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FIXED POINT AND MEAN ERGODIC THEOREMS FOR NEW NONLINEAR MAPPINGS IN HILBERT SPACES

TORU MARUYAMA, WATARU TAKAHASHI, AND MASAYUKI YAO

Abstract. In this paper, we first consider a broad class of nonlinear mappings containing the class of generalized hybrid mappings defined by Kocourek, Takahashi and Yao [11] in a Hilbert space. Then, we prove a fixed point theorem, a mean ergodic theorem of Baillon's type [2] and a weak convergence theorem of Mann's type [14] for these nonlinear mappings in a Hilbert space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$. Baillon [2] proved the following nonlinear mean ergodic theorem in a Hilbert space.

Theorem 1.1. *Let C be a nonempty closed convex subset of H and let $T : C \to C$ be nonexpansive. If $F(T) \neq \emptyset$, then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping F is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \le \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [4] and Goebel and Kirk [6]. It is known that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space; see, for instance, [3] and [5]. Recently, Kohsaka and Takahashi [13], and Takahashi [19] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \to C$ is called *nonspreading* [13] if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to C$ is called *hybrid* [19] if

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

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for all $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [12] and Iemoto and Takahashi [9]. Very recently, Takahashi and Yao [22] proved the following nonlinear ergodic theorem.

Theorem 1.2. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself such that $F(T)$ is nonempty. Suppose that T satisfies one of the following:*

(i) *T is nonspreading;*
(ii) *T is hybrid;*
(iii) $2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$

Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

Motivated by Theorems 1.1 and 1.2, Aoyama, Iemoto, Kohsaka and Takahashi [1] introduced a class of nonlinear mappings called λ-hybrid containing the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Kocourek, Takahashi and Yao [11] also introduced a more broad class of nonlinear mappings than the class of λ-hybrid mappings in a Hilbert space. A mapping $T : C \to C$ is called *generalized hybrid* [11] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping is called an (α, β)-*generalized hybrid* mapping.

In this paper, motivated by Kocourek, Takahashi and Yao [11], we introduce a broad class of nonlinear mappings containing the class of generalized hybrid mappings in a Hilbert space. Then, we prove a fixed point theorem, a mean ergodic theorem of Baillon's type [2] and a weak convergence theorem of Mann's type [14] for these nonlinear mappings in a Hilbert space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [18], we know the following basic equality. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.1}$$

We also know that for $x, y, u, v \in H$,

$$2 \langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.2}$$

From Opial [15], a Hilbert space H satisfies Opial's condition, i.e., for a sequence $\{x_n\}$ of H such that $x_n \rightharpoonup x$ and $x \neq y$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|. \tag{2.3}$$

Let C be a nonempty closed convex subset of H and let T be a mapping from C into itself. Then, we denote by $F(T)$ the set of fixed points of T. A mapping

$T : C \to C$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [10]. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Then, we have from (2.1) that

$$\begin{aligned}\|z - Tz\|^2 &= \|\alpha x + (1 - \alpha)y - Tz\|^2 \\ &= \alpha\|x - Tz\|^2 + (1 - \alpha)\|y - Tz\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\leq \alpha\|x - z\|^2 + (1 - \alpha)\|y - z\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha(1 - \alpha)^2\|x - y\|^2 + (1 - \alpha)\alpha^2\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 \\ &= 0.\end{aligned}$$

This implies $Tz = z$. So, $F(T)$ is convex.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n x_n \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For a proof of existence of a Banach limit and its other elementary properties, see [17]. Using Banach limits, Takahashi and Yao [22] proved the following fixed point theorem.

Theorem 2.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself. Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded and*

$$\mu_n\|T^n x - Ty\|^2 \leq \mu_n\|T^n x - y\|^2, \quad \forall y \in C$$

for some Banach limit μ. Then, T has a fixed point in C.

Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y\in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u\rangle \geq 0$$

for all $x \in H$ and $u \in C$; see [18] for more details. From Takahashi and Toyoda [21], we know the following result for metric projections in a Hilbert space.

