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REMOTAL POINTS AND A KREIN-MILMAN TYPE THEOREM

M. SABABHEH AND R. KHALI

Abstract. In this article we study the connection of remotal points, extreme points and exposed points. Namely, we prove that a uniquely remotal point is necessarily an exposed point but not vice versa. We give examples where some implications are not valid and we propose some questions regarding the problem. Then, we introduce a new class of points that play the role of extreme points and prove a Krein-Milman type Theorem.

1. Introduction

Let X be a Banach space and let E be a closed bounded subset of X. For $x \in X$ we denote $\sup\{\|x-e\| : e \in E\}$ by $D(x,E)$. The set E is said to be remotal in X (or remotal) if $D(x,E)$ is attained for all $x \in X$. That is, if for each $x \in X$ there exists $e \in E$ such that $D(x,E) = \|x-e\|$. The set at which $D(x,E)$ is attained will be denoted by $F(x,E)$. Thus, $F(x,E)$ denotes the set of points in E which are farthest from x. The set E will be said to be uniquely remotal if $F(x,E)$ is a singleton for each $x \in X$. On the other hand, a point $e \in E$ will be called remotal if $e \in F(x,E)$ for some $x \in X$. If e is remotal and if $F(x,E) = \{e\}$ for some $x \in X$ then e will be called a uniquely remotal point. Finally, if E is a convex set, then a point $e \in E$ will be called an extreme point of E if e is not a middle point of two points in E. That is, if e cannot be written as a convex combination $te_1 + (1-t)e_2$ with $0 < t < 1$ and $e_1, e_2 \in E$.

It can be easily seen that, see [2], if $e \in E$ is a uniquely remotal point then e is an extreme point of E. Our first objective of this article is to study the converse of this result. That is, if $e \in E$, a convex set, is extreme, must e be uniquely remotal? or remotal? We shall see that the answer is negative, see Proposition 2.2. Then, we shall study this question from different points of view to find out that extreme, and in fact boundary, points are remotal points under some equivalent norm, see Theorem 2.5.

Our second objective is to introduce a class of points that does the job of extreme points for compact convex sets. This gives some Krein-Milman Type Theorems.

2. Remotality of extreme and exposed points

Since every remotal point of E is necessarily a boundary point of E, one may ask about the nature of boundary points which are necessarily remotal points. In fact, at first glance, extreme points and exposed pints are the very first candidates for being remotal points. But, unfortunately, this hunch is not true as the following results show. First, a lemma.

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Lemma 2.1. *Let e be a boundary point of the closed bounded set E, a subset of the Banach space X. Then e is a remotal point if and only if there exists a ball $B(x,r)$ for some $x \in X$ and $r > 0$ such that $E \subset B(x,r)$ and $e \in E \cap S(x,r)$.*

The proof of this lemma is easily obtained and hence, is left to the reader.

Proposition 2.2. *There exist convex sets with extreme points that are not remotal points.*

Proof. Let $X = \mathbb{R}^2$. Let S_1 be the unit circle in X when endowed with the Euclidian norm and let S_2 be the unit circle in X when endowed with the sup norm. Let E be the convex hull of the union of $\{(x,y) \in S_1 : x \geq 0\}$ and $\{(x,y) \in S_2 : x \leq 0\}$. It is clear that the point $A(0,1)$ is an extreme point of E. We assert that A is not a remotal point. That is, we show that $A \notin F(x,E)$ for any $x \in X$, X being endowed with the Euclidian norm.
To do so, we use Lemma 2.1. Thus, suppose on the way of contrary that a circle $x^2 + y^2 + ax + by + c = 0$ exists such that A belongs to the circle and such that E is inside the circle. For A to be on the circle, we need to have $c = -b - 1$. Hence, the equation of the circle becomes $x^2 + y^2 + ax + by = b + 1$. Since E lies inside the circle, $B(-1,1), C(-1,-1)$ and $D(1,0)$ must be inside the circle. Now, for B to be inside the circle we must have $a \geq 1$ and for D to be inside the circle we need to have $a \leq b$.
Now let (x,y) be any point on the right half of the unit circle. That is $x^2 + y^2 = 1$ and $x \geq 0$. The question is "Does (x,y) lie inside the circle $x^2+y^2+ax+by = b+1?''$ That is "Does (x,y) satisfy $x^2 + y^2 + ax + by \leq b + 1?''$. Since $x^2 + y^2 = 1$, we need to check whether $ax + by \leq b$ or not. Thus the question reduces to: What is the maximum value of $ax+by$ subject to the conditions $x^2+y^2 = 1$ and $x \geq 0$? If we use the method of Lagrange multipliers we find that the point $\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$ is on the right half of the unit circle which lies outside the circle $x^2 + y^2 + ax + by = b + 1$. This shows that no circle passing through A can satisfy the conditions of Lemma 2.1. Hence, A is not a remotal point of E. □

