continuity of $f(x,\cdot) - \lambda^* g(x,\cdot)$, there exists $z_1 \in Y$ such that

$$\max_{z\in Y} \left(f(x,z) - \lambda^* g(x,z) \right) = f(x,z_1) - \lambda^* g(x,z_1) \geq 0.$$

It follows that

$$\frac{f(x,z_1)}{g(x,z_1)} \geq \lambda^* \quad \text{for all} \quad x \in X.$$

Whence,

$$\min_{x\in X} \max_{z\in Y} \frac{f(x,z)}{g(x,z)} \geq \lambda^*. \tag{2.9}$$

It remains to show the impossibility of

$$\min_{x\in X} \max_{z\in Y} \frac{f(x,z)}{g(x,z)} > \lambda^*. \tag{2.10}$$

Actually, as Y is a compact subset in $\Re^m$, there is a point z_2 in Y such that

$$\max_{z\in Y} \frac{f(x,z)}{g(x,z)} = \frac{f(x,z_2)}{g(x,z_2)} > \lambda^* \quad \text{for all} \quad x \in X$$

and

$$f(x,z_2) - \lambda^* g(x,z_2) > 0 \quad \text{for all} \quad x \in X.$$

It follows that

$$\max_{z\in Y} \left(f(x,z) - \lambda^* g(x,z) \right) \geq f(x,z_2) - \lambda^* g(x,z_2) > 0 \quad \text{for all} \quad x \in X,$$

and

$$\min_{x\in X} \max_{z\in Y} \left(f(x,z) - \lambda^* g(x,z) \right) > 0,$$

which contradicts the equality (2.8). Hence the inequality (2.10) is not true, and so (2.9) must be

$$\min_{x\in X} \max_{z\in Y} \frac{f(x,z)}{g(x,z)} = \lambda^*. \tag{2.11}$$

Since x_0 is an optimal solution of (P_{λ^*}) with optimal value $v(\lambda^*) = 0$, by the similar way, we can prove that

$$\min_{x\in X} \max_{z\in Y} \frac{f(x,z)}{g(x,z)} = \max_{z\in Y} \frac{f(x_0,z)}{g(x_0,z)} = \lambda^*. \tag{2.12}$$

Therefore, x_0 is an optimal solution of (P) with optimal value λ^*.

(b) The element y in $Y(x_0)$ means that

$$\frac{f(x_0,y)}{g(x_0,y)} = \max_{z\in Y} \frac{f(x_0,z)}{g(x_0,z)} = \lambda^* \quad \text{or} \quad f(x_0,y) - \lambda^* g(x_0,y) = 0. \tag{2.13}$$

Since x_0 is an optimal solution of (P), by part (a), we have

$$\max_{z\in Y} (f(x_0,z) - \lambda^* g(x_0,z)) = 0. \tag{2.14}$$

From the equations (2.13) and (2.14), we have

$$\max_{z\in Y} \left(f(x_0,z) - \lambda^* g(x_0,z) \right) = f(x_0,y) - \lambda^* g(x_0,y).$$

That is, $y \in Y_{\lambda^*}(x_0)$. Hence, $Y(x_0) \subset Y_{\lambda^*}(x_0)$. Conversely, if $y \in Y_{\lambda^*}(x_0)$,

$$f(x_0, y) - \lambda^* g(x_0, y) = \max_{z \in Y} \left(f(x_0, z) - \lambda^* g(x_0, z) \right) = 0.$$

and

$$\frac{f(x_0, y)}{g(x_0, y)} = \lambda^* = \max_{z \in Y} \frac{f(x_0, z)}{g(x_0, z)}.$$

This shows that $y \in Y(x_0)$, so $Y(x_0) \supset Y_{\lambda^*}(x_0)$. Consequently, $Y(x_0) = Y_{\lambda^*}(x_0)$. □

The following lemma will play an important role in the proof of the main result.

Lemma 2.3 ([6, page 204, Theorem 4]). *For each $\xi \in \partial(\max_{z \in Y} f(\cdot, z))(x_0)$, there exist a positive integer k, $k \le n+1$, and $\mu_i > 0$ with $\sum_{i=1}^{k} \mu_i = 1$ such that*

$$\xi = \mu_1 \xi_1 + \mu_2 \xi_2 + \cdots + \mu_k \xi_k$$

where $\xi_i \in \partial f(\cdot, y_i^*)(x_0)$, $y_i^* \in \{ y \in Y \mid f(x_0, y) = max_{z \in Y} f(x_0, z) \}$ for $i = 1, 2, \ldots, k$.