Lemma 2.2. *Let D be a nonempty closed convex subset of a Hilbert space H. Let P be the metric projection of H onto D and let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \le \|x_n - u\|$ for all $u \in D$ and $n \in \mathbb{N}$, then $\{Px_n\}$ converges strongly.*

3. Fixed point theorems

In this section, we start with introducing a broad class of nonlinear mappings containing the class of generalized hybrid mappings defined by Kocourek, Takahashi and Yao [11] in a Hilbert space. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then, a mapping $T : C \to C$ is called *2-generalized hybrid* if there are $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}(3.1)\qquad \alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \le \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2\end{aligned}$$

for all $x, y \in C$. We call such a mapping an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-*generalized hybrid* mapping. We observe that the mapping above covers several well-known mappings. For example, a $(0, \alpha_2, 0, \beta_2)$-generalized hybrid mapping is nonexpansive for $\alpha_2 = 1$ and $\beta_2 = 0$, nonspreading for $\alpha_2 = 2$ and $\beta_2 = 1$, and hybrid for $\alpha_2 = \frac{3}{2}$ and $\beta_2 = \frac{1}{2}$. A $(0, \alpha_2, 0, \beta_2)$-generalized hybrid mapping is an (α_2, β_2)-generalized hybrid mapping in the sense of Kocourek, Takahashi and Yao [11]. We can also show that if $x = Tx$, then for any $y \in C$,

$$\begin{aligned}\alpha_1\|x - Ty\|^2 + \alpha_2\|x - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \le \beta_1\|x - y\|^2 + \beta_2\|x - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2\end{aligned}$$

and hence $\|x - Ty\| \le \|x - y\|$. This means that a 2-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Now, we prove a fixed point theorem for 2-generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a 2-generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

Proof. Since $T : C \to C$ is a 2-generalized hybrid mapping, there are $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \le \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2\end{aligned}$$

for all $x, y \in C$. If $F(T) \neq \emptyset$, then $\{T^n z\} = \{z\}$ for $z \in F(T)$. So, $\{T^n z\}$ is bounded. We show the reverse. Take $z \in C$ such that $\{T^n z\}$ is bounded. Let μ be a Banach limit. Then, for any $y \in C$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}\alpha_1\|T^{n+2}z - Ty\|^2 + \alpha_2\|T^{n+1}z - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|T^n z - Ty\|^2 \\ \le \beta_1\|T^{n+2}z - y\|^2 + \beta_2\|T^{n+1}z - y\|^2 + (1 - \beta_1 - \beta_2)\|T^n z - y\|^2\end{aligned}$$

for any $y \in C$. Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Then, we have

$$\begin{aligned}&\mu_n(\alpha_1\|T^{n+2}z - Ty\|^2 + \alpha_2\|T^{n+1}z - Ty\|^2 + (1-\alpha_1-\alpha_2)\|T^n z - Ty\|^2)\\&\quad\le \mu_n(\beta_1\|T^{n+2}z - y\|^2 + \beta_2\|T^{n+1}z - y\|^2 + (1-\beta_1-\beta_2)\|T^n z - y\|^2).\end{aligned}$$

So, we obtain

$$\begin{aligned}&\alpha_1\mu_n\|T^{n+2}z - Ty\|^2 + \alpha_2\mu_n\|T^{n+1}z - Ty\|^2 + (1-\alpha_1-\alpha_2)\mu_n\|T^n z - Ty\|^2\\&\quad\le \beta_1\mu_n\|T^{n+2}z - y\|^2 + \beta_2\mu_n\|T^{n+1}z - y\|^2 + (1-\beta_1-\beta_2)\mu_n\|T^n z - y\|^2\end{aligned}$$

and hence

$$\begin{aligned}&\alpha_1\mu_n\|T^n z - Ty\|^2 + \alpha_2\mu_n\|T^n z - Ty\|^2 + (1-\alpha_1-\alpha_2)\mu_n\|T^n z - Ty\|^2\\&\quad\le \beta_1\mu_n\|T^n z - y\|^2 + \beta_2\mu_n\|T^n z - y\|^2 + (1-\beta_1-\beta_2)\mu_n\|T^n z - y\|^2.\end{aligned}$$

This implies

$$\mu_n\|T^n z - Ty\|^2 \le \mu_n\|T^n z - y\|^2$$

for all $y \in C$. By Theorem 2.1, we have a fixed point in C. □

As a direct consequence of Theorem 3.1, we have the following result.