It is worth to remark that the point A above is a remotal point of E, as above, if $\mathbb{R}^2$ is endowed with the sup norm. Thus, the above example is an example of a point $A \in E$ which is not remotal when $E \subset (\mathbb{R}^2, \| \|_2)$ but is remotal when $E \subset (\mathbb{R}^2, \| \|_\infty)$ although $\| \|_2$ and $\| \|_\infty$ are equivalent.

Thus, equivalence of norms does not preserve the remotality of points.

In fact, Proposition 2.2 opens a wide door of questions about the nature of extreme points which must be remotal. Recall that an exposed point e of a convex set E is a boundary point of E such that a functional $f \in X^*$ exists with the properties

$$f(e) = \alpha \text{ and } f(x) < \alpha \; \forall x \in E \backslash \{e\}.$$

In words, it means that a hyperplane H exists such that e is the only element of E that belongs to H and E lies entirely in one side of H.
It is clear that an exposed point is necessarily an extreme point but not vice versa. The example of Proposition 2.2 is an example of an extreme point which is not exposed and not remotal.

In fact, even exposed points in very nice spaces are not necessarily remotal points. The following proposition gives an example.

Proposition 2.3. *There are closed bounded convex sets in Hilbert spaces whose exposed points are not necessarily remotal points.*

Proof. Let X be $\mathbb{R}^2$ endowed with the standard norm and let

$$E_0 = \left\{ \left(\pm\frac{1}{n}, \frac{1}{n^3} \right) : n \in \mathbb{N} \right\}.$$

If E is the closed convex hull of E_0, then clearly E lies above the $x-$axis which touches E uniquely at $(0,0)$. That is, $(0,0)$ is an exposed point of E.
We assert that $(0,0)$ is not a remotal point for E. Observe that if $(x,y) \in X$ then

$$\begin{aligned} \|(x,y) - (\pm 1/n, 1/n^3)\|^2 &= (x^2+y^2) + \left(\frac{1}{n^2} + \frac{1}{n^6}\right) + \left(\mp\frac{2}{n}x - \frac{2}{n^3}y\right) \\ &= (x^2+y^2) + \left(\frac{1}{n^6} \mp \frac{2}{n}x\right) + \left(\frac{1}{n^2} - \frac{2}{n^3}y\right). \end{aligned}$$

Observe that $n \in \mathbb{N}$ can be chosen so that $\left(\frac{1}{n^2} - \frac{2}{n^3}y\right) > 0$ for a given y. As for $\left(\frac{1}{n^6} \mp \frac{2}{n}x\right)$, one can manage to have the plus or the minus n, according to whether x is negative or positive, in order to make $\left(\frac{1}{n^6} \mp \frac{2}{n}x\right) > 0$. These observations together tell us that

$$\|(x,y) - (\pm 1/n, 1/n^3)\|^2 > x^2 + y^2 = \|(x,y) - (0,0)\|.$$

That is, $(0,0)$ cannot be a farthest point in E from (x,y). In other words, $(0,0)$ is not a remotal point of E. □

These observations about extreme and exposed points foster the question:
Question: Describe Banach spaces in which every extreme point (or exposed point) is a remotal point.
A strongly related question is:
Question: Given a Banach space X and a closed bounded subset $E \subset X$. What conditions must E satisfy in order to make every extreme point (or exposed point) a remotal point?

