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THE EXISTENCE AND STRUCTURE OF APPROXIMATE SOLUTIONS OF DYNAMIC DISCRETE TIME ZERO-SUM GAMES

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Abstract. In this paper we establish turnpike results for a class of dynamic discrete-time two-player zero-sum games. These results describe the structure of approximate solutions which is independent of the length of the interval, for all sufficiently large intervals. We also show that for each initial state there exists a pair of overtaking equilibria strategies over an infinite horizon.

1. Introduction

The study of the existence and the structure of (approximate) solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research [4-7, 9-11, 13, 17-20, 27, 28, 30]. These problems arise in engineering [1, 31], in models of economic growth [8, 14, 16, 24, 25, 30], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3, 26] and in the theory of thermodynamical equilibrium for materials [12, 15].

In this paper we study the existence and structure of solutions for a class of dynamic discrete-time two-player zero-sum games and establish a turnpike result. This result describes the structure of approximate solutions which is independent of the length of the interval, for all sufficiently large intervals. We also show that for each initial state there exists a pair of overtaking equilibria strategies over an infinite horizon.

Denote by $||\cdot||$ the Euclidean norm in R^m. Let $X \subset R^{m_1}$ and $Y \subset R^{m_2}$ be nonempty convex compact sets. Denote by $\mathcal{M}$ the set of all continuous functions $f : X \times X \times Y \times Y \to R^1$ such that:

for each $(y_1, y_2) \in Y \times Y$ the function $(x_1, x_2) \to f(x_1, x_2, y_1, y_2)$, $(x_1, x_2) \in X \times X$ is convex;

for each $(x_1, x_2) \in X \times X$ the function $(y_1, y_2) \to f(x_1, x_2, y_1, y_2)$, $(y_1, y_2) \in Y \times Y$ is concave.

For the set $\mathcal{M}$ we define a metric $\rho : \mathcal{M} \times \mathcal{M} \to R^1$ by

$$\rho(f, g) = \sup\{|f(x_1, x_2, y_1, y_2) - g(x_1, x_2, y_1, y_2)| : x_1, x_2 \in X, \quad y_1, y_2 \in Y\}, \quad f, g \in \mathcal{M}. \tag{1.1}$$

Clearly $(\mathcal{M}, \rho)$ is a complete metric space.

Given $f \in \mathcal{M}$ and an integer $n \geq 1$ we consider a discrete-time two-player zero-sum game over the interval $[0, n]$. For this game $\{\{x_i\}_{i=0}^n : x_i \in X,\ i = 0, \dots n\}$ is

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the set of strategies for the first player, $\{\{y_i\}_{i=0}^n : y_i \in Y,\ i = 0, \dots n\}$ is the set of strategies for the second player, and the cost for the first player associated with the strategies $\{x_i\}_{i=0}^n$, $\{y_i\}_{i=0}^n$ is given by $\sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_i, y_{i+1})$.

Let $f \in \mathcal{M}$, $n \geq 1$ be an integer and let $M \in [0, \infty)$. A pair of sequences $\{\bar{x}_i\}_{i=0}^n \subset X$, $\{\bar{y}_i\}_{i=0}^n \subset Y$ is called (f, M)-good if the following properties hold:

(i) for each sequence $\{x_i\}_{i=0}^n \subset X$ satisfying $x_0 = \bar{x}_0$, $x_n = \bar{x}_n$,

$$M + \sum_{i=0}^{n-1} f(x_i, x_{i+1}, \bar{y}_i, \bar{y}_{i+1}) \geq \sum_{i=0}^{n-1} f(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1}); \tag{1.2}$$

(ii) for each sequence $\{y_i\}_{i=0}^n \subset Y$ satisfying $y_0 = \bar{y}_0$, $y_n = \bar{y}_n$,

$$M + \sum_{i=0}^{n-1} f(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1}) \geq \sum_{i=0}^{n-1} f(\bar{x}_i, \bar{x}_{i+1}, y_i, y_{i+1}). \tag{1.3}$$

If a pair of sequences $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ is $(f, 0)$-good then it is called (f)-optimal.

In this paper we study the turnpike property of good pairs of sequences. To have this property means, roughly speaking, that the good pairs of sequences are determined mainly by the objective function, and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints. Turnpike properties are well known in mathematical economics and optimal control (see [14-16, 24, 27-31] and the references mentioned there).

Consider any $f \in \mathcal{M}$. We say that the function f has the *turnpike property* if there exists a unique pair $(x_f, y_f) \in X \times Y$ for which the following assertion holds:

For each $\epsilon > 0$ there exist an integer $n_0 \geq 2$ and a number $\delta > 0$ such that for each integer $n \geq 2n_0$ and each (f, δ)-good pair of sequences $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ the relations $||x_i - x_f||$, $||y_i - y_f|| \leq \epsilon$ holds for all integers $i \in [n_0, n - n_0]$.

In [29] we showed that the turnpike property holds for a generic $f \in \mathcal{M}$. Namely, in [29] we proved the existence of a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}$ such that each $f \in \mathcal{F}$ has the turnpike property. Results of this kind for classes of single-player control systems have been established in [27, 28, 30]. Thus, instead of considering the turnpike property for a single objective function, we investigate it for a space of all such functions equipped with some natural metric, and show that this property holds for most of these functions.

Note that the generic approach of [29] is not limited to the turnpike property, but is also applicable to other problems in Mathematical Analysis [21-23].

In [29] and in the present paper we also study the existence of equilibria over an infinite horizon and employ the following version of the overtaking optimality criterion [8, 25, 30, 31].

Let $f \in \mathcal{M}$. A pair of sequences $\{\bar{x}_i\}_{i=0}^\infty \subset X$, $\{\bar{y}_i\}_{i=0}^\infty \subset Y$ is called (f)-*overtaking optimal* if the following properties hold:

for each sequence $\{x_i\}_{i=0}^\infty \subset X$ satisfying $x_0 = \bar{x}_0$

$$\limsup_{T \to \infty} [\sum_{i=0}^{T-1} f(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1}) - \sum_{i=0}^{T-1} f(x_i, x_{i+1}, \bar{y}_i, \bar{y}_{i+1})] \leq 0; \tag{1.4}$$

for each sequence $\{y_i\}_{i=0}^{\infty} \subset Y$ satisfying $y_0 = \bar{y}_0$

$$\limsup_{T\to\infty} \Big[\sum_{i=0}^{T-1} f(\bar{x}_i, \bar{x}_{i+1}, y_i, y_{i+1}) - \sum_{i=0}^{T-1} f(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1}) \Big] \le 0. \tag{1.5}$$

In [29] we showed that for a generic $f \in \mathcal{M}$ and each $(x, y) \in X \times Y$ there exists an (f)-overtaking optimal pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{y_i\}_{i=0}^{\infty} \subset Y$ such that $x_0 = x$, $y_0 = y$.

According to the results of [29] we know that for most functions $f \in \mathcal{M}$ the turnpike property holds and that (f)-overtaking optimal pairs of sequences exist. Nevertheless it is very important to have conditions on $f \in \mathcal{M}$ which imply the turnpike property and guarantee the existence of (f)-overtaking optimal pairs of sequences. These conditions will be found in the present paper.

The paper is organized as follows. In Section 2 we consider minimal pairs of sequences. In Section 3 we present our main (strict convexity-concavity) assumptions of $f \in \mathcal{M}$ and state Theorems 3.1 and 3.2 which establish the turnpike property and the existence of (f)-overtaking optimal pairs of sequences respectively. Section 4 contains preliminary results. Auxiliary results for Theorem 3.1 are proved in Section 5. Theorem 3.1 is proved in Section 6. Auxiliary results for Theorem 3.2 are proved in Section 7. Section 8 contains the proof of Theorem 3.2.

2. Minimal pairs of sequences

Let $f \in \mathcal{M}$. Define a function $\bar{f} : X \times Y \to R^1$ by

$$\bar{f}(x, y) = f(x, x, y, y), \quad x \in X,\ y \in Y. \tag{2.1}$$

Then there exists a saddle point $(x_f, y_f) \in X \times Y$ for $\bar{f}$ [2, 26] such that

$$\sup_{y\in Y} \bar{f}(x_f, y) = \bar{f}(x_f, y_f) = \inf_{x\in X} \bar{f}(x, y_f). \tag{2.2}$$

Set

$$\mu(f) = \bar{f}(x_f, y_f). \tag{2.3}$$

A pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{y_i\}_{i=0}^{\infty} \subset Y$ is called (f)-minimal if for each integer $n \ge 2$ the pair of sequences $\{x_i\}_{i=0}^{n}$, $\{y_i\}_{i=0}^{n}$ is (f)-optimal.

The following results were established in [29].

Proposition 2.1 (29, Proposition 5.1.)**.** *Let $n \ge 2$ be an integer and*

$$\bar{x}_i = x_f, \quad \bar{y}_i = y_f, \quad i = 0, \dots n.$$

Then the pair of sequences $\{\bar{x}_i\}_{i=0}^{n}$, $\{\bar{y}_i\}_{i=0}^{n}$ is (f)-optimal.