Theorem 3.2. *Let C be nonempty bounded closed convex subset of a Hilbert space H and let T be a 2-generalized hybrid mapping from C to itself. Then T has a fixed point.*

Using Theorem 3.1, we can also prove the following well-known fixed point theorems. We first prove a fixed point theorem for nonexpansive mappings in a Hilbert space.

Theorem 3.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.1, a $(0, 1, 0, 0)$-generalized hybrid mapping of C into itself is nonexpansive. By Theorem 3.1, T has a fixed point in C. □

The following is a fixed point theorem for nonspreading mappings in a Hilbert space.

Theorem 3.4 ([13])**.** *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.1, a $(0, 2, 0, 1)$-generalized hybrid mapping of C into itself is nonspreading. By Theorem 3.1, T has a fixed point in C. □

The following is a fixed point theorem for hybrid mappings by Takahashi [19] in a Hilbert space.

Theorem 3.5 ([19])**.** *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a hybrid mapping, i.e.,*

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.1, a $(0, \frac{3}{2}, 0, \frac{1}{2})$-generalized hybrid mapping of C into itself is hybrid in the sense of Takahashi [19]. By Theorem 3.1, T has a fixed point in C. □

We can also prove the following fixed point theorem in a Hilbert space.

Theorem 3.6. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping such that*

$$2\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.1, a $(0, 1, 0, \frac{1}{2})$-generalized hybrid mapping of C into itself is the mapping in our theorem. By Theorem 3.1, T has a fixed point in C. □

Finally, we prove the following fixed point theorem in a Hilbert space.

Theorem 3.7. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping such that*

$$\|T^2 x - Ty\|^2 + \|Tx - Ty\|^2 + \|x - Ty\|^2 \le 3\|x - y\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.1, consider a $(\frac{1}{3}, \frac{1}{3}, 0, 0)$-generalized hybrid mapping T of C into itself. Then, we have that

$$\frac{1}{3}\|T^2 x - Ty\|^2 + \frac{1}{3}\|Tx - Ty\|^2 + \frac{1}{3}\|x - Ty\|^2 \le \|x - y\|^2, \quad \forall x, y \in C.$$

This is equivalent to the mapping in our theorem:

$$\|T^2 x - Ty\|^2 + \|Tx - Ty\|^2 + \|x - Ty\|^2 \le 3\|x - y\|^2, \quad \forall x, y \in C.$$

By Theorem 3.1, T has a fixed point in C. □

Remark 3.8. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $n \in \mathbb{N}$. Then, a mapping $T : C \to C$ is called *n-generalized hybrid* if there are $\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n \in \mathbb{R}$ such that

$$\begin{aligned}\sum_{k=1}^{n} \alpha_k \|T^{n+1-k} x - Ty\|^2 &+ (1 - \sum_{k=1}^{n} \alpha_k)\|x - Ty\|^2 \\ &\le \sum_{k=1}^{n} \beta_k \|T^{n+1-k} x - y\|^2 + (1 - \sum_{k=1}^{n} \beta_k)\|x - y\|^2 \end{aligned} \tag{3.2}$$

for all $x, y \in C$. We call such a mapping an $(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n)$-*generalized hybrid* mapping. As in the proof of Theorem 3.1, we can prove a fixed point theorem for n-generalized hybrid mappings in a Hilbert space.