3. Necessary and sufficient conditions

We will use the following notation. For each $x \in X$, we define

$$\begin{aligned} K_\lambda(x) = \{ (k, \mu, y) \in \aleph \times \Re_+^k \times \Re^{mk} \mid 1 \le k \le n+1;\ \mu = (\mu_1, \ldots, \mu_k) \in \Re_+^k, \\ \sum_{i=1}^{k} \mu_i = 1 \quad \text{and} \quad \mu_i > 0; \quad y = (y_1, \ldots, y_k) \\ \text{with} \quad y_i \in Y_\lambda(x) \quad \text{for all} \quad i = 1, \ldots, k \} \end{aligned} \tag{3.0}$$

and denote

$$F_\lambda(x) = \max_{y \in Y} \left(f(x, y) - \lambda g(x, y) \right).$$

In this section, using a parametric approach, we establish the following necessary and sufficient optimality conditions for the nondifferentiable minimax fractional programming problem (P).

Theorem 3.1 (Necessary and Sufficient Conditions). *Suppose that problem (P) satisfies a regularity condition, that is, there exists $x' \in X$ such that $h_j(x') < 0$ for all $j = 1, 2, \ldots, p$ and $f(\cdot, y), -g(\cdot, y)$ and h_j are real-valued convex subdifferentiable functions. Then x_0 is an optimal solution of (P) if and only if there exist $\lambda^* \in \Re_+$, $(k^*, \mu^*, y^*) \in K_{\lambda^*}(x_0)$ and $\beta^* \in \Re_+^p$ satisfying the Kuhn Tucker type conditions as follows:*

$$0 \in \sum_{i=1}^{k^*} \mu_i^* \left(\partial f(x_0, y_i^*) + \lambda^* \partial \left(-g(x_0, y_i^*) \right) \right) + \sum_{j=1}^{p} \beta_j^* \partial h_j(x_0) + N(x_0 / K); \tag{3.1}$$

$$f(x_0, y_i^*) - \lambda^* g(x_0, y_i^*) = 0 \quad \textit{for all} \quad i = 1, 2, \ldots, k^*, \tag{3.2}$$

$$\sum_{j=1}^{p} \beta_j^* h_j(x_0) = 0, \tag{3.3}$$

and the value $\lambda^* = \max_{y \in Y} f(x_0, y)/g(x_0, y)$ is an optimal value of (P).

Proof. If x_0 is an optimal solution of (P) with optimal value $\lambda^* = \max_{y \in Y} f(x_0, y)/g(x_0, y)$, by Lemma 2.2, x_0 is also an optimal solution of (P_{λ^*}). Since $F_{\lambda^*}(x)$ is a convex continuous subdifferentiable function, by Lemma 2.1, there exists a Lagrange multiplier vector $\beta^* \in \Re^p_+$ such that

$$0 \in \partial F_{\lambda^*}(x_0) + \sum_{j=1}^{p} \beta_j^* \, \partial h_j(x_0) + N(x_0 \, / \, K)$$

and

$$\sum_{j=1}^{p} \beta_j^* \, h_j(x_0) = 0.$$

It follows from Lemma 2.3 that there exists $(\, k^*, \, \mu^*, \, y^* \,) \in K_{\lambda^*}(x_0)$ defined by (3.0) such that

$$0 \in \sum_{i=1}^{k^*} \mu_i^* \, (\, \partial f(x_0, y_i^*) + \lambda^* \, \partial \, (\, -g(x_0, y_i^*) \,) \,) + \sum_{j=1}^{p} \beta_j^* \, \partial h_j(x_0) + N(x_0 \, / \, K)$$

and

$$f(x_0, y_i^*) - \lambda^* g(x_0, y_i^*) = 0 \quad \text{for all} \quad i = 1, 2, \ldots, k^*.$$