Proposition 2.2 (29, Proposition 5.2.)**.** *Let $n \ge 2$ be an integer and let*

$$(\{x_i^{(k)}\}_{i=0}^{n}, \{y_i^{(k)}\}_{i=0}^{n}) \subset X \times Y, \quad k = 1, 2, \dots$$

be a sequence of (f)-optimal pairs. Assume that

$$\lim_{k\to\infty} x_i^{(k)} = x_i, \quad \lim_{k\to\infty} y_i^{(k)} = y_i, \quad i = 0, 1, 2, \dots, n.$$

Then the pair of sequences $(\{x_i\}_{i=0}^{n}, \{y_i\}_{i=0}^{n})$ is (f)-optimal.

Proposition 2.3 (29, Proposition 5.3.)**.** *Let $x \in X$, $y \in Y$. Then there exists an (f)-minimal pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{y_i\}_{i=0}^{\infty} \subset Y$ such that $x_0 = x$, $y_0 = y$.*

Let $n \geq 1$ be an integer and let $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in X \times X \times Y \times Y$. Define

$$\Lambda_X(\xi, n) = \{\{x_i\}_{i=0}^{n} \subset X : \quad x_0 = \xi_1,\ x_n = \xi_2\}, \tag{2.4}$$

$$\Lambda_Y(\xi, n) = \{\{y_i\}_{i=0}^{n} \subset Y : \quad y_0 = \xi_3,\ y_n = \xi_4\}, \tag{2.5}$$

$$f^{(\xi,n)}((x_0, \dots, x_i, \dots, x_n), (y_0, \dots, y_i, \dots y_n)) = \sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_i, y_{i+1}), \tag{2.6}$$

$$\{x_i\}_{i=0}^{n} \in \Lambda_X(\xi, n), \quad \{y_i\}_{i=0}^{n} \in \Lambda_Y(\xi, n).$$

3. Main results

Let $f \in \mathcal{M}$. Then there exists $(x_f, y_f) \in X \times Y$ such that [2, 29]

$$\sup_{y \in Y} f(x_f, x_f, y, y) = f(x_f, x_f, y_f, y_f) = \inf_{x \in X} f(x, x, y_f, y_f). \tag{3.1}$$

In this paper we suppose that the following assumptions hold:
(A1) for each $x \in X \setminus \{x_f\}$ and each $x' \in X$

$$f(2^{-1}(x_f + x), 2^{-1}(x_f + x'), y_f, y_f) < 2^{-1} f(x_f, x_f, y_f, y_f) + 2^{-1} f(x, x', y_f, y_f);$$

(A2) for each $y \in Y \setminus \{y_f\}$ and each $y' \in Y$

$$f(x_f, x_f, 2^{-1}(y_f + y), 2^{-1}(y' + y_f)) > 2^{-1} f(x_f, x_f, y_f, y_f) + 2^{-1} f(x_f, x_f, y, y').$$

Choose a number

$$D_0 \geq \sup\{|f(x_1, x_2, y_1, y_2)| : x_1, x_2 \in X,\ y_1, y_2 \in Y\}. \tag{3.2}$$

In this section we present our main results.

Theorem 3.1. *Let $\epsilon \in (0,1)$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$, an integer $n_1 \geq 4$ and a number $\delta \in (0, \epsilon)$ such that for each $g \in \mathcal{U}$, each integer $n \geq 2n_1$ and each (g, δ)-good pair of sequences $\{x_i\}_{i=0}^{n} \subset X$, $\{y_i\}_{i=0}^{n} \subset Y$ the relation*

$$||x_i - x_f||,\ ||y_i - y_f|| \leq \epsilon \tag{3.3}$$

holds for all integers $i \in [n_1, n - n_1]$. Moreover, if $||x_0 - x_f||$, $||y_0 - y_f|| \leq \delta$ then (3.3) holds for all integers $i \in [0, n - n_1]$, and if $||x_n - x_f||$, $||y_n - y_f|| \leq \delta$ then (3.3) is valid for all integers $i \in [n_1, n]$.

Theorem 3.2. *For each $x \in X$ and each $y \in Y$ there exists an (f)-overtaking optimal pair of sequences $\{x_i\}_{i=0}^{\infty} \subset X$, $\{y_i\}_{i=0}^{\infty} \subset Y$ such that $x_0 = x$, $y_0 = y$.*

4. Preliminary results

Let M, N be nonempty sets and let $f : M \times N \to R^1$. Set

$$f^a(x) = \sup_{y \in N} f(x,y),\ x \in M, \quad f^b(y) = \inf_{x \in M} f(x,y),\ y \in N, \tag{4.1}$$

$$v_f^a = \inf_{x \in M} \sup_{y \in N} f(x,y), \quad v_f^b = \sup_{y \in N} \inf_{x \in M} f(x,y). \tag{4.2}$$

Clearly

$$v_f^b \le v_f^a. \tag{4.3}$$

We have the following result (see [2, Chapter 6, Section 2, Proposition 1]).

Proposition 4.1. *Let $f : M \times N \to R^1$, $\bar{x} \in M$, $\bar{y} \in N$. Then*

$$\sup_{y \in N} f(\bar{x}, y) = f(\bar{x}, \bar{y}) = \inf_{x \in M} f(x, \bar{y}) \tag{4.4}$$

if and only if

$$v_f^a = v_f^b,\ \sup_{y \in N} f(\bar{x}, y) = v_f^a,\ \inf_{x \in M} f(x, \bar{y}) = v_f^b. \tag{4.5}$$

Let $f : M \times N \to R^1$. If $(\bar{x}, \bar{y}) \in M \times N$ satisfies (4.4), then it is called a saddle point (for f). We have the following result (see [2, Chapter 6, Section 2, Theorem 8]).

Proposition 4.2. *Let $M \subset R^m$, $N \subset R^n$ be convex compact sets and let $f : M \times N \to R^1$ be a continuous function. Assume that for each $y \in N$ the function $x \to f(x,y)$, $x \in M$ is convex and for each $x \in M$ the function $y \to f(x,y)$, $y \in N$ is concave. Then there exists a saddle point for f.*

Proposition 4.3 (29, Proposition 4.3.)**.** *Let M, N be nonempty sets, $f : M \times N \to R^1$ and let*

$$-\infty < v_f^a = v_f^b < +\infty,\ x_0 \in M,\ y_0 \in N,\ \Delta_1, \Delta_2 \in [0, \infty), \tag{4.6}$$

$$\sup_{y \in N} f(x_0, y) \le v_f^a + \Delta_1, \quad \inf_{x \in M} f(x, y_0) \ge v_f^b - \Delta_2. \tag{4.7}$$

Then

$$\sup_{y \in N} f(x_0, y) - \Delta_1 - \Delta_2 \le f(x_0, y_0) \le \inf_{x \in M} f(x, y_0) + \Delta_1 + \Delta_2. \tag{4.8}$$

Proposition 4.4 (29, Proposition 4.4.)**.** *Let M, N be nonempty sets and let $f : M \times N \to R^1$. Assume that (4.6) is valid, $x_0 \in M$, $y_0 \in N$, Δ_1, $\Delta_2 \in [0, \infty)$,*

$$\sup_{y \in N} f(x_0, y) - \Delta_2 \le f(x_0, y_0) \le \inf_{x \in M} f(x, y_0) + \Delta_1. \tag{4.9}$$

Then

$$\sup_{y \in N} f(x_0, y) \le v_f^a + \Delta_1 + \Delta_2, \quad \inf_{x \in M} f(x, y_0) \ge v_f^b - \Delta_1 - \Delta_2.$$

5. Auxiliary results for Theorem 3.1

We use all the definitions, notation and assumptions made in Section 3. In particular we suppose that assumptions (A1) and (A2) hold.