4. Nonlinear ergodic theorem

In this section, using the technique developed by Takahashi [16], we prove a nonlinear ergodic theorem of Baillon's type [2] for generalized hybrid mappings in a Hilbert space.

Theorem 4.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a 2-generalized hybrid mapping with $F(T) \neq \emptyset$ and let P be the mertic projection of H onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element p of $F(T)$, where $p = \lim_{n\to\infty} PT^n x$.

Proof. Since $T : C \to C$ is a 2-generalized hybrid mapping, there are $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} \alpha_1 \|T^2x - Ty\|^2 + &\alpha_2 \|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ &\leq \beta_1 \|T^2x - y\|^2 + \beta_2 \|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2 \end{aligned}$$

for all $x, y \in C$. Since T is an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping, T is quasi-nonexpansive. So, we have that $F(T)$ is closed and convex. Let $x \in C$ and let P be the metric projection of H onto $F(T)$. Then, we have

$$\begin{aligned} \|PT^n x - T^n x\| &\leq \|PT^{n-1}x - T^n x\| \\ &\leq \|PT^{n-1}x - T^{n-1}x\|. \end{aligned}$$

This implies that $\{\|PT^n x - T^n x\|\}$ is nonincreasing. We also know that for any $v \in C$ and $u \in F(T)$, $\langle v - Pv, Pv - u\rangle \geq 0$ and hence

$$\|v - Pv\|^2 \leq \langle v - Pv, v - u\rangle.$$

So, we get

$$\begin{aligned} \|Pv - u\|^2 &= \|Pv - v + v - u\|^2 \\ &= \|Pv - v\|^2 - 2\langle Pv - v, u - v\rangle + \|v - u\|^2 \\ &\leq \|v - u\|^2 - \|Pv - v\|^2. \end{aligned}$$

Let $m, n \in \mathbb{N}$. Putting $v = T^m x$ and $u = PT^n x$, we have

$$\begin{aligned} \|PT^m x - PT^n x\|^2 &\leq \|T^m x - PT^n x\|^2 - \|PT^m x - T^m x\|^2 \\ &\leq \|T^n x - PT^n x\|^2 - \|PT^m x - T^m x\|^2. \end{aligned}$$

So, $\{PT^n x\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{PT^n x\}$ converges strongly to an element p of $F(T)$. Take $u \in F(T)$. Then we obtain that for any $n \in \mathbb{N}$,

$$\|S_n x - u\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x - u\| \leq \|x - u\|.$$

So, $\{S_nx\}$ is bounded and hence there exists a weakly convergent subsequence $\{S_{n_i}x\}$ of $\{S_nx\}$. If $S_{n_i}x \rightharpoonup v$, then we have $v \in F(T)$. In fact, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have that

$$\begin{aligned}
0 &\le \beta_1\|T^{k+2}x-y\|^2+\beta_2\|T^{k+1}x-y\|^2+(1-\beta_1-\beta_2)\|T^kx-y\|^2\\
&\quad -\alpha_1\|T^{k+2}x-Ty\|^2-\alpha_2\|T^{k+1}x-Ty\|^2-(1-\alpha_1-\alpha_2)\|T^kx-Ty\|^2\\
&=\beta_1(\|T^{k+2}x-Ty\|^2+2\langle T^{k+2}x-Ty,Ty-y\rangle+\|Ty-y\|^2)\\
&\quad +\beta_2(\|T^{k+1}x-Ty\|^2+2\langle T^{k+1}x-Ty,Ty-y\rangle+\|Ty-y\|^2)\\
&\quad +(1-\beta_1-\beta_2)(\|T^kx-Ty\|^2+2\langle T^kx-Ty,Ty-y\rangle+\|Ty-y\|^2)\\
&\quad -\alpha_1\|T^{k+2}x-Ty\|^2-\alpha_2\|T^{k+1}x-Ty\|^2-(1-\alpha_1-\alpha_2)\|T^kx-Ty\|^2\\
&=\|Ty-y\|^2+2\langle \beta_1T^{k+2}x+\beta_2T^{k+1}x+(1-\beta_1-\beta_2)T^kx-Ty,Ty-y\rangle\\
&\quad +(\beta_1-\alpha_1)(\|T^{k+2}x-Ty\|^2-\|T^kx-Ty\|^2)\\
&\quad +(\beta_2-\alpha_2)(\|T^{k+1}x-Ty\|^2-\|T^kx-Ty\|^2).
\end{aligned}$$