In fact, easy computations show that the point $(0,0)$ is a remotal point of the set E, in proposition 2.3 if $\mathbb{R}^2$ is endowed with the infinity norm! This together with the observation following proposition 2.2 do not mean that exposed or extreme points are always remotal points in the infinity norm. One can look at the point $P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ as an exposed point of the unit disk in $\mathbb{R}^2$. In this example, we can show that the point P is a remotal point if $\mathbb{R}^2$ is endowed with the standard norm but is not remotal if the infinity norm is endowed.

The next result tells us that an exposed point must be a uniquely remotal point under some equivalent norm. First, a definition:

Definition 2.4. Let E be a closed bounded convex subset of the normed space $(X, \|\ \|)$ and let e be a boundary point of E. We say that e is a nice boundary point of E if there exists a linear functional $f : X \to \mathbb{R}$, of norm 1, an element $a \in X$ of norm 1 and a number α such that

$$f(e) = \alpha, f(e) \leq \alpha \; \forall e \in E \text{ and } f(a) = -1.$$

Observe that every boundary point is a nice boundary point in reflexive spaces. Also, observe that the condition that f has norm 1 is artificial because f can be replaced by $f/\|f\|$. One last observation, the condition that an element $a \in X$ exists with the properties $\|a\| = 1$ and $f(a) = -1$ means that f attains its norm.

Theorem 2.5. *Let $(X, \| \,\|)$ be a normed space. Let e be a nice boundary point of the closed bounded convex set $E \subset X$. Then, there exists a norm $\| \,\|'$ on X, which is equivalent to $\| \,\|$, such that e is a remotal point of E in $(X, \| \,\|')$.*

Proof. Without loss of generality we can assume that $e = 0$. If not, replace E by $E - \{e\}$ and α by 0. Thus, there exists a linear functional $f : X \to \mathbb{R}$ such that

$$\|f\| = 1, \; f(0) = 0, \; f(e) \leq 0 \; \forall e \in E \text{ and } f(a) = -1 \text{ for some } \|a\| = 1.$$

On X, define the norm

$$\|x\|' = \max\{\, |f(x)|, \; \|x + f(x)a\| \,\}.$$

It is easy to check that $\| \,\|'$ is a norm on X. We first prove that $\| \,\|$ and $\| \,\|'$ are equivalent. Observe that

$$\begin{aligned} \|x + f(x)a\| &\leq \|x\| + |f(x)| \, \|a\| \\ &\leq \|x\| + \|x\|; \text{ because } \|f\| = 1 \\ &= 2\|x\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|x\|' &\leq \max\{|f(x)|, \; 2\|x\|\} \\ &\leq \max\{\|x\|, \; 2\|x\|\} \\ &= 2\|x\|. \end{aligned}$$

On the other hand, adding $\|x\|' \geq |f(x)|$ and $\|x\|' \geq \|x + f(x)a\| \geq \|x\| - |f(x)|$ yields $2\|x\|' \geq \|x\|$. Thus, we have shown that

$$\frac{1}{2}\|x\| \leq \|x\|' \leq 2\|x\|$$

which means that $\| \,\|$ and $\| \,\|'$ are equivalent.

It remains to show that 0 is a remotal point of E in $(X, \| \,\|')$. For this, let $n \in \mathbb{N}$ be such that

$$n \geq 3\|e\|, \; \forall e \in E,$$

where such an n exists because E is bounded. Observe that $\|na - 0\|' = n$. Moreover, for any $e \in E$, we have

$$\begin{aligned} \|e - na\|' &= \max\{|f(e - na)|, \; \|e - na + f(e - na)a\|\} \\ &= \{|f(e) - nf(a)|, \; \|e - na + f(e)a - nf(a)a\|\}. \end{aligned} \tag{2.1}$$