Lemma 5.1. *Let $\epsilon \in (0,1)$. Then there exists a number $\delta \in (0,\epsilon)$ such that for each integer $n \geq 2$ and each (f,δ)-good pair of sequences $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ satisfying*

$$x_n, x_0 = x_f, \quad y_n, y_0 = y_f \tag{5.1}$$

the following relations hold:

$$||x_i - x_f||,\ ||y_i - y_f|| \leq \epsilon, \quad i = 0, \dots n. \tag{5.2}$$

Proof. By (A1), (A2) and continuity of f there exists a positive number γ such that the following properties hold:

(P1) for each $x \in X$ and each $x' \in X$ satisfying $||x - x_f|| \geq \epsilon$,

$$-f(2^{-1}(x_f + x), 2^{-1}(x_f + x'), y_f, y_f) + 2^{-1} f(x_f, x_f, y_f, y_f) + 2^{-1} f(x, x', y_f, y_f) \geq \gamma;$$

(P2) for each $y \in Y$ and each $y' \in Y$ satisfying $||y - y_f|| \geq \epsilon$,

$$f(x_f, x_f, 2^{-1}(y_f + y), 2^{-1}(y' + y_f)) - 2^{-1} f(x_f, x_f, y_f, y_f) - 2^{-1} f(x_f, x_f, y, y') \geq \gamma.$$

Choose a positive number δ such that

$$\delta < \gamma/4,\ \delta < 8^{-1}\epsilon. \tag{5.3}$$

Assume that an integer $n \geq 2$, $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ is an (f,δ)-good pair of sequences and that (5.1) is valid. Set

$$\xi_1, \xi_2 = x_f, \quad \xi_3, \xi_4 = y_f, \quad \xi = (\xi_1, \xi_2, \xi_3, \xi_4). \tag{5.4}$$

Consider the sets $\Lambda_X(\xi, n)$, $\Lambda_Y(\xi, n)$ and the functions $f^{(\xi,n)}$ (see (2.4)-(2.6)). It follows from (5.1) and Proposition 2.1 that

$$\sup\Big\{ \sum_{i=0}^{n-1} f(x_f, x_f, u_i, u_{i+1}) :\ \{u_i\}_{i=0}^n \in \Lambda_Y(\xi, n) \Big\} = n f(x_f, x_f, y_f, y_f) \tag{5.5}$$

$$= \inf \Big\{ \sum_{i=0}^{n-1} f(p_i, p_{i+1}, y_f, y_f) :\ \{p_i\}_{i=0}^n \in \Lambda_X(\xi, n) \Big\}.$$

By (5.1) and (5.4),

$$\{x_i\}_{i=0}^n \in \Lambda_X(\xi, n)\},\ \{y_i\}_{i=0}^n \in \Lambda_Y(\xi, n). \tag{5.6}$$

Since $(\{x_i\}_{i=0}^n, \{y_i\}_{i=0}^n)$ is an (f,δ)-good pair of sequences we conclude that

$$\sup\Big\{ \sum_{i=0}^{n-1} f(x_i, x_{i+1}, u_i, u_{i+1}) :\ \{u_i\}_{i=0}^n \in \Lambda_Y(\xi, n) \Big\} - \delta \leq \sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_i, y_{i+1})$$

$$\leq \inf \Big\{ \sum_{i=0}^{n-1} f(p_i, p_{i+1}, y_i, y_{i+1}) :\ \{p_i\}_{i=0}^n \in \Lambda_X(\xi, n) \Big\} + \delta. \tag{5.7}$$

It follows from (5.4)-(5.7) that

$$\begin{aligned} nf(x_f,x_f,y_f,y_f) &\le \sum_{i=0}^{n-1} f(x_i,x_{i+1},y_f,y_f) \le \sum_{i=0}^{n-1} f(x_i,x_{i+1},y_i,y_{i+1}) + \delta \\ &\le \sum_{i=0}^{n-1} f(x_f,x_f,y_i,y_{i+1}) + 2\delta \le nf(x_f,x_f,y_f,y_f) + 2\delta. \end{aligned} \tag{5.8}$$

By (5.8),

$$\Big|\sum_{i=0}^{n-1} f(x_i,x_{i+1},y_i,y_{i+1}) - nf(x_f,x_f,y_f,y_f)\Big| \le \delta, \tag{5.9}$$

$$\sum_{i=0}^{n-1} f(x_i,x_{i+1},y_f,y_f) - nf(x_f,x_f,y_f,y_f) \in [0,2\delta], \tag{5.10}$$

$$\sum_{i=0}^{n-1} f(x_f,x_f,y_i,y_{i+1}) - nf(x_f,x_f,y_f,y_f) \in [-2\delta,0]. \tag{5.11}$$

Set

$$\tilde{x}_i = 2^{-1}(x_i + x_f),\ \tilde{y}_i = 2^{-1}(y_i + y_f),\ i = 0,\dots,n. \tag{5.12}$$

By (5.5), (5.6) and (5.12),

$$\sum_{i=0}^{n-1} f(\tilde{x}_i,\tilde{x}_{i+1},y_f,y_f) \ge nf(x_f,x_f,y_f,y_f) \ge \sum_{i=0}^{n-1} f(x_f,x_f,\tilde{y}_i,\tilde{y}_{i+1}). \tag{5.13}$$

In order to complete the proof of the lemma it is sufficient to show that (5.2) holds.

Assume the contrary. Then in view of (5.1) there is an integer $k \in [1, n-1]$ such that

$$\max\{||x_k - x_f||,\ ||y_k - y_f||\} > \epsilon. \tag{5.14}$$

By (5.12) for all $i = 0,\dots,n-1$,

$$f(\tilde{x}_i,\tilde{x}_{i+1},y_f,y_f) \le 2^{-1}f(x_i,x_{i+1},y_f,y_f) + 2^{-1}f(x_f,x_f,y_f,y_f), \tag{5.15}$$

$$f(x_f,x_f,\tilde{y}_i,\tilde{y}_{i+1}) \ge 2^{-1}f(x_f,x_f,y_i,y_{i+1}) + 2^{-1}f(x_f,x_f,y_f,y_f). \tag{5.16}$$

By (5.12), (5.14)-(5.16), (P1) and (P2),

$$\begin{aligned} &2^{-1}f(x_k,x_{k+1},y_f,y_f) + 2^{-1}f(x_f,x_f,y_f,y_f) - f(\tilde{x}_k,\tilde{x}_{k+1},y_f,y_f) \\ &+f(x_f,x_f,\tilde{y}_k,\tilde{y}_{k+1}) - 2^{-1}f(x_f,x_f,y_k,y_{k+1}) - 2^{-1}f(x_f,x_f,y_f,y_f) \ge \gamma. \end{aligned} \tag{5.17}$$

By (5.10), (5.11), (5.13) and (5.15)-(5.17),

$$\begin{aligned} \gamma &\le \sum_{i=0}^{n-1}[2^{-1}f(x_i,x_{i+1},y_i,y_{i+1}) + 2^{-1}f(x_f,x_f,y_f,y_f) - f(\tilde{x}_i,\tilde{x}_{i+1},y_f,y_f)] \\ &\quad + \sum_{i=0}^{n-1}[f(x_f,x_f,\tilde{y}_i,\tilde{y}_{i+1}) - 2^{-1}f(x_f,x_f,y_i,y_{i+1}) - 2^{-1}f(x_f,x_f,y_f,y_f)] \end{aligned}$$

$$= \sum_{i=0}^{n-1}[f(x_f, x_f, \tilde{y}_i, \tilde{y}_{i+1}) - f(\tilde{x}_i, \tilde{x}_{i+1}, y_f, y_f)$$
$$+2^{-1}f(x_i, x_{i+1}, y_f, y_f) - 2^{-1}f(x_f, x_f, y_i, y_{i+1})] \leq \delta.$$

This contradicts (5.3). The contradiction we have reached proves (5.2). This completes the proof of Lemma. 5.1. □

We can easily prove the following result.

Lemma 5.2. *Let $n \geq 2$ be an integer, M be a positive number and let $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ be an (f, M)-good pair of sequences. Then the pair of sequences $\{\bar{x}_i\}_{i=0}^n \subset X$, $\{\bar{y}_i\}_{i=0}^n \subset Y$ defined by*

$$\bar{x}_i = x_i, \ \bar{y}_i = y_i, \ i = 1, \dots n-1, \quad \bar{x}_0, \bar{x}_n = x_f, \quad \bar{y}_0, \bar{y}_n = y_f$$

is $(f, M + 8D_0)$-good.

By using the uniform continuity of the function $f : X \times X \times Y \times Y \to R^1$ we can easily prove the following lemma.

Lemma 5.3. *Let ϵ be a positive number. Then there exists a number $\delta > 0$ such that for each integer $n \geq 2$ and each sequences*

$$\{x_i\}_{i=0}^n, \ \{\bar{x}_i\}_{i=0}^n \subset X, \ \{y_i\}_{i=0}^n, \ \{\bar{y}_i\}_{i=0}^n \subset Y$$

which satisfy

$$(5.18) \qquad ||\bar{x}_j - x_j||, \ ||\bar{y}_j - y_j|| \leq \delta, \ j = 0, n, \quad x_j = \bar{x}_j, \ y_j = \bar{y}_j, \ j = 1, \dots n-1$$

the following relation holds:

$$\Big|\sum_{i=0}^{n-1}[f(x_i, x_{i+1}, y_i, y_{i+1}) - f(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1})]\Big| \leq \epsilon.$$

Lemma 5.3 implies the following result.

Lemma 5.4. *Assume that $\epsilon > 0$. Then there exists a number $\delta > 0$ such that for each integer $n \geq 2$, each (f, ϵ)-good pair of sequences $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ and each pair of sequences $\{\bar{x}_i\}_{i=0}^n \subset X$, $\{\bar{y}_i\}_{i=0}^n \subset Y$ the following assertion holds: if (5.18) is valid then the pair of sequences $(\{\bar{x}_i\}_{i=0}^n, \{\bar{y}_i\}_{i=0}^n)$ is $(f, 2\epsilon)$-good.*

Lemmas 5.4 and 5.1 imply the following auxiliary result.