Summing up these inequalities with respect to $k = 0, 1, \dots, n-1$,

$$\begin{aligned}
0 &\le n\|Ty-y\|^2\\
&\quad +2\Big\langle \sum_{k=0}^{n-1}T^kx+\beta_1(T^{n+1}x+T^nx-x-Tx)+\beta_2(T^nx-x)-nTy,Ty-y\Big\rangle\\
&\quad +(\beta_1-\alpha_1)(\|T^{n+1}x-Ty\|^2+\|T^nx-Ty\|^2-\|x-Ty\|^2-\|Tx-Ty\|^2)\\
&\quad +(\beta_2-\alpha_2)(\|T^nx-Ty\|^2-\|x-Ty\|^2).
\end{aligned}$$

Deviding this inequality by n, we have

$$\begin{aligned}
0 &\le \|Ty-y\|^2\\
&\quad +2\Big\langle S_nx+\frac{1}{n}\beta_1(T^{n+1}x+T^nx-x-Tx)+\frac{1}{n}\beta_2(T^nx-x)-Ty,Ty-y\Big\rangle\\
&\quad +\frac{1}{n}(\beta_1-\alpha_1)(\|T^{n+1}x-Ty\|^2+\|T^nx-Ty\|^2-\|x-Ty\|^2-\|Tx-Ty\|^2)\\
&\quad +\frac{1}{n}(\beta_2-\alpha_2)(\|T^nx-Ty\|^2-\|x-Ty\|^2),
\end{aligned}$$

where $S_nx = \frac{1}{n}\sum_{k=0}^{n-1}T^kx$. Replacing n by n_i and letting $n_i \to \infty$, we obtain from $S_{n_i}x \rightharpoonup v$ that

$$0 \le \|Ty-y\|^2 + 2\langle v-Ty, Ty-y\rangle.$$

Putting $y = v$, we have $0 \le -\|Tv-v\|^2$ and hence $Tv = v$. To complete the proof, it is sufficient to show that if $S_{n_i}x \rightharpoonup v$, then $v = p$. We have that

$$\langle T^kx-PT^kx, PT^kx-u\rangle \ge 0$$

for all $u \in F(T)$. Since $\{\|T^kx-PT^kx\|\}$ is nonincreasing, we have

$$\begin{aligned}
\langle u-p, T^kx-PT^kx\rangle &\le \langle PT^kx-p, T^kx-PT^kx\rangle\\
&\le \|PT^kx-p\|\cdot\|T^kx-PT^kx\|\\
&\le \|PT^kx-p\|\cdot\|x-Px\|.
\end{aligned}$$

Adding these inequalities from $k = 0$ to $k = n - 1$ and dividing n, we have

$$\langle u - p, S_n x - \frac{1}{n}\sum_{k=0}^{n-1} PT^k x\rangle \leq \frac{\|x - Px\|}{n}\sum_{k=0}^{n-1}\|PT^k x - p\|.$$

Since $S_{n_i}x \rightharpoonup v$ and $PT^k x \to p$, we have

$$\langle u - p, v - p\rangle \leq 0.$$

We know $v \in F(T)$. So, putting $u = v$, we have $\langle v - p, v - p\rangle \leq 0$ and hence $\|v - p\|^2 \leq 0$. So, we obtain $v = p$. This completes the proof. □

Remark 4.2. As in the proof of Theorem 4.1, we can prove a nonlinear ergodic theorem of Baillon's type for n-generalized hybrid mappings in a Hilbert space.