Conversely, if there exist $(\, k^*, \, \mu^*, \, y^* \,) \in K_{\lambda^*}(x_0)$ and $\beta^* \in \Re^p_+$ that satisfy relations (3.1) and (3.2), then there exist $\xi_i \in \partial f(x_0, y_i^*)$, $\eta_i \in \partial \, (\, -g(x_0, y_i^*) \,)$ for $i = 1, 2, \ldots, k^*$, $\zeta_j \in \partial h_j(x_0)$ for $j = 1, 2, \ldots, p$, and $d \in N(x_0 \, / \, K)$ such that the $n-$vector

$$\sum_{i=1}^{k^*} \mu_i^* (\, \xi_i + \lambda^* \eta_i \,) + \sum_{j=1}^{p} \beta_j^* \zeta_j + d = 0.$$

Using the characterization of the subgradients, we obtain

$$f(x, y_i^*) \geq f(x_0, y_i^*) + \xi_i^T (x - x_0) \qquad 1 \leq i \leq k^*, \tag{3.4}$$

$$-g(x, y_i^*) \geq -g(x_0, y_i^*) + \eta_i^T (x - x_0) \qquad 1 \leq i \leq k^*, \tag{3.5}$$

$$h_j(x) \geq h_j(x_0) + \zeta_j^T (x - x_0) \qquad j = 1, 2, \ldots, p, \tag{3.6}$$

$$d^T (x - x_0) \leq 0, \quad \text{for all} \quad x \in K.$$

Now, multiplying (3.4) by μ_i^*, (3.5) by $\lambda^* \mu_i^*$, (3.6) by β_j^*, and then by summing up these inequalities, we obtain

$$\begin{aligned}
&\sum_{i=1}^{k^*} \mu_i^* \, (\, f(x, y_i^*) + \lambda^* \, (\, -g(x, y_i^*) \,) \,) + \sum_{j=1}^{p} \beta_j^* h_j(x) \\
&\quad - \sum_{i=1}^{k^*} \mu_i^* \, (\, f(x_0, y_i^*) + \lambda^* \, (\, -g(x_0, y_i^*) \,) \,) - \sum_{j=1}^{p} \beta_j^* h_j(x_0) \\
&\quad \geq (x - x_0)^T \left(\sum_{i=1}^{k^*} \mu_i^* (\, \xi_i + \lambda^* \eta_i \,) + \sum_{j=1}^{p} \beta_j^* \zeta_j \right) \\
&\quad = -(x - x_0)^T d \geq 0.
\end{aligned}$$

Using the feasibility of x for (P), $\beta^* \in \Re^p_+$, and the equality (3.3), we have

$$(3.7)\quad \sum_{i=1}^{k^*} \mu_i^* \left(f(x, y_i^*) + \lambda^* \left(-g(x, y_i^*) \right) \right) \geq \sum_{i=1}^{k^*} \mu_i^* \left(f(x_0, y_i^*) + \lambda^* \left(-g(x_0, y_i^*) \right) \right).$$

Suppose on the contrary that x_0 is not an optimal solution of (P). Then by Lemma 2.2, it is not an optimal solution of (P_{λ^*}). Hence there exists a (P)-feasible point x_1 in X such that

$$\begin{aligned} f(x_1, y_i^*) + \lambda^* (-g(x_1, y_i^*)) &= \max_{y \in Y} (f(x_1, y) + \lambda^* (-g(x_1, y))) \\ &< \max_{y \in Y} (f(x_0, y) + \lambda^* (-g(x_0, y))) \\ &= f(x_0, y_i^*) + \lambda^* (-g(x_0, y_i^*)) \end{aligned}$$

for all $i = 1, 2, \ldots, k^*$ and $y_i^* \in Y_{\lambda^*}(x_0) \subset Y$. From $\sum_{i=1}^{k^*} \mu_i^* = 1$, $\mu_i > 0$ for $i = 1, 2, \ldots, k^*$, we have

$$\sum_{i=1}^{k^*} \mu_i^* \left(f(x_1, y_i^*) + \lambda^* \left(-g(x_1, y_i^*) \right) \right) < \sum_{i=1}^{k^*} \mu_i^* \left(f(x_0, y_i^*) + \lambda^* \left(-g(x_0, y_i^*) \right) \right),$$

which contradicts the inequality (3.7). Hence the proof is complete. □

4. Parametric dual model

From the optimality conditions for problem (P) in the proceeding section, we can formulate the following parametric dual problem which is a dual problem (Theorem 3.1) with respect to the minimax problem (P):