Lemma 5.5. *Let $\epsilon \in (0, 1)$. Then there exists a number $\delta \in (0, \epsilon)$ such that for each integer $n \geq 2$ and each (f, δ)-good pair of sequences $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ which satisfies $||x_j - x_f||$, $||y_j - y_f|| \leq \delta$, $j = 0, n$ the following relations hold: $||x_i - x_f||$, $||y_i - y_f|| \leq \epsilon$, $i = 0, \dots n$.*

Denote by Card(E) the cardinality of a set E.

Lemma 5.6. *Let M be a positive number and let $\epsilon \in (0, 1)$. Then there exists an integer $n_0 \geq 4$ such that for each (f, M)-good pair of sequences $\{x_i\}_{i=0}^{n_0} \subset X$, $\{y_i\}_{i=0}^{n_0} \subset Y$ which satisfies*

$$(5.19) \qquad x_0, x_{n_0} = x_f, \quad y_0, y_{n_0} = y_f$$

there is $j \in \{1, \dots n_0 - 1\}$ *for which*

$$||x_j - x_f||, \ ||y_j - y_f|| \le \epsilon. \tag{5.20}$$

Proof. By (A1), (A2) and continuity of f there exists a positive number γ such that:

for each $x \in X$ and each $x' \in X$ satisfying $||x - x_f|| \ge \epsilon$,

$$-f(2^{-1}(x_f + x), 2^{-1}(x_f + x'), y_f, y_f) + 2^{-1} f(x_f, x_f, y_f, y_f) + 2^{-1} f(x, x', y_f, y_f) \ge \gamma; \tag{5.21}$$

for each $y \in Y$ and each $y' \in Y$ satisfying $||y - y_f|| \ge \epsilon$,

$$f(x_f, x_f, 2^{-1}(y_f + y), 2^{-1}(y' + y_f)) - 2^{-1} f(x_f, x_f, y_f, y_f) - 2^{-1} f(x_f, x_f, y, y') \ge \gamma. \tag{5.22}$$

Choose a natural number

$$n_0 > 8 + 2(\gamma)^{-1} M. \tag{5.23}$$

Set

$$\xi_1, \xi_2 = x_f, \quad \xi_3, \xi_4 = y_f, \quad \xi = \{\xi_i\}_{i=1}^4. \tag{5.24}$$

Assume that $\{x_i\}_{i=0}^{n_0} \subset X$, $\{y_i\}_{i=0}^{n_0} \subset Y$ is an (f, M)-good pair of sequences and that (5.19) holds. We show that there is an integer $j \in \{1, \dots, n_0 - 1\}$ such that (5.20) holds. Assume the contrary. Then

$$\max\{||x_j - x_f||, \ ||y_j - y_f||\} > \epsilon, \ j = 1, \dots, n_0 - 1. \tag{5.25}$$

It follows from (5.19), (5.24) and Proposition 2.1 that

$$\sup\Big\{ \sum_{i=0}^{n_0-1} f(x_f, x_f, u_i, u_{i+1}) : \ \{u_i\}_{i=0}^{n_0} \in \Lambda_Y(\xi, n_0)\Big\} = n_0 f(x_f, x_f, y_f, y_f)$$

$$= \inf\Big\{ \sum_{i=0}^{n_0-1} f(p_i, p_{i+1}, y_f, y_f) : \ \{p_i\}_{i=0}^{n_0} \in \Lambda_X(\xi, n_0)\Big\}. \tag{5.26}$$

By (5.19) and (5.24),

$$\{x_i\}_{i=0}^{n_0} \in \Lambda_X(\xi, n_0), \ \{y_i\}_{i=0}^{n_0} \in \Lambda_Y(\xi, n_0). \tag{5.27}$$

Since $(\{x_i\}_{i=0}^{n_0}, \{y_i\}_{i=0}^{n_0})$ is an (f, M)-good pair of sequences we conclude that

$$\sup\Big\{ \sum_{i=0}^{n_0-1} f(x_i, x_{i+1}, u_i, u_{i+1}) : \ \{u_i\}_{i=0}^{n_0} \in \Lambda_Y(\xi, n_0)\Big\} - M \le \sum_{i=0}^{n_0-1} f(x_i, x_{i+1}, y_i, y_{i+1})$$

$$\le \inf\Big\{ \sum_{i=0}^{n_0-1} f(p_i, p_{i+1}, y_i, y_{i+1}) : \ \{p_i\}_{i=0}^{n_0} \in \Lambda_X(\xi, n_0)\Big\} + M. \tag{5.28}$$

It follows from (5.24), (5.26)-(5.28) that

$$\begin{aligned} n_0 f(x_f, x_f, y_f, y_f) \ &\le \ \sum_{i=0}^{n_0-1} f(x_i, x_{i+1}, y_f, y_f) \le \sum_{i=0}^{n_0-1} f(x_i, x_{i+1}, y_i, y_{i+1}) + M \\ &\le \ \sum_{i=0}^{n_0-1} f(x_f, x_f, y_i, y_{i+1}) + 2M \le n_0 f(x_f, x_f, y_f, y_f) + 2M. \end{aligned} \tag{5.29}$$

By (5.29),

$$\left| n_0 f(x_f, x_f, y_f, y_f) - \sum_{i=0}^{n_0-1} f(x_i, x_{i+1}, y_i, y_{i+1}) \right| \le M, \tag{5.30}$$

$$\sum_{i=0}^{n_0-1} f(x_i, x_{i+1}, y_f, y_f) - n_0 f(x_f, x_f, y_f, y_f) \in [0, 2M], \tag{5.31}$$

$$\sum_{i=0}^{n_0-1} f(x_f, x_f, y_i, y_{i+1}) - n_0 f(x_f, x_f, y_f, y_f) \le [-2M, 0]. \tag{5.32}$$

Set

$$\tilde{x}_i = 2^{-1}(x_i + x_f),\ \tilde{y}_i = 2^{-1}(y_i + y_f),\ i = 0, 1, \dots, n_0. \tag{5.33}$$

By (5.24), (5.26), (5.27) and (5.33),

$$\sum_{i=0}^{n_0-1} f(\tilde{x}_i, \tilde{x}_{i+1}, y_f, y_f) \ge n_0 f(x_f, x_f, y_f, y_f) \ge \sum_{i=0}^{n_0-1} f(x_f, x_f, \tilde{y}_i, \tilde{y}_{i+1}). \tag{5.34}$$

In view of (5.33) for $i = 0, \dots, n_0 - 1$,

$$f(\tilde{x}_i, \tilde{x}_{i+1}, y_f, y_f) \le 2^{-1} f(x_i, x_{i+1}, y_f, y_f) + 2^{-1} f(x_f, x_f, y_f, y_f), \tag{5.35}$$

$$f(x_f, x_f, \tilde{y}_i, \tilde{y}_{i+1}) \ge 2^{-1} f(x_f, x_f, y_i, y_{i+1}) + 2^{-1} f(x_f, x_f, y_f, y_f). \tag{5.36}$$

By (5.35), (5.25), the choice of γ (see (5.21) and (5.22)) and (5.33) for each $i = 1, \dots, n_0 - 1$ at least one of the following inequalities holds:

$$2^{-1} f(x_i, x_{i+1}, y_f, y_f) + 2^{-1} f(x_f, x_f, y_f, y_f) - f(\tilde{x}_i, \tilde{x}_{i+1}, y_f, y_f) \ge \gamma,$$

$$f(x_f, x_f, \tilde{y}_i, \tilde{y}_{i+1}) - 2^{-1} f(x_f, x_f, y_i, y_{i+1}) - 2^{-1} f(x_f, x_f, y_f, y_f) \ge \gamma.$$

Combined with (5.35) and (5.36) this implies that for each $i = 1, \dots, n_0 - 1$

$$\begin{aligned} &2^{-1} f(x_i, x_{i+1}, y_f, y_f) + 2^{-1} f(x_f, x_f, y_f, y_f) - f(\tilde{x}_i, \tilde{x}_{i+1}, y_f, y_f) \\ &\quad + f(x_f, x_f, \tilde{y}_i, \tilde{y}_{i+1}) - 2^{-1} f(x_f, x_f, y_i, y_{i+1}) - 2^{-1} f(x_f, x_f, y_f, y_f) \ge \gamma. \end{aligned}$$

