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# ALTERNATING PROJECTIONS AND ORTHOGONAL DECOMPOSITIONS

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Abstract. We present a new proof of von Neumann's classical convergence theorem regarding alternating orthogonal projections in Hilbert space. Our argument is based on an orthogonal decomposition lemma and on the construction of a Tietze-type potential.

## 1. Introduction

A few years ago we presented [7] an elementary geometric proof of von Neumann's classical convergence theorem regarding alternating orthogonal projections in Hilbert space. In a subsequent note [8] we presented another geometric proof of this seminal result. In this paper we present a new proof of von Neumann's theorem. This time our argument is based on an orthogonal decomposition lemma (which has already been used in [8]) and on the construction of a Tietze-type potential.

Let $S_1$ and $S_2$ be two closed subspaces of a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$, and let $P_1 : H \mapsto S_1$ and $P_2 : H \mapsto S_2$ be the corresponding orthogonal projections of $H$ onto $S_1$ and $S_2$, respectively. Denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of nonnegative integers. Let $x_0$ be an arbitrary point in $H$, and define the sequence $\{x_i : i \in \mathbb{N}\}$ of alternating projections by

$$x_{2i+1} = P_1 x_{2i} \quad \text{and} \quad x_{2i+2} = P_2 x_{2i+1}, \tag{1.1}$$

where $i \in \mathbb{N}$.

**Theorem 1.1.** *The sequence $\{x_i : i \in \mathbb{N}\}$ defined by (1.1) converges in norm as $i \to \infty$ to $P_S x_0$, where $P_S : H \mapsto S$ is the orthogonal projection of $H$ onto the intersection $S = S_1 \cap S_2$.*

This is von Neumann's classical theorem [10, p. 475]. It was rediscovered by several other authors; see, for example, [1], [9] and [13]. More information regarding this theorem and its diverse applications can be found in [3] and the references mentioned therein. Other proofs of Theorem 1.1 can be found, for instance, in [4], [11], [2], [7] and [8].

We begin the next section of our paper with an orthogonal decomposition lemma [6, Lemma 1.2] and then continue by recalling a special case of a Tietze-type extension theorem [6, Theorem 2.4]. In Section 3 we use these two results to construct a

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differentiable Tietze-type potential with a derivative the Lipschitz constant of which is *independent* of the dimension. The existence of such a potential leads in Section 4 to a key estimate which, in its turn, yields a proof of Theorem 1.1 itself. We denote by $|\cdot|$ the norm induced in $H$ by its inner product $\langle\cdot,\cdot\rangle$. The real line and the $d$-dimensional Euclidean space are denoted by $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

## 2. Decompositions and extensions

We first present an orthogonal decomposition lemma [6, Lemma 1.2]. It shows that any two finite-dimensional subspaces $X, Y \subset H$ with $1 \leq \dim X = m \leq n = \dim Y$ possess orthonormal bases $\{e_j\}_{j=1}^m$ and $\{f_j\}_{j=1}^n$, respectively, so that $X + Y$ can be written as a sum of the following pairwise orthogonal, at most two-dimensional, subspaces defined by the basis vectors:

$$\operatorname{span}\{e_1, f_1\} \oplus \cdots \oplus \operatorname{span}\{e_m, f_m\} \oplus \operatorname{span}\{f_{m+1}\} \oplus \cdots \oplus \operatorname{span}\{f_n\}. \tag{2.1}$$

**Lemma 2.1.** *Let $X$ and $Y$ be two subspaces of $H$ with $1 \leq \dim X = m \leq n = \dim Y$. Then there exist orthonormal bases $\{e_j\}_{j=1}^m$ and $\{f_j\}_{j=1}^n$ of $X$ and $Y$, respectively, and a nonnegative integer $0 \leq k \leq m$ so that*

(i) *$e_j = f_j$ if and only if $j \leq k$;*
(ii) *the at most two-dimensional spaces $\operatorname{span}\{e_j\}$, $j \leq k$, $\operatorname{span}\{e_j, f_j\}$, $k < j \leq m$, and $\operatorname{span}\{f_j\}$, $m < j \leq n$, are all pairwise orthogonal.*

Next we recall a special case of [6, Theorem 2.4]. This is a Tietze-type extension theorem which yields a differentiable potential. More precisely, given $K$ subspaces and two points $a$ and $b$ in $\mathbb{R}^d$ with $|b - a| = 1$, there is a differentiable function $\Phi$ such that $\Phi(b) - \Phi(a) = 1$ and on the $K$ given subspaces, the derivative of $\Phi$ belongs to these subspaces. Moreover, the Lipschitz constant of $\Phi'$ only depends on $K$ and $d$. The proof of this theorem involves a rather intricate application of a Whitney-type extension theorem [12, page 177].

**Proposition 2.2.** *Let $L_1, L_2, \ldots, L_K$ be $K$ subspaces of $\mathbb{R}^d$ and let $a, b \in \mathbb{R}^d$ be two points with $|b - a| = 1$. Then there exists a differentiable function $\Phi : \mathbb{R}^d \to \mathbb{R}$ such that*

(i) *$\Phi(b) - \Phi(a) = 1$;*
(ii) *$\Phi'(L_i) \subset L_i$ for $i = 1, \ldots, K$;*
(iii) *the derivative $\Phi' : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz with a constant $c(K, d)$ which only depends on $K$ and $d$.*

## 3. Potentials

In this section we use Proposition 2.2 in the case $K = d = 2$ to construct a differentiable Tietze-type potential with a derivative the Lipschitz constant of which is *independent* of the dimension. We remark in passing that this particular case of Proposition 2.2 can be proved directly by using an elementary geometric argument (*cf.* [5]).