$$(D_P) \qquad \max_{(k, \mu, y) \in K_\lambda(u)} \; \max_{(u, \beta, \lambda) \in H(k, \mu, y)} \lambda$$

where $H(k,\, \mu,\, y)$ denotes the set of all triples $(u,\, \beta,\, \lambda) \in \Re^n \times \Re^p_+ \times R_+$ which satisfies the Kuhn-Tucker type conditions (3.1) $\sim$ (3.3) of Theorem 3.1 in the alternative form:

$$(4.1)\qquad 0 \in \sum_{i=1}^{k} \mu_i \left(\partial f(u, y_i) + \lambda \, \partial \left(-g(u, y_i) \right) \right) + \sum_{j=1}^{p} \beta_j \, \partial h_j(u) + N(u \,/\, K),$$

$$(4.2)\qquad \sum_{j=1}^{k} \mu_i \left(f(u, y_i) + \lambda \left(-g(u, y_i) \right) \right) + \sum_{j=1}^{p} \beta_j h_j(u) \geq 0$$

where (4.1) and (4.2) are deduced from the optimality conditions (3.1), (3.2) and (3.3). If for a triple $(k,\, \mu,\, y) \in K_\lambda(u)$, the set $H(k,\, \mu,\, y)$ is empty, then we define the supremum over it to be $-\infty$.

Theorem 4.1 (Weak Duality). *Let x and $(u,\, \beta,\, \lambda,\, k,\, \mu,\, y)$ be (P)-feasible and (D_P)-feasible,respectively. Then*

$$F(x) = \max_{y \in Y} f(x, y)/g(x, y) \geq \lambda.$$

Proof. Suppose on the contrary that

$$\max_{y\in Y} \frac{f(x,y)}{g(x,y)} < \lambda. \tag{4.3}$$

Then

$$f(x,y) - \lambda g(x,y) < 0 \quad \text{for all} \quad y \in Y.$$

Since $\mu_i > 0$ with $\sum_{i=1}^{k} \mu_i = 1$ and $y_i \in Y_\lambda(u) \subset Y$, the above inequality follows

$$\mu_i\,(\, f(x,y_i) - \lambda g(x,y_i)\,) < 0 \quad \text{for all} \quad i = 1,2,\ldots,k,$$

and the sum over the index $i = 1,2,\ldots,k$ is

$$\sum_{i=1}^{k} \mu_i\,(\, f(x,y_i) + \lambda\,(\,-g(x,y_i)\,)\,) < 0. \tag{4.4}$$

By (4.1), there exist $\xi_i \in \partial f(u,y_i)$, $\eta_i \in \partial\,(-g(u,y_i))$ for $i = 1,2,\ldots,k$, $zeta_j \in \partial h_j(u)$ for $j = 1,2,\ldots,p$, and $d \in N(u\,/\,K)$ such that

$$\sum_{i=1}^{k} \mu_i(\xi_i + \lambda\eta_j) + \sum_{j=1}^{p} \beta_j\zeta_j + d = 0.$$

By the subdifferentiability of f, $-g$, and h_j for $j = 1,2,\ldots,p$, we have

$$f(x,y_i) \geq f(u,y_i) + \xi_i^T(x-u) \qquad 1 \leq i \leq k, \tag{4.5}$$

$$-g(x,y_i) \geq -g(u,y_i) + \eta_i^T(x-u) \qquad 1 \leq i \leq k, \tag{4.6}$$

$$h_j(x) \geq h(u) + \zeta_j^T(x-u) \qquad j = 1,2,\ldots,p, \tag{4.7}$$

$$d^T(x-u) \leq 0 \quad \text{for all} \quad x \in K.$$

Now, multiplying (4.5) by μ_i, (4.6) by $\lambda\mu_i$, (4.7) by β_j, and then by summing up these inequalities, we obtain