Together with (5.35), (5.36) and (5.34) this implies that

$$\begin{aligned} \gamma(n_0 - 1) \le &\sum_{i=0}^{n_0-1} [2^{-1} f(x_i, x_{i+1}, y_f, y_f) + 2^{-1} f(x_f, x_f, y_f, y_f) - f(\tilde{x}_i, \tilde{x}_{i+1}, y_f, y_f) \\ &\quad + f(x_f, x_f, \tilde{y}_i, \tilde{y}_{i+1}) - 2^{-1} f(x_f, x_f, y_i, y_{i+1}) - 2^{-1} f(x_f, x_f, y_f, y_f)] \\ = &\sum_{i=0}^{n_0-1} [f(x_f, x_f, \tilde{y}_i, \tilde{y}_{i+1}) - f(\tilde{x}_i, \tilde{x}_{i+1}, y_f, y_f) \\ &\quad + 2^{-1} f(x_i, x_{i+1}, y_f, y_f) - 2^{-1} f(x_f, x_f, y_i, y_{i+1}]) \\ \le &n_0 f(x_f, x_f, y_f, y_f) - n_0 f(x_f, x_f, y_f, y_f) + 2^{-1} n_0 f(x_f, x_f, y_f, y_f) + M \\ &\quad - 2^{-1} n_0 f(x_f, x_f, y_f, y_f) + M \le 2M, \\ \gamma(n_0 - 1) \le &2M. \end{aligned}$$

This contradicts (5.23). The contradiction we have reached proves that there is an integer $j \in \{1, \dots, n_0 - 1\}$ such that (5.20) holds. Lemma 5.6 is proved. □

Lemmas 5.6 and 5.2 imply the following auxiliary result.

Lemma 5.7. *Let $\epsilon \in (0,1)$, $M \in (0,\infty)$. Then there exists an integer $n_0 \geq 4$ such that for each (f,M)-good pair of sequences $\{x_i\}_{i=0}^{n_0} \subset X$, $\{y_i\}_{i=0}^{n_0} \subset Y$ there is $j \in \{1, \dots, n_0 - 1\}$ for which $||x_f - x_j||$, $||y_f - y_j|| \leq \epsilon$.*

Lemma 5.8. *Let $\epsilon \in (0,1)$, $M \in (0,\infty)$. Then there exists an integer $n_0 \geq 4$ and a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that for each $g \in \mathcal{U}$ and each (g,M)-good pair of sequences $\{x_i\}_{i=0}^{n_0} \subset X$, $\{y_i\}_{i=0}^{n_0} \subset Y$ there is $j \in \{1, \dots n_0 - 1\}$ for which*

$$||x_f - x_j||,\ ||y_f - y_j|| \leq \epsilon. \tag{5.37}$$

Proof. By Lemma 5.7 there is an integer $n_0 \geq 4$ such that for each $(f, M+8)$-good pair of sequences $\{x_i\}_{i=0}^{n_0} \subset X$, $\{y_i\}_{i=0}^{n_0} \subset Y$ there is $j \in \{1, \dots n_0 - 1\}$ for which (5.37) is valid. Set

$$\mathcal{U} = \{g \in \mathcal{M} :\ \rho(f,g) \leq (16n_0)^{-1}\}. \tag{5.38}$$

Assume that $g \in \mathcal{U}$ and $\{x_i\}_{i=0}^{n_0} \subset X$, $\{y_i\}_{i=0}^{n_0} \subset Y$ is a (g,M)-good pair of sequences. By (5.38) the pair of sequences $\{x_i\}_{i=0}^{n_0}$, $\{y_i\}_{i=0}^{n_0}$ is $(f, M+8)$-good. It follows from the definition of n_0 that there exists $j \in \{1, \dots n_0 - 1\}$ for which (5.37) is valid. The lemma is proved. □

6. Proof of Theorem 3.1

By Lemma 5.5 there exists $\delta_0 \in (0, \epsilon)$ such that the following property holds:

(P3) for each integer $n \geq 2$ and each (f, δ_0)-good pair of sequences $\{x_i\}_{i=0}^{n} \subset X$, $\{y_i\}_{i=0}^{n} \subset Y$ satisfying

$$||x_j - x_f||,\ ||y_j - y_f|| \leq \delta_0, \quad j = 0, n$$

we have

$$||x_j - x_f||,\ ||y_j - y_f|| \leq \epsilon \tag{6.1}$$

for all $i = 0, \dots, n$.

By Lemma 5.8 there exists an integer $n_0 \geq 4$ and a neighborhood $\mathcal{U}_0$ of f in $\mathcal{M}$ such that the following property holds:

(P4) for each $g \in \mathcal{U}_0$ and each $(g, 8)$-good pair of sequences $\{x_i\}_{i=0}^{n_0} \subset X$, $\{y_i\}_{i=0}^{n_0} \subset Y$ there is $j \in \{1, \dots n_0 - 1\}$ for which

$$||x_j - x_f||,\ ||y_j - y_f|| \leq \delta_0.$$

Fix an integer

$$n_1 \geq 4n_0 \tag{6.2}$$

and a number

$$\delta \in (0, 4^{-1}\delta_0). \tag{6.3}$$

Define

$$\mathcal{U} = \mathcal{U}_0 \cap \{g \in \mathcal{M} : \quad \rho(g,f) \leq 16^{-1}\delta n_1^{-1}\}. \tag{6.4}$$

Assume that $g \in \mathcal{U}$, an integer $n \geq 2n_1$ and $\{x_i\}_{i=0}^n \subset X$, $\{y_i\}_{i=0}^n \subset Y$ is a (g, δ)-good pair of sequences. It follows from (6.2)-(6.4) and (P4) that there exists a sequence of integers $\{t_i\}_{i=1}^k \subset [0, n]$ such that

$$t_1 \leq n_0, \ t_{i+1} - t_i \in [n_0, 3n_0], \ i = 1, \dots k-1, \tag{6.5}$$

$$n - t_k \leq n_0, \quad ||x_{t_i} - x_f||, \ ||y_{t_i} - y_f|| \leq \delta_0, \ i = 1, \dots k \tag{6.6}$$

and, moreover, if $||x_0 - x_f||, \ ||y_0 - y_f|| \leq \delta$ then $t_1 = 0$, and if $||x_n - x_f||, \ ||y_n - y_f|| \leq \delta$ then $t_k = n$. Clearly $k \geq 2$. Fix $q \in \{1, \dots k-1\}$. To complete the proof of the theorem it is sufficient to show that for each integer $i \in [t_q, t_{q+1}]$ the relation (6.1) holds.

Define sequences $\{x_i^{(q)}\}_{i=0}^{t_{q+1}-t_q} \subset X$, $\{y_i^{(q)}\}_{i=0}^{t_{q+1}-t_q} \subset Y$ by

$$x_i^{(q)} = x_{i+t_q}, \ y_i^{(q)} = y_{i+t_q}, \quad i \in [0, t_{q+1} - t_q]. \tag{6.7}$$

It is easy to see that $\{x_i^{(q)}\}_{i=0}^{t_{q+1}-t_q}$, $\{y_i^{(q)}\}_{i=0}^{t_{q+1}-t_q}$ is a (g, δ)-good pair of sequences. Together with (6.2)-(6.6) this implies that the pair $\{x_i^{(q)}\}_{i=0}^{t_{q+1}-t_q}$, $\{y_i^{(q)}\}_{i=0}^{t_{q+1}-t_q}$ is (f, δ_0)-good. It follows from (6.5), (6.6) and (P3) that

$$||x_i^{(q)} - x_f||, \ ||y_i^{(q)} - y_f|| \leq \epsilon, \quad i = 0, \dots t_{q+1} - t_q.$$

Together with (6.7) this implies that $||x_i - x_f||, \ ||y_i - y_f|| \leq \epsilon, \ i = t_q, \dots t_{q+1}$. This completes the proof of the Theorem 3.1.

7. Preliminary lemmas for Theorem 3.2

Let $f \in \mathcal{M}$. $x_f \in X$, $y_f \in Y$ satisfy (3.1). We use all the definitions, notation and assumptions made in Section 3. In particular we suppose that assumptions (A1) and (A2) hold.

For each metric space K denote by $C(K)$ the space of all continuous functions on K with the topology of uniform convergence ($||\phi|| = \sup\{|\phi(z)| : z \in K\}$, $\phi \in C(K)$).