5. Weak convergence theorem of Mann's type

In this section, we prove a weak convergence theorem of Mann's type [14] for 2-generalized hybrid mappings in a Hilbert space. Before proving the theorem, we need the following two lemmas.

Lemma 5.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a 2-generalized hybrid mapping. Then, $x_n \rightharpoonup z$, $x_n - Tx_n \to 0$ and $x_n - T^2x_n \to 0$ imply $z \in F(T)$.*

Proof. Since T is 2-generalized hybrid, there are $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|T^2x - Ty\|^2 &+ \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ &\leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2\end{aligned}$$

for all $x, y \in C$. Suppose $x_n \rightharpoonup z$, $x_n - Tx_n \to 0$ and $x_n - T^2x_n \to 0$. Since $x_n \rightharpoonup z$, we know from [17] that $\{x_n\}$ is bounded. Since $x_n - Tx_n \to 0$ and $x_n - T^2x_n \to 0$, we have also that $\{Tx_n\}$ and $\{T^2x_n\}$ are bounded. Next, let us consider

$$\begin{aligned}\alpha_1\|T^2x_n - Tz\|^2 &+ \alpha_2\|Tx_n - Tz\|^2 + (1 - \alpha_1 - \alpha_2)\|x_n - Tz\|^2 \\ &\leq \beta_1\|T^2x_n - z\|^2 + \beta_2\|Tx_n - z\|^2 + (1 - \beta_1 - \beta_2)\|x_n - z\|^2.\end{aligned}$$

From this inequality, we have

$$\begin{aligned}&\alpha_1(\|T^2x_n - x_n\|^2 + 2\langle T^2x_n - x_n, x_n - Tz\rangle + \|x_n - Tz\|^2) \\ &\quad + \alpha_2(\|Tx_n - x_n\|^2 + 2\langle Tx_n - x_n, x_n - Tz\rangle + \|x_n - Tz\|^2) \\ &\quad + (1 - \alpha_1 - \alpha_2)\|x_n - Tz\|^2 \\ &\leq \beta_1(\|T^2x_n - x_n\|^2 + 2\langle T^2x_n - x_n, x_n - z\rangle + \|x_n - z\|^2) \\ &\quad + \beta_2(\|Tx_n - x_n\|^2 + 2\langle Tx_n - x_n, x_n - z\rangle + \|x_n - z\|^2) \\ &\quad + (1 - \beta_1 - \beta_2)\|x_n - z\|^2.\end{aligned}$$

We apply a Banach limit μ to both sides of the inequality. Then, we have

$$\begin{aligned}\alpha_1(\mu_n\|T^2x_n-x_n\|^2&+2\mu_n\langle T^2x_n-x_n,x_n-Tz\rangle+\mu_n\|x_n-Tz\|^2)\\&+\alpha_2(\mu_n\|Tx_n-x_n\|^2+2\mu_n\langle Tx_n-x_n,x_n-Tz\rangle+\mu_n\|x_n-Tz\|^2)\\&+(1-\alpha_1-\alpha_2)\mu_n\|x_n-Tz\|^2\\\le&\ \beta_1(\mu_n\|T^2x_n-x_n\|^2+2\mu_n\langle T^2x_n-x_n,x_n-z\rangle+\mu_n\|x_n-z\|^2)\\&+\beta_2(\mu_n\|Tx_n-x_n\|^2+2\mu_n\langle Tx_n-x_n,x_n-z\rangle+\mu_n\|x_n-z\|^2)\\&+(1-\beta_1-\beta_2)\mu_n\|x_n-z\|^2\end{aligned}$$

and hence

$$\begin{aligned}\alpha_1\mu_n\|x_n-Tz\|^2&+\alpha_2\mu_n\|x_n-Tz\|^2+(1-\alpha_1-\alpha_2)\mu_n\|x_n-Tz\|^2\\&\le\beta_1\mu_n\|z-x_n\|^2+\beta_2\mu_n\|x_n-z\|^2+(1-\beta_1-\beta_2)\mu_n\|x_n-z\|^2.\end{aligned}$$