But $f(a) = -1$ and $f(e) \leq 0$. Hence, when $e \in E$, (2.1) becomes

$$\|e - na\|' = \max\{n - |f(e)|, \; \|e + f(e)a\|\}$$

$$\leq \quad n - |f(e)| \tag{2.2}$$
$$\leq \quad n, \tag{2.3}$$

where inequality (2.2) is a result of the fact that $|f(e)| + \|e + f(e)a\| \leq 3\|e\| \leq n \Rightarrow n - |f(e)| \geq \|e + f(e)a\|$.
Thus, we have shown that $\|na - 0\|' = n$ and that $\|na - e\|' \leq n$ for all $e \in E$. This means that 0 is a remotal point of E in $(X, \| \|')$. This completes the proof of the theorem. □

It is worth to remark that the norm $\| \|'$ coincides with the infinity norm of $\mathbb{R}^2$ in both propositions 2.2 and 2.3.

Observe that the condition that E is convex is of no importance. But usually we are interested in convex sets when we speak of hyperplanes and separation.

In fact, the proof of Theorem 2.5 can be imitated step by step to see that in the case of an exposed point, inequality (2.3) is a strict inequality. Thus, we get

Theorem 2.6. *Let e be a nice exposed point of the closed convex bounded set E in the normed space $(X, \| \|)$. Then, under the norm $\| \|'$, e is a uniquely remotal point of E.*

A stronger relation between exposed points and uniquely remotal points is available. Namely, we know that a uniquely remotal point is necessarily an extreme point, see [2]. In fact, we have the following stronger result.

Theorem 2.7. *Let e be a uniquely remotal point of the closed convex bounded set E in a normed space X. Then, e is an exposed point of E.*

Proof. Let $x \in X$ be such that $F(x, E) = \{e\}$ and let $r = D(x, E)$. By virtue of Lemma 2.1, the ball $B(x, r)$ and the sphere $S(x, r)$ satisfy the following

$$E \cap S(x, r) = \{e\} \text{ and } E \subset B(x, r).$$

Let H be a hyperplane that supports $S(x, r)$ at e. Then, by the definition of a supporting hyperplane, H does not contain any point of the interior of $S(x, r)$, see [4], hence $E \cap H = \{e\}$. Thus, H supports E uniquely at e. That is, e is an exposed point of E. □

Thus, so far we have seen that an exposed point is not necessarily a remotal point but it is uniquely remotal under some equivalent norm and that a uniquely remotal point is necessarily an exposed point. Our last remark about the connection of exposed and uniquely remotal points is to say that even if an exposed point is remotal, it does not follow that this point is a uniquely remotal point. This can be seen on taking $X = \mathbb{R}^2$ equipped with the $\| \|_\infty$, E to be the unit ball in X and e to be the corner $(1, 1)$. Then clearly, e is exposed and remotal but is not uniquely remotal.

Before proceeding to the next concept, namely the concept of extremely remotal points, we would like to give some properties of the new norm $\| \|'$ and the new space $(X, \| \|')$.

Proposition 2.8. *Let e be a nice boundary point of the closed bounded convex set E in a normed space X. Let $f \in X^*$ be such that*

$$f(e) = \alpha,\ f(e') \leq \alpha\ \forall e' \in E,\ \|f\| = 1 \textit{ and } f(a) = -1 \textit{ for some } \|a\| = 1.$$

Then

$$\{x \in X : f(x) \leq \alpha\} = \bigcup_{n\in\mathbb{N}} B'(e+na,n)$$

where $B'(e+na,n)$ is the ball centered at $e+na$ with radius n in $(X, \|\ \|')$.

Proof. Let x be such that $f(x) \leq \alpha$ and let $n \in \mathbb{N}$ be such that

$$n \geq (\alpha - f(x)) + \|x - e + f(x)a - \alpha a\|.$$

Then

$$\begin{aligned} \|x-(e+na)\|' &= \max\{|f(x-e-na)|,\ \|x-e-na+f(x-e-na)a\|\} \\ &= \max\{|f(x)-\alpha+n|,\ \|x-e+f(x)a-\alpha a\|\} \\ &= \max\{n-(\alpha-f(x)),\ \|x-e+f(x)a-\alpha a\|\} \end{aligned}$$

where the last inequality is a consequence of the fact that $f(x) \leq \alpha$. But since $n \geq (\alpha - f(x)) + \|x-e+f(x)a-\alpha a\|$ we see that

$$\begin{aligned} \|x-(e+na)\|' &= n-(\alpha-f(x)) \\ &\leq n;\ \text{because } f(x) \leq \alpha. \end{aligned}$$