Define functions $f^{(X)} : X \times X \to R^1$, $f^{(Y)} : Y \times Y \to R^1$ by

$$f^{(X)}(x_1, x_2) = f(x_1, x_2, y_f, y_f), \quad x_1, x_2 \in X, \tag{7.1}$$

$$f^{(Y)}(y_1, y_2) = f(x_f, x_f, y_1, y_2), \quad y_1, y_2 \in Y. \tag{7.2}$$

Lemma 7.1. *Let $\epsilon \in (0,1)$. Then there exists a number $\delta \in (0, \epsilon)$ for which the following assertion holds:*

Assume that an integer $n \geq 2$,

$$\{x_i\}_{i=0}^n \subset X, \quad x_0, x_n = x_f \tag{7.3}$$

and for each $\{z_i\}_{i=0}^n \subset X$ satisfying

$$z_0 = x_0, \quad z_n = x_n \tag{7.4}$$

the relation

$$\sum_{i=0}^{n-1} f^{(X)}(x_i, x_{i+1}) \leq \sum_{i=0}^{n-1} f^{(X)}(z_i, z_{i+1}) + \delta \tag{7.5}$$

holds. Then

$$||x_i - x_f|| \leq \epsilon, \quad i = 0, \dots n. \tag{7.6}$$

Proof. By (A1) and continuity of f there exists s positive number γ such that the following property holds:

(P5) for each $x \in X$ and each $x' \in X$ satisfying $||x - x_f|| \geq \epsilon$,

$$-f(2^{-1}(x_f + x), 2^{-1}(x_f + x'), y_f, y_f) + 2^{-1}f(x_f, x_f, y_f, y_f) + 2^{-1}f(x, x', y_f, y_f) \geq \gamma.$$

Choose a positive number δ such that

$$\delta < \min\{8^{-1}\epsilon, \ \gamma/4\}. \tag{7.7}$$

Assume that an integer $n \geq 2$, $\{x_i\}_{i=0}^n \subset X$, (7.3) is valid and for each sequence $\{z_i\}_{i=0}^n \subset X$ satisfying (7.4), relation (7.5) holds. We show that (7.6) holds. Assume the contrary. Then there is an integer $j \in \{1, \dots, n-1\}$. such that

$$||x_j - x_f|| > \epsilon. \tag{7.8}$$

Since (7.5) holds with $z_i = x_f$, $i = 0, \dots, n$ it follows from Proposition 2.1 that

$$\sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_f, y_f) \leq nf(x_f, x_f, y_f, y_f) + \delta \leq \sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_f, y_f) + \delta. \tag{7.9}$$

Set

$$\tilde{x}_i = 2^{-1}(x_i + x_f), \ i = 0, \dots, n-1. \tag{7.10}$$

By (7.1) and (7.10) for $i = 0, \dots, n-1$,

$$f^{(X)}(\tilde{x}_i, \tilde{x}_{i+1}) \leq 2^{-1}f^{(X)}(x_i, x_{i+1}) + 2^{-1}f^{(X)}(x_f, x_f). \tag{7.11}$$

By (7.8), (7.9) and (P5),

$$f(\tilde{x}_j, \tilde{x}_{j+1}, y_f, y_f) \leq 2^{-1}f(x_j, x_{j+1}, y_f, y_f) + 2^{-1}f(x_f, x_f, y_f, y_f) - \gamma. \tag{7.12}$$

In view of (7.3), (7.9)-(7.12) and Proposition 2.1,

$$\begin{aligned} nf(x_f, x_f, y_f, y_f) &\leq \sum_{i=0}^{n-1} f(\tilde{x}_i, \tilde{x}_{i+1}, y_f, y_f) \\ &\leq 2^{-1}\sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_f, y_f) + 2^{-1}nf(x_f, x_f, y_f, y_f) - \gamma \\ &\leq nf(x_f, x_f, y_f, y_f) + \delta - \gamma. \end{aligned}$$

This contradicts (7.7). The contradiction we have reached proves that (7.6) holds. Lemma 7.1 is proved. □

Analogously to Lemma 7.1 we can establish the following auxiliary result.

Lemma 7.2. *Let $\epsilon \in (0, 1)$. Then there exists a number $\delta \in (0, \epsilon)$ for which the following assertion holds:*

Assume that an integer $n \geq 2$,

$$\{y_i\}_{i=0}^n \subset Y, \quad y_0, y_n = y_f$$

and for each $\{z_i\}_{i=0}^n \subset Y$ *satisfying*

$$z_0 = y_0, \quad z_n = y_n \tag{7.13}$$

the relation

$$\sum_{i=0}^{n-1} f^{(Y)}(y_i, y_{i+1}) \geq \sum_{i=0}^{n-1} f^{(Y)}(z_i, z_{i+1}) - \delta \tag{7.14}$$

holds. Then

$$||y_i - y_f|| \leq \epsilon, \quad i = 0, \dots n. \tag{7.15}$$

Let $g \in C(X \times X)$, $n \geq 1$ be an integer and let $M \in [0, \infty)$. A sequence $\{\bar{x}_i\}_{i=0}^n \subset X$ is called (g, X, M)-good if for each sequence $\{x_i\}_{i=0}^n \subset X$ satisfying $x_0 = \bar{x}_0$, $x_n = \bar{x}_n$ the relation $M + \sum_{i=0}^{n-1} g(x_i, x_{i+1}) \geq \sum_{i=0}^{n-1} g(\bar{x}_i, \bar{x}_{i+1})$ is valid.

Let $g \in C(Y \times Y)$, $n \geq 1$ be an integer and let $M \in [0, \infty)$. A sequence $\{\bar{y}_i\}_{i=0}^n \subset Y$ is called (g, Y, M)-good if for each sequence $\{y_i\}_{i=0}^n \subset Y$ satisfying $y_0 = \bar{y}_0$, $y_n = \bar{y}_n$ the relation $\sum_{i=0}^{n-1} g(y_i, y_{i+1}) \leq M + \sum_{i=0}^{n-1} g(\bar{y}_i, \bar{y}_{i+1})$ is valid.

Let $n_1 \geq 0$, $n_2 > n_1$ be integers, and let $\{g_i\}_{i=n_1}^{n_2-1} \subset C(X \times X)$, $M \in [0, \infty)$. A sequence $\{\bar{x}_i\}_{i=n_1}^{n_2} \subset X$ is called $(\{g_i\}_{i=n_1}^{n_2-1}, X, M)$-good if for each sequence $\{x_i\}_{i=n_1}^{n_2} \subset X$ satisfying $x_{n_1} = \bar{x}_{n_1}$, $x_{n_2} = \bar{x}_{n_2}$

$$M + \sum_{i=n_1}^{n_2-1} g_i(x_i, x_{i+1}) \geq \sum_{i=n_1}^{n_2-1} g_i(\bar{x}_i, \bar{x}_{i+1}).$$

Let $n_1 \geq 0$, $n_2 > n_1$ be integers, and let $\{g_i\}_{i=n_1}^{n_2-1} \subset C(Y \times Y)$, $M \in [0, \infty)$. A sequence $\{\bar{y}_i\}_{i=n_1}^{n_2} \subset Y$ is called $(\{g_i\}_{i=n_1}^{n_2-1}, Y, M)$-good if for each sequence $\{y_i\}_{i=n_1}^{n_2} \subset Y$ satisfying $y_{n_1} = \bar{y}_{n_1}$, $y_{n_2} = \bar{y}_{n_2}$

$$\sum_{i=n_1}^{n_2-1} g_i(y_i, y_{i+1}) \leq \sum_{i=n_1}^{n_2-1} g_i(\bar{y}_i, \bar{y}_{i+1}) + M.$$

By using Lemmas 7.1 and 5.3 we can easily deduce the following auxiliary result.

Lemma 7.3. *Let* $\epsilon \in (0, 1)$. *Then there exists a number* $\delta > 0$ *such that for each integer* $n \geq 2$ *and each* $(f^{(X)}, X, \delta)$*-good sequence* $\{x_i\}_{i=0}^n \subset X$ *satisfying* $||x_0 - x_f||$, $||x_n - x_f|| \leq \delta$ *the following relation holds:* $||x_i - x_f|| \leq \epsilon$, $i = 0, \dots n$.

By using Lemmas 7.2 and 5.3 we can easily deduce the following auxiliary result.

Lemma 7.4. *Let* $\epsilon \in (0, 1)$. *Then there exists a number* $\delta > 0$ *such that for each integer* $n \geq 2$ *and each* $(f^{(Y)}, Y, \delta)$*-good sequence* $\{y_i\}_{i=0}^n \subset Y$ *satisfying* $||y_0 - y_f||$, $||y_n - y_f|| \leq \delta$ *the following relation holds:* $||y_i - y_f|| \leq \epsilon$, $i = 0, \dots n$.