So, we have

$$\mu_n\|x_n-Tz\|^2\le\mu_n\|x_n-z\|^2.$$

Since $\mu_n\|x_n-Tz\|^2=\mu_n\|x_n-z\|^2+2\mu_n\langle x_n-z,z-Tz\rangle+\mu_n\|z-Tz\|^2$, we have from $x_n\rightharpoonup z$ that

$$\mu_n\|x_n-z\|^2+\mu_n\|z-Tz\|^2\le\mu_n\|x_n-z\|^2.$$

Then we have $\|z-Tz\|^2\le 0$ and hence $Tz=z$. This completes the proof. □

Lemma 5.2. *Let H be a Hilbert space. Let $x,y,z\in H$ and let α, β and γ be real numbers such that $\alpha+\beta+\gamma=1$. Then,*

$$\begin{aligned}\|\alpha x+&\beta y+\gamma z\|^2\\&=\alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\beta\gamma\|y-z\|^2-\alpha\gamma\|x-z\|^2.\end{aligned}$$

Proof. We have that

$$\begin{aligned}\|\alpha x+&\beta y+\gamma z\|^2=\langle\alpha x+\beta y+\gamma z,\alpha x+\beta y+\gamma z\rangle\\&=\alpha^2\|x\|^2+\beta^2\|y\|^2+\gamma^2\|z\|^2+2\alpha\beta\langle x,y\rangle+2\beta\gamma\langle y,z\rangle+2\alpha\gamma\langle x,z\rangle.\end{aligned}$$

Since $2\langle u,v\rangle=\|u\|^2+\|v\|^2-\|u-v\|^2$ for all $u,v\in H$ and $\alpha+\beta+\gamma=1$, we have

$$\begin{aligned}\|\alpha x+&\beta y+\gamma z\|^2=\alpha^2\|x\|^2+\beta^2\|y\|^2+\gamma^2\|z\|^2+\alpha\beta(\|x\|^2+\|y\|^2-\|x-y\|^2)\\&+\beta\gamma(\|y\|^2+\|z\|^2-\|y-z\|^2)+\alpha\gamma(\|x\|^2+\|z\|^2-\|x-z\|^2)\\&=\alpha(\alpha+\beta+\gamma)\|x\|^2+\beta(\alpha+\beta+\gamma)\|y\|^2+\gamma(\alpha+\beta+\gamma)\|z\|^2\\&-\alpha\beta\|x-y\|^2-\beta\gamma\|y-z\|^2-\alpha\gamma\|x-z\|^2\\&=\alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\beta\gamma\|y-z\|^2-\alpha\gamma\|x-z\|^2.\end{aligned}$$

This completes the proof. □

Using Lemmas 5.1 and 5.2, we can prove the following weak convergence theorem for 2- generalized hybrid mappings in a Hilbert space.

Theorem 5.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and let $T : C \to C$ be an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping with $F(T) \neq \emptyset$. Let P be the mertic projection of H onto $F(T)$ and let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers such that $0 < a \leq a_n, b_n, c_n \leq b < 1$ and $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n T^2 x_n, \quad n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element v of $F(T)$, where $v = \lim_{n\to\infty} P x_n$.