This shows that

$$\{x \in X : f(x) \leq \alpha\} \subseteq \bigcup_{n\in\mathbb{N}} B'(e+na,n).$$

For the reversed inclusion, let $x \in X$ be such that $\|x-(e+na)\|' \leq n$ for some $n \in \mathbb{N}$. Then, for this n,

$$\begin{aligned} & |f(x-e-na)| \leq n \text{ by the definition of } \|\ \|' \\ \Rightarrow\ & n + f(x) - \alpha \leq n \\ \Rightarrow\ & f(x) \leq \alpha. \end{aligned}$$

This completes the proof of the proposition. □

To illustrate the importance of this proposition and the new norm, we consider the following example: Let $X = \mathbb{R}^2$ be endowed with the infinity norm, let $E = \{(x,y) : x^2+y^2 = 1\}$ and let $e = (1/\sqrt{2}, 1/\sqrt{2})$. If $f \in X^*$ is such that

$$f(a) = -1 \text{ for some } \|a\| = 1, f(e) = \alpha,\ f(x,y) \leq \alpha\ \forall (x,y) \in E \text{ and } \|f\| = 1,$$

then it is easy to see that

$$\{f \leq \alpha\} \neq \bigcup_{n\in\mathbb{N}} B(e+na,n)$$

where $B(e+na,n)$ is the ball centered at $e+na$ with radius n in the infinity norm.

Having introduced the result of proposition 2.8, we must remark that the truth of

$$\{f \leq \alpha\} = \bigcup_{n\in\mathbb{N}} B(e+na,n)$$

does not imply that e is a remotal point. This can be seen by the example of proposition 2.3.

One more property of $\|\ \|'$ is the following.

Proposition 2.9. *Let e be a nice boundary point of the closed bounded convex set E in the normed space X and let f, α and a be as definition 2 suggests. Then*

$$A := \{x \in H : \|x - e\| \leq t\} = B := \left\{x \in H : \|x - (e + ta)\|' = t\right\}$$

for every $t > 0$.

Proof. Let $x \in A$, then $f(x) = \alpha$ and $\|x - e\| \leq t$. Now,

$$\begin{aligned} \|x - (e + ta)\|' &= \max\{|f(x - e - ta)|,\ \|x - e - ta + f(x)a - f(e)a - tf(a)a\|\} \\ &= \max\{t, \|x - e\|\} \\ &= t \text{ because } \|x - e\| \leq t. \end{aligned}$$

But this means that $A \subseteq B$. For the other inclusion, let $x \in B$, then following similar computations as above yields $\max\{t, \|x-e\|\} = t$, which implies $\|x-e\| \leq t$. Hence $B \subseteq A$. This completes the proof. □

3. Extremely remotal points and Krein-Milman Type Theorems

It happens that some remotal points are more interesting than other ones. These points are defined in the following definition.

Definition 3.1. Let E be a closed bounded convex subset of the Banach space X. A point $e \in E$ will be called extremely remotal if either e is an extreme point or if $e \in F(x, E)$ for some $x \in \partial E$, the boundary being taken in the space generated by E, with the property that $[e, x] \cap E^\circ = \phi$. Here, $[e, x] = \{te_1 + (1 - t)x : 0 \leq t \leq 1\}$ and E° is the interior of E.

Since remotal points and extreme points are boundary points, we easily see that an extremely remotal point of E is necessarily a boundary point of E.
We know that every point of the unit sphere of $L^p, 1 < p < \infty$ is an extreme point of the unit ball of L^p. This makes the study of extremely remotal points of the unit ball of $L^p, 1 < p < \infty$ not interesting. For this reason, we shall study extremely remotal points for the unit ball of spaces in which the unit ball has no extreme points such as L^1 and the space of continuous functions on $[0, 1]$. We give the following example of extremely remotal points:

Example 3.2. Let $X := L^1[0, 1]$ and let B_1 be the unit ball of X. If $f = 2\chi_{[0,1/2]}$ then $f \in S_1$, the unit sphere of L^1. Now, if $g = 2\chi_{[1/2,1]}$ then $\|f-g\| = 2 = D(g, B_1)$. Hence f is a remotal point of B_1. Further, if $0 < t < 1$ then

$$\|tf + (1 - t)g\| = 1 \Rightarrow tf + (1 - t)g \in S_1.$$

This implies that f is extremely remotal of B_1.