$$\begin{aligned}
&\sum_{i=1}^{k}\mu_i\,(\, f(x,y_i) + \lambda\,(\,-g(x,y_i)\,)\,) + \sum_{j=1}^{p} \beta_j h_j(x) \\
&\quad - \sum_{i=1}^{k} \mu_i\,(\, f(u,y_i) + \lambda\,(\,-g(u,y_i)\,)\,) - \sum_{j=1}^{p} \beta_j h_j(u) \\
&\quad \geq (x-u)^T \left(\sum_{i=1}^{k} \mu_i(\xi_i + \lambda\eta_i) + \sum_{j=1}^{p} \beta_j\zeta_j \right) \\
&\quad = -(x-u)^T d \geq 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{k}\mu_i\,(\, f(x,y_i) + \lambda\,(\,-g(x,y_i)\,)\,) + \sum_{j=1}^{p} \beta_j h_j(x) \\
&\quad \geq \sum_{i=1}^{k} \mu_i\,(\, f(u,y_i) + \lambda\,(\,-g(u,y_i)\,)\,) + \sum_{j=1}^{p} \beta_j h_j(u).
\end{aligned} \tag{4.8}$$

From relations (4.2), (4.8), and $\sum_{j=1}^{p} \beta_j h_j(x) \leq 0$, we have

$$\sum_{i=1}^{k} \mu_i \left(f(x, y_i) + \lambda \left(-g(x, y_i) \right) \right) \geq 0.$$

This contradicts the inequality (4.4). Thus, the inequality (4.3) is not true. This completes the proof. □

Theorem 4.2 (Strong Duality)**.** *Suppose that problem (P) satisfies a regularity condition. If x_0 is an optimal solution of (P), then there exist $\lambda^* \in \Re_+$, $(k^*, \mu^*, y^*) \in K_{\lambda^*}(x_0)$ and $\beta^* \in \Re^p_+$ such that $(x_0, \beta^*, \lambda^*, k^*, \mu^*, y^*)$ is an optimal solution of (D_P). Moreover the optimal values of (P) and (D_P) are equal, that is,* $\min(P) = \max(D_P)$.

Proof. By Theorem 3.1, there exist $\lambda^* \in \Re_+$, $(k^*, \mu^*, y^*) \in K_{\lambda^*}(x_0)$ and $\beta^* \in \Re^p_+$ such that $(x_0, \beta^*, \lambda^*, k^*, \mu^*, y^*)$ is a feasible solution of (D_P). Since

$$\lambda^* = \max_{y \in Y} \frac{f(x_0, y)}{g(x_0, y)},$$

the optimality of the feasible solution $(x_0, \beta^*, \lambda^*, k^*, \mu^*, y^*)$ for (D_P) reduces to be the maximum value of (D_P). This fact follows from Theorem 4.1. □

Theorem 4.3 (Strict Converse Duality)**.** *Suppose that problem (P) satisfies a regularity condition. Let x' and $(x_0, \beta^*, \lambda^*, k^*, \mu^*, y^*)$ be optimal solutions of (P) and (D_P), respectively. If one of the functions $f(\cdot, y)$, $-g(\cdot, y)$, and $h_j(\cdot)$ for $j = 1, 2, \ldots, p$ is strictly convex at x_0, then $x' = x_0$, that is, x_0 is an optimal solution of (P) and* $\max_{y \in Y} f(x', y)/g(x', y) = \lambda^*$.

Proof. Suppose on the contrary that $x' \neq x_0$. From Theorem 4.2, we know that there exist $\lambda' \in \Re_+$, $(k', \mu', y') \in K_{\lambda'}(x')$ and $\beta' \in \Re^m_+$ such that $(x', \beta', k', \mu', y')$ becomes an optimal solution of (D_P) with the optimal value

$$\max_{y \in Y} \frac{f(x', y)}{g(x', y)} = \lambda'.$$

Now as in the proof of Theorem 4.1 after replacing x by x' and $(u, \beta, \lambda, k, \mu, y)$ by $(x_0, \beta^*, \lambda^*, k^*, \mu^*, y^*)$, we will arrive at the strict inequality

$$\max_{y \in Y} \frac{f(x', y)}{g(x', y)} > \lambda^*.$$

This contradicts the fact that

$$\max_{y \in Y} \frac{f(x', y)}{g(x', y)} = \lambda' = \lambda^*.$$

Therefore, we conclude that $x' = x_0$, and

$$\max_{y \in Y} \frac{f(x', y)}{g(x', y)} = \lambda^*.$$

□

5. Remarks for further development

If the functions in an optimization problem are nonconvex but Lipschitz continuous, one can consult the generalized subgradient in Clarke [5]. The other hand. we can relax convexity to some kind of generalized convexity; for example, pseudoconvexity, quasiconvexity or (F, ρ)-convexity [3,13, 14, 18]. In addition, one can study problem (P) in complex variable, e.g., [2]:

$$\min_{\zeta\in X} \max_{\eta\in Y} \frac{Re\, f(\zeta,\eta)}{Re\, g(\zeta,\eta)}$$
$$\text{subject to} \quad \zeta \in S^0 = \left\{ \zeta \in C^{2n} \mid -h(\zeta) \in S \right\}.$$