Lemma 7.5. *Let* $\epsilon \in (0, 1)$ *and* M *be a positive number. Then there exists an integer* $n_0 \geq 4$ *such that for each* $(f^{(X)}, X, M)$*-good sequence* $\{x_i\}_{i=0}^{n_0} \subset X$ *satisfying*

$$x_0 = x_f, \ x_{n_0} = x_f \tag{7.16}$$

there is $j \in \{1, \dots n_0 - 1\}$ *for which*

$$||x_j - x_f|| \leq \epsilon. \tag{7.17}$$

Proof. By (A1) there exists s positive number γ such that the following property holds:

(P6) for each $x \in X$ and each $x' \in X$ satisfying $||x - x_f|| \geq \epsilon$,

$$-f(2^{-1}(x_f + x), 2^{-1}(x_f + x'), y_f, y_f) + 2^{-1}f(x_f, x_f, y_f, y_f) + 2^{-1}f(x, x', y_f, y_f) \geq \gamma.$$

Choose a natural number

$$n_0 > 8 + M\gamma^{-1}. \tag{7.18}$$

Assume that an $(f^{(X)}, X, M)$-good sequence $\{x_i\}_{i=0}^{n_0} \subset X$ satisfies (7.16). We show that there is an integer $j \in \{1, \ldots, n_0 - 1\}$ such that (7.17) holds. Assume the contrary. Then

$$||x_i - x_f|| > \epsilon, \ i = 1, \ldots, n_0 - 1. \tag{7.19}$$

Set

$$\tilde{x}_i = 2^{-1}x_i + 2^{-1}x_f, \ i = 0, \ldots, n_0. \tag{7.20}$$

By (7.20) for $i = 0, \ldots, n_0 - 1$,

$$f(\tilde{x}_i, \tilde{x}_{i+1}, y_f, y_f) \leq 2^{-1}f(x_i, x_{i+1}, y_f, y_f) + 2^{-1}f(x_f, x_f, y_f, y_f). \tag{7.21}$$

It follows from (7.19), (7.20) and (P6) that for $i = 1, \ldots, n_0 - 1$

$$f(\tilde{x}_i, \tilde{x}_{i+1}, y_f, y_f) \leq 2^{-1}f(x_i, x_{i+1}, y_f, y_f) + 2^{-1}f(x_f, x_f, y_f, y_f) - \gamma. \tag{7.22}$$

By (7.16), (7.20), (7.22), (7.24) and Proposition 2.1

$$\begin{aligned} n_0 f(x_f, x_f, y_f, y_f) &\leq \sum_{i=0}^{n_0-1} f(\tilde{x}_i, \tilde{x}_{i+1}, y_f, y_f) \\ &\leq 2^{-1} \sum_{i=0}^{n_0-1} f(x_i, x_{i+1}, y_f, y_f) \\ &\quad +2^{-1}n_0 f(x_f, x_f, y_f, y_f) - (n_0 - 1)\gamma \\ &\leq 2^{-1}(M + n_0 f(x_f, x_f, y_f, y_f)) \\ &\quad +2^{-1}n_0 f(x_f, x_f, y_f, y_f) - (n_0 - 1)\gamma, \\ (n_0 - 1)\gamma &\leq 2^{-1}M. \end{aligned}$$

This contradicts (7.18). The contradiction we have reached proves that there is an integer $j \in \{1, \ldots, n_0 - 1\}$ such that (7.17) holds. Lemma 7.5 is proved. □

Analogously to Lemma 7.5 we can establish the following auxiliary result.

Lemma 7.6. *Let $\epsilon \in (0, 1)$ and M be a positive number. Then there exists an integer $n_0 \geq 4$ such that for each $(f^{(Y)}, Y, M)$-good sequence $\{y_i\}_{i=0}^{n_0} \subset Y$ satisfying $y_0 = y_f$, $y_{n_0} = y_f$ there is $j \in \{1, \ldots n_0 - 1\}$ for which $||y_j - y_f|| \leq \epsilon$.*

We can easily prove the following result.

Lemma 7.7.

1. *Assume that* $n \geq 2$ *is an integer,* M *is a positive number, a sequence* $\{x_i\}_{i=0}^n \subset X$ *is* $(f^{(X)}, X, M)$*-good and* $\bar{x}_0 = x_f,\ \bar{x}_n = x_f,\ \bar{x}_i = x_i, \quad i = 1, \dots n-1$. *Then the sequence* $\{\bar{x}_i\}_{i=0}^n$ *is* $(f^{(X)}, X, M + 8D_0)$*-good.*

2. *Assume that* $n \geq 2$ *is an integer,* M *is a positive number, a sequence* $\{y_i\}_{i=0}^n \subset Y$ *is* $(f^{(Y)}, Y, M)$*-good and* $\bar{y}_0 = y_f,\ \bar{y}_n = y_f,\ \bar{y}_i = y_i,\ i = 1, \dots n-1$. *Then the sequence* $\{\bar{y}_i\}_{i=0}^n$ *is* $(f^{(Y)}, Y, M + 8D_0)$*-good.*

Lemmas 7.5, 7.6 and 7.7 imply the following two results.

Lemma 7.8. *Let* $\epsilon \in (0,1)$ *and* M *be a positive number. Then there exists an integer* $n_0 \geq 4$ *such that for each* $(f^{(X)}, X, M)$*-good sequence* $\{x_i\}_{i=0}^{n_0} \subset X$ *there is* $j \in \{1, \dots n_0 - 1\}$ *for which* $||x_j - x_f|| \leq \epsilon$.

Lemma 7.9. *Let* $\epsilon \in (0,1)$ *and* M *be a positive number. Then there exists an integer* $n_0 \geq 4$ *such that for each* $(f^{(Y)}, Y, M)$*-good sequence* $\{y_i\}_{i=0}^{n_0} \subset Y$ *there is* $j \in \{1, \dots n_0 - 1\}$ *for which* $||y_j - y_f|| \leq \epsilon$.

By using Lemmas 7.8 and 7.9, analogously to the proof of Lemma 5.8, we can establish the following two results.

Lemma 7.10. *Let* $\epsilon \in (0,1)$, $M \in (0,\infty)$. *Then there exists an integer* $n_0 \geq 4$ *and a neighborhood* $\mathcal{U}$ *of* $f^{(X)}$ *in* $C(X \times X)$ *such that for each* $\{g_i\}_{i=0}^{n_0-1} \subset \mathcal{U}$ *and each* $(\{g_i\}_{i=0}^{n_0-1}, X, M)$*-good sequence* $\{x_i\}_{i=0}^{n_0} \subset X$ *there is* $j \in \{1, \dots n_0 - 1\}$ *for which* $||x_f - x_j|| \leq \epsilon$.

Lemma 7.11. *Let* $\epsilon \in (0,1)$, $M \in (0,\infty)$. *Then there exists an integer* $n_0 \geq 4$ *and a neighborhood* $\mathcal{U}$ *of* $f^{(Y)}$ *in* $C(Y \times Y)$ *such that for each* $\{g_i\}_{i=0}^{n_0-1} \subset \mathcal{U}$ *and each* $(\{g_i\}_{i=0}^{n_0-1}, Y, M)$*-good sequence* $\{y_i\}_{i=0}^{n_0} \subset Y$ *there is* $j \in \{1, \dots n_0 - 1\}$ *for which* $||y_f - y_j|| \leq \epsilon$.

Lemma 7.12. *Let* $\epsilon \in (0,1)$. *Then there exist a neighborhood* $\mathcal{U}$ *of* $f^{(X)}$ *in* $C(X \times X)$, *a number* $\delta \in (0,\epsilon)$ *and an integer* $n_1 \geq 4$ *such that for each integer* $n \geq 2n_1$, *each* $\{g_i\}_{i=0}^{n-1} \subset \mathcal{U}$ *and each* $(\{g_i\}_{i=0}^{n-1}, X, \delta)$*-good sequence* $\{x_i\}_{i=0}^n \subset X$ *the relation*

$$||x_i - x_f|| \leq \epsilon \tag{7.23}$$

holds for all integers $i \in [n_1, n - n_1]$. *Moreover if* $||x_0 - x_f|| \leq \delta$ *then (7.23) holds for all integers* $i \in [0, n - n_1]$, *and if* $||x_n - x_f|| \leq \delta$ *then (7.23) is valid for all integers* $i \in [n_1, n]$.

Proof. By Lemma 7.3 there exists $\delta_0 \in (0,\epsilon)$ such that the following property holds:

(P7) for each integer $n \geq 2$ and each $(f^{(X)}, X, \delta_0)$-good sequence $\{x_i\}_{i=0}^n \subset X$ satisfying $||x_0 - x_f||$, $||x_n - x_f|| \leq \delta_0$ relation (7.23) is valid for $i = 0, \dots n$.

By Lemma 7.10 there exist an integer $n_0 \geq 4$ and a neighborhood $\mathcal{U}_0$ of $f^{(X)}$ in $C(X \times X)$ such that the following property holds:

(P8) for each $\{g_i\}_{i=0}^{n_0-1} \subset \mathcal{U}_0$ and each $(\{g_i\}_{i=0}^{n_0-1}, X, 8)$-good sequence $\{x_i\}_{i=0}^n \subset X$ there is $j \in \{1, \dots n_0 - 1\}$ for which $||x_j - x_f|| \leq \delta_0$.