The following example shows that some spaces do not have extremely remotal points except the extreme points.

Example 3.3. Let $X = L^2[0, 1]$ and let $f \in S_1$; the unit sphere of X. We know that f is an extreme point of B_1; the unit ball of X. However, we assert that f cannot be extremely remotal in the other sense! That is, we show that no function $g \in S_1$ satisfies

$$\|f - g\| = 2, \text{ and } \|tf + (1 - t)g\| = 1,\ \forall 0 < t < 1.$$

Observe that the condition $\|tf+(1-t)g\|=1,\ \forall 0<t<1$ cannot be true.

Extremely remotal points are of special interest in spaces where the unit ball has no extreme points. The easiest example of such a space is $L^1[0,1]$. In the following result, we describe all extremely remotal points of the unit ball of $L^1[0,1]$.

Theorem 3.4. *Let f be in the unit sphere of $L^1[0,1]$. Then, f is extremely remotal if and only if* $0<\mu\,(\textit{support}(f))<1$.

Proof. Denote the support of f by E and suppose that $0<\mu(E)<1$. We assert that f is extremely remotal. So, let

$$g=\frac{1}{\mu(E^c)}\chi_{E^c}\Rightarrow \|g\|=1.$$

Then, $\|tf+(1-t)g\|=1$ for any $0<t<1$. That is, $[f,g]$ lies on the boundary of B_1. Moreover, $\|f-g\|=2$. This implies that f is an extremely remotal point of B_1.

Conversely, suppose that f is an extremely remotal point of B_1. We want to show that $0<\mu\,(\text{support}(f))<1$. Let $g\in L^1[0,1]$ be such that

$$\|g\|=1,\ f\in F(g,B_1) \text{ and } \|tf+(1-t)g\|=1,$$

where such g exists because f is extremely remotal. Since $D(g,B_1)=2$ we see that $\|f-g\|=2$. Now,

$$1=\|tf+(1-t)g\|\le t\|f\|+(1-t)\|g\|=1.$$

This means that

$$\|tf+(1-t)g\|=t\|f\|+(1-t)\|g\|.$$

But this happens only if f and g have the same sign. In this context, we consider 0 to be of any sign we desire.

Let $E=\{x\in[0,1]: f(x)\neq 0\neq g(x)\}$. Now, if f and g have the same sign we would have

$$\begin{aligned}\|f-g\| &= \int_0^1 |\,|f(x)|-|g(x)|\,|\ dx\\ &= \int_E |\,|f(x)|-|g(x)|\,|\ dx+\int_{E^c} |\,|f(x)|-|g(x)|\,|\ dx.\end{aligned}$$

Since either $f(x)=0$ or $g(x)=0$ when $x\in E^c$ we have

$$\int_{E^c} |\,|f(x)|-|g(x)|\,|\ dx=\int_{E^c}|f(x)|\ dx+\int_{E^c}|g(x)|\ dx.$$

Now, if $\mu(E)>0$ then

$$\int_{E} |\,|f(x)|-|g(x)|\,|\ dx<\int_{E}|f(x)|\ dx+\int_{E}|g(x)|\ dx.$$

Consequently, if $\mu(E)>0$,

$$\begin{aligned}2 &= \|f-g\|\\ &= \int_E |\,|f(x)|-|g(x)|\,|\ dx+\int_{E^c} |\,|f(x)|-|g(x)|\,|\ dx\end{aligned}$$

$$
\begin{aligned}
&< \int_E |f(x)|\,dx + \int_E |g(x)|\,dx + \int_{E^c} |f(x)|\,dx + \int_{E^c} |g(x)|\,dx \\
&= \int_0^1 |f(x)|\,dx + \int_0^1 |g(x)|\,dx \\
&= 2;
\end{aligned}
$$

which is a contradiction. Hence, we must have $\mu(E) = 0$. But this means that f and g have disjoint supports. Finally, since $\|g\| = 1$, we see that $\mu(\text{support}(g)) > 0$ and since $\|f\| = 1$, we have $\mu(\text{support}(f)) > 0$. These facts together imply that $0 < \mu\,(\text{support}(f)) < 1$. □