References

[1] S. Chandra and V. Kumar, *Duality in fractional minimax programing*, Journal of the Australian Mathematical Society **Ser.A 58** (1995), 376–386 .

[2] J. C. Chen, H. C. Lai and S. Schaible, *Complex fractional programming and Charnes-Cooper transformation*, Journal of Optimization Theory and Applications **126** (2005), 203–213.

[3] J. C. Chen and H. C. Lai, *Fractional programming for variational problem with (F, ρ, θ)-invexity*, Journal of Nonlinear and Convex Analysis **4** (2003), 25–41.

[4] A. Chinchuluun, P. M. Pardalos, A. Migdalas and L. Pitsoulis, *Pareto Optimality, Game Theory and Equilibria*, Springer, Berlin, 2008.

[5] F. H. Clarke, *Optimization Analysis, 2en Edition*, Classics in Applied Mathematics (SIAM), Philadephia, PA, 1990.

[6] A. D. Ioffe and V. M. Tihomirov, *Theory of Extremal Problems*, Elsevier North-Holland, Amsterdam, New York, 1979.

[7] P. Kanniappan and S. M. Sastry, *Duality theorems and optimality conditions for nondifferentiable convex programming*, Journal of the Australian Mathematical Society **Ser. A32** (1982), 369–378.

[8] H. C. Lai, J. C. Liu and K. Tanaka, *Necessary and sufficient conditions for minimax fractional programming*, Journal of Mathematical Analysis and Applications **230** (1999), 311–328.

[9] H. C. Lai, *On a dynamic fractional game*, Journal of Mathematical Analysis and Applications **294** (2004), 644–654.

[10] H. C. Lai and J. C. Liu, *Minimax fractional programming involving generalized invex functions*, ANZIAM Journal **44** (2003), 339–354.

[11] H. C. Lai and J. C. Lee, *On duality for a nondifferentiable minimax fractional programming*, Journal of Computational and Applied Mathematics **146** (2002), 115-126.

[12] J. C. Liu, C. S. Wu and R. L. Sheu, *Duality for fractional minimax programming*, Optimization **41** (1997), 117–133.

[13] J. C. Liu and C. S. Wu, *On minimax fractional optimality conditions with invexity*, Journal of Mathematical Analysis and Applications **219** (1998), 21–35.

[14] J. C. Liu and C. S. Wu, *On minimax fractional optimality conditions with (F, ρ) convexity*, Journal of Mathematical Analysis and Applications **219** (1998), 36–51.

[15] W. E. Schmittendorf, *Necessary conditions and sufficient conditions for static minimax programming*, Journal of Mathematical Analysis and Applications **57** (1977), 683–693.

[16] C. Singh, *Optimality conditions for fractional minimax programming*, Journal of Mathematical Analysis and Applications **100** (1984), 409–415.

[17] S. R. Yadav and R. N. Mukherjee, *Duality for fractional minimax programming problems*, Journal of the Australian Mathematical Society **Ser.B 31** (1990), 484–492.

[18] D. H. Yuan, X. L. Liu , A. Chinchuluun and P. M. Pardalos, *Nondifferentiable minimax fractional programming problems with $(C; \alpha, \rho, d)$-convexity*, Journal of Optimization Theory and Applications, **129** (2006),185-199.

Manuscript received December 6, 2010
revised December 17, 2010

Hang-Chin Lai
Department of Applied Mathematics, Chung Yuan Christian Univeristy, Taiwan and, Department of Mathematics, National Tsing Hua University, Taiwan,
E-mail address: hclai@cycu.edu.tw and laihc@mx.nthu.edu.tw

Jen-Chwan Liu
National Taiwan Normal University, Linkou Campus, Taiwan
E-mail address: liujc@ntnu.edu.tw