Proof. Since T is an $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid mapping with $F(T) \neq \emptyset$, T is quasi-nonexpansive. So, for $z \in F(T)$, we have that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|a_n x_n + b_n T x_n + c_n T^2 x_n - z\|^2 \\ &\leq a_n \|x_n - z\|^2 + b_n \|T x_n - z\|^2 + c_n \|T^2 x_n - z\|^2 \\ &\leq \alpha_n \|x_n - z\|^2 + b_n \|x_n - z\|^2 + c_n \|x_n - z\|^2 \\ &= \|x_n - z\|^2\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \|x_n - z\|^2$ exists. Then, we have that $\{x_n\}$ is bounded. We also have from Lemma 5.2 that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|a_n x_n + b_n T x_n + c_n T^2 x_n - z\|^2 \\ &= a_n \|x_n - z\|^2 + b_n \|T x_n - z\|^2 + c_n \|T^2 x_n - z\|^2 \\ &\quad - a_n b_n \|x_n - T x_n\|^2 - a_n c_n \|x_n - T^2 x_n\|^2 - b_n c_n \|T x_n - T^2 x_n\|^2 \\ &\leq a_n \|x_n - z\|^2 + b_n \|x_n - z\|^2 + c_n \|x_n - z\|^2 \\ &\quad - a_n b_n \|x_n - T x_n\|^2 - a_n c_n \|x_n - T^2 x_n\|^2 - b_n c_n \|T x_n - T^2 x_n\|^2 \\ &\leq \|x_n - z\|^2 \\ &\quad - a_n b_n \|x_n - T x_n\|^2 - a_n c_n \|x_n - T^2 x_n\|^2 - b_n c_n \|T x_n - T^2 x_n\|^2.\end{aligned}$$

So, we have

$$a_n b_n \|x_n - T x_n\|^2 + a_n c_n \|x_n - T^2 x_n\|^2 + b_n c_n \|T x_n - T^2 x_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Since $\lim_{n\to\infty} \|x_n - z\|^2$ exists and $0 < a \leq a_n, b_n, c_n \leq b < 1$, we have $\|T x_n - x_n\|^2 \to 0$, $\|T^2 x_n - x_n\|^2 \to 0$ and $\|T x_n - T^2 x_n\|^2 \to 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. From Lemma 5.1 we obtain $v \in F(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. Then, we show $v_1 = v_2$. We know $v_1, v_2 \in F(T)$ and hence $\lim_{n\to\infty} \|x_n - v_1\|^2$ and $\lim_{n\to\infty} \|x_n - v_2\|^2$ exist. Suppose $v_1 \neq v_2$. Since H satisfies

Opial's condition, we have from (2.3) that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - v_1\| &= \lim_{i\to\infty}\|x_{n_i} - v_1\| \\ &< \lim_{i\to\infty}\|x_{n_i} - v_2\| \\ &= \lim_{n\to\infty}\|x_n - v_2\| \\ &= \lim_{j\to\infty}\|x_{n_j} - v_2\| \\ &< \lim_{j\to\infty}\|x_{n_j} - v_1\| \\ &= \lim_{n\to\infty}\|x_n - v_1\|.\end{aligned}$$

This is a contradiction. So, we have $v_1 = v_2$. This implies that $\{x_n\}$ converges weakly to some point of $F(T)$. Since $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $z \in F(T)$ and $n \in \mathbb{N}$, we obtain from Lemma 2.2 that $\{Px_n\}$ converges strongly to an element p of $F(T)$. On the other hand, we have from the property of P that

$$\langle x_n - Px_n, Px_n - u\rangle \geq 0$$

for all $u \in F(T)$ and $n \in \mathbb{N}$. Since $x_n \rightharpoonup v$ and $Px_n \to p$, we obtain

$$\langle v - p, p - u\rangle \geq 0$$

for all $u \in F(T)$. Putting $u = v$, we obtain $-\|v - p\|^2 \geq 0$ and hence $p = v$. This means $v = \lim_{n\to\infty} Px_n$. This completes the proof. □

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Toru Maruyama
Department of Economics, Keio University, Mita 2–15–45, Minato-ku, Tokyo 108-8345, Japan
E-mail address: maruyama@econ.keio.ac.jp

Wataru Takahashi
Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan and Department of Economics, Keio University, Mita 2–15–45, Minato-ku, Tokyo 108-8345, Japan
E-mail address: wataru@is.titech.ac.jp

Masayuki Yao
Graduate School of Economics, Keio University, Mita 2–15–45, Minato-ku, Tokyo 108-8345, Japan
E-mail address: myao@gs.econ.keio.ac.jp