In fact, following similar ideas as above we can easily prove

Theorem 3.5. *Let $x = (x_n) \in S_1$, the unit sphere of ℓ^1. Then, x is extremely remotal for the unit ball of ℓ^1 if and only if there exists $n \in \mathbb{N}$ such that $x_n = 0$. That is, supp$(x) \neq \mathbb{N}$.*

The next result describes all extremely remotal points of the unit ball of the space of continuous functions $C[0,1]$.

Theorem 3.6. *Let f be an element of S_1; the unit sphere of $C[0,1]$. Then, f is an extremely remotal point of B_1;the unit ball of $C[0,1]$, if and only if f attains its norm at more than one point.*

Proof. Let $f \in C[0,1]$ be such that $\|f\| = 1$ and let $x_1, x_2 \in [0,1]$ be such that $|f(x_1)| = |f(x_2)| = 1$. We assert that f is extremely remotal by finding a function $g \in C[0,1]$ such that

$$\|g\| = 1,\ \|f - g\| = 2 \text{ and } \|tf + (1-t)g\| = 1.$$

Let g be a continuous function defined on $[0,1]$ with $\|g\| = 1$, $g(x_1) = -f(x_1)$ and $g(x_2) = f(x_2)$. It is clear that $\|f - g\| = 2$. Hence, $f \in F(g, B_1)$. Now, for $0 < t < 1$ we have

$$
\begin{aligned}
\|tf + (1-t)g\| &\geq |tf(x_2) + (1-t)g(x_2)|; \text{ recall that } \|\ \| = \|\ \|_\infty \\
&\geq |g(x_2)| - t|f(x_2) - g(x_2)| \\
&= 1.
\end{aligned}
$$

On the other hand,

$$\|tf + (1-t)g\| \leq t\|f\| + (1-t)\|g\| = 1.$$

Consequently, $\|tf + (1-t)g\| = 1$. This means that $[f, g]$ lies entirely on the boundary of B_1. Thus, f is an extremely remotal point of B_1.
Conversely, suppose that f is an extremely remotal point of B_1. Our goal is to show that f attains its norm at least twice. Let $g \in C[0,1]$ be such that

$$\|g\| = 1,\ \|f - g\| = 2 \text{ and } \|tf + (1-t)g\| = 1,$$

where such a function g exists because f is extremely remotal of B_1. Let $x_1 \in [0,1]$ be such that $|f(x_1)| = 1$ and suppose that $|f(x)| < 1$ for all $x \neq x_1$. Since $\|f - g\| = 2$ and $|f(x)|, |g(x)| \leq 1 \forall x \in [0,1]$, we must have $\|f - g\|$ attained when $|f(x)| = 1$ and $|g(x)| = 1$. But $|f(x)| = 1$ only when $x = x_1$. Consequently, we must have $g(x_1) = -f(x_1)$. Now, for any $t \in (0,1)$ we have $|tf(x_1) + (1-t)g(x_1)| = |f(x_1)|\ |2t - 1| <$

1, $\forall 0 < t < 1$. But since both f and g attain their norms at x_1, $tf + (1-t)g$ must attain its norm at x_1. That is, $\|tf + (1-t)g\| < 1$ which contradicts the fact that $[f, g]$ does not intersect the interior of B_1. This contradiction is a result of the assumption that x_1 is the unique point at which f attains its norm. That is, f must attain its norm at least twice in order to be an extremely remotal point of B_1.
It is worth to remark that the only extreme points of the unit ball of $C[0,1]$ are those functions f such that $|f| = 1$. These functions attain their norms at least twice. □