Choose an integer $n_1 \geq 4n_0$ and a number $\delta \in (0, 4^{-1}\delta_0)$. Define

$$\mathcal{U} = \mathcal{U}_0 \cap \{g \in C(X \times X) : ||g - f^{(X)}|| \leq (16n_1)^{-1}\delta\}.$$

Assume that an integer $n \geq 2n_1$, $\{g_i\}_{i=0}^{n-1} \subset \mathcal{U}$ and a sequence $\{x_i\}_{i=0}^{n} \subset X$ is $(\{g_i\}_{i=0}^{n-1}, X, \delta)$-good. Arguing as in the proof of Theorem 3.1 we complete the proof of Lemma 7.12. □

Analogously to Lemma 7.12 we can prove the following

Lemma 7.13. *Let $\epsilon \in (0,1)$. Then there exist a neighborhood $\mathcal{U}$ of $f^{(Y)}$ in $C(Y \times Y)$, a number $\delta \in (0,\epsilon)$ and an integer $n_1 \geq 4$ such that for each integer $n \geq 2n_1$, each $\{g_i\}_{i=0}^{n-1} \subset \mathcal{U}$ and each $(\{g_i\}_{i=0}^{n-1}, Y, \delta)$-good sequence $\{y_i\}_{i=0}^{n} \subset Y$ the relation*

$$||y_i - y_f|| \leq \epsilon \tag{7.24}$$

holds for all integers $i \in [n_1, n - n_1]$. Moreover if $||y_0 - y_f|| \leq \delta$ then (7.24) holds for all integers $i \in [0, n - n_1]$, and if $||y_n - y_f|| \leq \delta$ then (7.24) is valid for all integers $i \in [n_1, n]$.

8. Proof of Theorems 3.2

Let $x \in X$ and $y \in Y$. By Proposition 2.3 there is an (f)-minimal pair of sequences $\{\bar{x}_j\}_{j=0}^{\infty} \subset X$, $\{\bar{y}_j\}_{j=0}^{\infty} \subset Y$ for which

$$\bar{x}_0 = x, \quad \bar{y}_0 = y. \tag{8.1}$$

We will show that the pair of sequences $(\{\bar{x}_j\}_{j=0}^{\infty}, \{\bar{y}_j\}_{j=0}^{\infty})$ is (f)-overtaking optimal. Theorem 3.1 implies that

$$\bar{x}_j \to x_f, \quad \bar{y}_j \to y_f \text{ as } j \to \infty. \tag{8.2}$$

Let $\{x_i\}_{i=0}^{\infty} \subset X$ and $x_0 = x$. We will show that

$$\limsup_{T\to\infty} \Big[\sum_{j=0}^{T-1} f(\bar{x}_j, \bar{x}_{j+1}, \bar{y}_j, \bar{y}_{j+1}) - \sum_{j=0}^{T-1} f(x_j, x_{j+1}, \bar{y}_j, \bar{y}_{j+1}) \Big] \leq 0. \tag{8.3}$$

Assume the contrary. Then there exists a number $\Gamma_0 > 0$ and a strictly increasing sequence of natural numbers $\{T_k\}_{k=1}^{\infty}$ such that for all integers $k \geq 1$

$$\sum_{j=0}^{T_k-1} f(\bar{x}_j, \bar{x}_{j+1}, \bar{y}_j, \bar{y}_{j+1}) - \sum_{j=0}^{T_k-1} f(x_j, x_{j+1}, \bar{y}_j, \bar{y}_{j+1}) \geq \Gamma_0. \tag{8.4}$$

We will show that

$$x_j \to x_f \text{ as } j \to \infty.$$

For $j = 0, 1, \ldots$ define a function $g_j : X \times X \to R^1$ by

$$g_j(u_1, u_2) = f(u_1, u_2, \bar{y}_j, \bar{y}_{j+1}), \quad u_1, u_1 \in X. \tag{8.5}$$

By (8.2),

$$\lim_{j\to\infty} ||g_j - f^{(X)}|| = 0. \tag{8.6}$$

Since the pair of sequences $(\{\bar{x}_j\}_{j=0}^{\infty}, \{\bar{y}_j\}_{j=0}^{\infty})$ is (f)-minimal there exists a constant $c_0 > 0$ such that for each integer $T \geq 1$

$$\sum_{i=0}^{T-1} f(\bar{x}_j, \bar{x}_{j+1}, \bar{y}_j, \bar{y}_{j+1}) \leq \inf\{\sum_{j=0}^{T-1} f(u_j, u_{j+1}, \tilde{y}_j, \tilde{u}_{j+1}) :$$

$$\{u_j\}_{j=0}^{T} \subset X,\ u_0 = z\} + c_0. \tag{8.7}$$

Equations (8.4), (8.5) and (8.7) imply that the following property holds:

(P9) For each $\Delta > 0$ there exists an integer $j(\Delta) \geq 1$ such that for each pair of integers $n_1 \geq j(\Delta)$, $n_2 > n_1$ the sequence $\{x_j\}_{j=n_1}^{n_2}$ is $(\{g_j\}_{j=n_1}^{n_2-1}, X, \Delta)$-good.

By (P9) and Theorem 3.1

$$\lim_{j\to\infty} x_j = x_f. \tag{8.8}$$

There exists a number $\epsilon_0 > 0$ such that for each $z_1,\ z_2,\ \bar{z}_1,\ \bar{z}_2 \in X$ and each $\xi_1,\ \xi_2,\ \bar{\xi}_1,\ \bar{\xi}_2 \in Y$ which satisfy

$$||z_j - \bar{z}_j||,\ ||\xi_j - \bar{\xi}_j|| \leq 2\epsilon_0, \quad j = 1, 2 \tag{8.9}$$

the following relation holds:

$$|f(z_1, z_2, \xi_1, \xi_2) - f(\bar{z}_1, \bar{z}_2, \bar{\xi}_1, \bar{\xi}_2)| \leq 8^{-1}\Gamma_0. \tag{8.10}$$

By (8.8) and (8.2) there exists an integer $j_0 \geq 8$ such that for all integers $j \geq j_0$

$$||x_j - x_f|| \leq 2^{-1}\epsilon_0, \quad ||\bar{x}_j - x_f|| \leq 2^{-1}\epsilon_0. \tag{8.11}$$

There exists an integer $s \geq 1$ such that

$$T_s > j_0. \tag{8.12}$$

Define a sequence $\{x_j^*\}_{j=0}^{T_s} \subset X$ by

$$x_j^* = x_j,\ j = 0, \dots T_s - 1, \quad x_{T_s}^* = \bar{x}_{T_s}. \tag{8.13}$$

Since the pair of sequences $(\{\bar{x}_j\}_{j=0}^{\infty}, \{\bar{y}_j\}_{j=0}^{\infty})$ is (f)-minimal we conclude that

$$\sum_{j=0}^{T_s-1} f(\bar{x}_j, \bar{x}_{j+1}, \bar{y}_j, \bar{y}_{j+1}) - \sum_{j=0}^{T_s-1} f(x_j^*, x_{j+1}^*, \bar{y}_j, \bar{y}_{j+1}) \leq 0. \tag{8.14}$$

On the other hand it follows from (8.4), (8.11)-(8.13) and the definition of ϵ_0 (see (8.9), (8.10)) that

$$\begin{aligned}
&\sum_{j=0}^{T_s-1} f(\bar{x}_j, \bar{x}_{j+1}, \bar{y}_j, \bar{y}_{j+1}) - \sum_{j=0}^{T_s-1} f(x_j^*, x_{j+1}^*, \bar{y}_j, \bar{y}_{j+1}) \\
&= \sum_{j=0}^{T_s-1} f(\bar{x}_j, \bar{x}_{j+1}, \bar{y}_j, \bar{y}_{j+1}) - \sum_{j=0}^{T_s-1} f(x_j, x_{j+1}, \bar{y}_j, \bar{y}_{j+1}) \\
&\quad + f(x_{T_s-1}, x_{T_s}, \bar{y}_{T_s-1}, \bar{y}_{T_s}) - f(x_{T_s-1}^*, x_{T_s}^*, \bar{y}_{T_s-1}, \bar{y}_{T_s}) \\
&\geq \Gamma_0 + f(x_{T_s-1}, x_{T_s}, \bar{y}_{T_s-1}, \bar{y}_{T_s}) - f(x_{T_s-1}, \bar{x}_{T_s}, \bar{y}_{T_s-1}, \bar{y}_{T_s}) \\
&\geq \Gamma_0 - 8^{-1}\Gamma_0.
\end{aligned}$$

This is contradictory to (8.14). The obtained contradiction proves that (8.3) holds.

Analogously we can show that for each sequence $\{y_j\}_{j=0}^{\infty} \subset Y$ satisfying $y_0 = y$

$$\limsup_{T\to\infty} \Big[\sum_{j=0}^{T-1} f(\bar{x}_j, \bar{x}_{j+1}, y_j, y_{j+1}) - \sum_{j=0}^{T-1} f(\bar{x}_j, \bar{x}_{j+1}, \bar{y}_j, \bar{y}_{j+1})\Big] \leq 0.$$

This implies that the pair of sequences $(\{\bar{x}_j\}_{j=0}^{\infty}, \{\bar{y}_j\}_{j=0}^{\infty})$ is (f)-overtaking optimal. This completes the proof of Theorem 3.2.

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