Similar ideas as above yield the following result:

Theorem 3.7. *Let $x = (x_n)$ be in the unit sphere of c_0. Then, x is extremely remotal for the unit ball of c_0 if and only if $\|x\|_\infty$ is attained twice. That is, if there are at least two indices n_1 and n_2 such that $|x_{n_1}| = |x_{n_2}| = 1$.*

The Krein-Milman theorem asserts that a compact convex subset of a locally convex topological space is the closed convex hull of its extreme points. On the other hand, it is shown [1] that the mazur intersection property is equivalent to the fact that every closed bounded convex set is the closed convex hull of its remotal points under some restrictions.

Now, we give a Krein-Milman type theorem. Observe that the unit ball of $L^1[0,1]$ has no extreme points neither it is compact. Therefore, the Krein-Milman theorem cannot be applied. As for the unit ball of $L^1[0,1]$, every point is a remotal point. Therefore, it is trivial that the unit ball is the closed convex hull of its remotal points.
Having introduced the new concept of extremely remotal points, we present the following result.

Theorem 3.8. *The unit ball B_1 of $L^1[0,1]$ is the closed convex hull of its extremely remotal points.*

Proof. Observe first that B_1 is the convex hull of S_1; the unit sphere of L^1. Thus, it suffices to show that every element of S_1 is in the convex hull of extremely remotal points of B_1. So, let $f \in S_1$ be such that $\mu(\text{support}(f)) = 1$. Let

$$f_1(x) = \frac{1}{\int_0^{1/2} |f|\, dx} f(x)\chi_{[0,1/2]} \text{ and } f_2(x) = \frac{1}{\int_{1/2}^{1} |f|\, dx} f(x)\chi_{(1/2,1]}.$$

Then clearly, $f_1, f_2 \in S_1$. Moreover, since $\|f\| = 1$ we have $\int_{1/2}^1 |f|\, dx = 1 - \int_0^{1/2} |f|\, dx$. Now, let $t = \int_0^{1/2} |f|\, dx$. Then $0 < t < 1$ and clearly $f = tf_1 + (1-t)f_2$. By virtue of Theorem 3.4, both f_1 and f_2 are extremely remotal points of B_1. This ends the proof. □

Our last result is a Krein-Milman type Theorem for the space c_0:

Theorem 3.9. *The unit ball of c_0 is the closed convex hull of its extremely remotal points.*

Proof. Let $x = (x_n) \in c_0$ be such that $\|x\| = 1$ and suppose that this norm is attained uniquely. Let $k \in \mathbb{N}$ be the index at which $\|x\|$ is attained. That is, $|x_k| = 1$. Let k' be any other index than k. Then, $-1 < x_{k'} < 1$ because $\|x\| = 1$

is attained uniquely at x_k. Let $t_0 \in (0,1)$ be such that $x_{k'} = t_0(-1) + (1-t_0)(1)$. Now define two elements $y = (y_n)$ and $z = (z_n)$ in c_0 in the following way

$$y_n = \begin{cases} x_n, & n \neq k, k' \\ -1, & n = k' \\ x_k, & n = k \end{cases} \quad \text{and} \quad z_n = \begin{cases} x_n, & n \neq k, k' \\ 1, & n = k' \\ x_k, & n = k \end{cases}.$$

Then, clearly y and z are extremely remotal points of the unit ball of c_0 because both norms $\|y\|$ and $\|z\|$ are attained twice, at least. Moreover, it is clear that $x = t_0 y + (1-t_0)z$.

Thus, we have shown that any boundary point of the unit ball of c_0 is in the convex hull of the extremely remotal points of the unit ball. This implies the result of the Theorem.

□

We conclude our paper by the following question: Can we prove such Krein-Milman type Theorems for other Banach spaces?

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M. Sababheh
Department of Science and Humanities, Princess Sumaya University For Technology, Amman 11941, Jordan
E-mail address: sababheh@yahoo.com

R. Khalil
Department of Mathematics, University of Jordan, Amman 11942, Jordan
E-mail address: roshdi@ju.edu.jo