**Proposition 3.1.** *Let $X$ and $Y$ be two finite-dimensional subspaces of $H$, $Z = X + Y$, and let $a, b \in Z$ be two points with $|b - a| = 1$. Then there exists a differentiable function $\Psi : Z \to \mathbb{R}$ such that*

(i) $\Psi(b) - \Psi(a) = 1$;
(ii) $\Psi'(X) \subset X$ *and* $\Psi'(Y) \subset Y$;
(iii) *the derivative* $\Psi' : Z \to Z$ *is Lipschitz with a universal Lipschitz constant* $C$ *which is independent of the dimensions of* $X$ *and* $Y$.

*Proof.* Assume without loss of generality that $m = \dim X \le \dim Y = n$, and let

$$Z = X + Y = E_1 \oplus E_2 \oplus \cdots \oplus E_m \oplus \cdots \oplus E_n \tag{3.1}$$

be the decomposition obtained in Lemma 2.1. For each $1 \le j \le n$, let $Q_j : H \to E_j$ be the orthogonal projection of $H$ onto $E_j$, $X_j := Q_j(X) = X \cap E_j$, $Y_j := Q_j(Y) = Y \cap E_j$, $a_j := Q_j(a)$ and $b_j := Q_j(b)$. Since $|b_j - a_j| \le |b - a| = 1$, we may apply Proposition 2.2 to $X_j, Y_j, E_j, a_j$ and $b_j$ to obtain, for each $1 \le j \le n$, a potential $\Phi_j : E_j \to \mathbb{R}$ such that

(iv) $\Phi_j(b_j) - \Phi_j(a_j) = |b_j - a_j|^2$;
(v) $\Phi_j'(X_j) \subset X_j$ and $\Phi_j'(Y_j) \subset Y_j$;
(vi) the derivative $\Phi_j'$ is $C$-Lipschitz with $C \le c(2, 2)$.

Using the potentials $\Phi_j$, we now define the potential $\Psi : Z \to \mathbb{R}$ by

$$\Psi(z) := \sum_{j=1}^{n} \Phi_j(Q_j z). \tag{3.2}$$

Claim 1. $\Psi(b) - \Psi(a) = 1$.

*Proof.* Indeed, using (iv), we obtain

$$\begin{aligned} \Psi(b) - \Psi(a) &= \sum_{j=1}^{n} \Phi_j(Q_j b) - \sum_{j=1}^{n} \Phi_j(Q_j a) \\ &= \sum_{j=1}^{n} [\Phi_j(b_j) - \Phi_j(a_j)] \\ &= \sum_{j=1}^{n} |b_j - a_j|^2 \\ &= |b - a|^2 = 1, \end{aligned}$$

as claimed. □

Claim 2. $\Psi'(X) \subset X$ and $\Psi'(Y) \subset Y$.

*Proof.* Let the point $z$ belong to $Z$. Using the chain rule, we obtain

$$\Psi'(z) = \sum_{j=1}^{n} \Phi_j'(Q_j z) \circ Q_j, \; z \in Z. \tag{3.3}$$

If $z \in X$, then $Q_j z \in X_j$ for each $1 \le j \le n$ and hence $\Phi_j'(Q_j z) \in X_j$ by (v). It follows that $\Psi'(z) \in X$, as claimed. An analogous argument shows that $\Psi'(z) \in Y$ whenever $z \in Y$. □

Claim 3. The derivative $\Psi'$ has Lipschitz constant $C$.

*Proof.* Let the points $z_1$, $z_2$ and $w$ belong to $Z$. Then we have by (3.2), (3.3) and (vi),

$$\begin{aligned}
|(\Psi'(z_1) - \Psi'(z_2))(w)|^2 &= |\sum_{j=1}^{n}[\Phi_j'(Q_j z_1) - \Phi_j'(Q_j z_2)](Q_j w)|^2 \\
&\le [\sum_{j=1}^{n}|\Phi_j'(Q_j z_1) - \Phi_j'(Q_j z_2)||(Q_j w)|]^2 \\
&\le [\sum_{j=1}^{n}|\Phi_j'(Q_j z_1) - \Phi_j'(Q_j z_2)|^2][\sum_{j=1}^{n}|Q_j w|^2] \\
&\le C^2\sum_{j=1}^{n}|Q_j z_1 - Q_j z_2|^2|w|^2 \\
&= C^2|z_1 - z_2|^2|w|^2,
\end{aligned}$$

so that $|\Psi'(z_1) - \Psi'(z_2)| \le C|z_1 - z_2|$, as claimed. □

Combining these three claims, we conclude that the potential $\Psi$ indeed satisfies (i), (ii) and (iii), as required. This completes the proof of Proposition 3.1. □

## 4. Alternating projections

In this section we first use Proposition 3.1 to obtain a key estimate and then use this estimate to prove Theorem 1.1.

**Proposition 4.1.** *Let the sequence $\{x_i : i \in \mathbb{N}\}$ be defined by (1.1), and let $p$ and $r$ belong to $\mathbb{N}$. Then*

$$|x_r - x_{r+p}|^2 \le M(|x_r|^2 - |x_{r+p}|^2), \tag{4.1}$$

*where $M$ is a universal constant.*

*Proof.* We may assume without any loss of generality that $x_{r+1} \in S_1$. Assume first that $p = 2k$ for some $k \ge 1$. Let

$$X := \operatorname{span}\{x_{r+1}, x_{r+3}, \ldots, x_{r+2k+1}\} \subset S_1 \tag{4.2}$$

and

$$Y := \operatorname{span}\{x_{r+2}, x_{r+4}, \ldots, x_{r+2k}\} \subset S_2. \tag{4.3}$$

Then $x_{r+2k} = (P_2P_1)^k x_r = (P_Y P_X)^k x_r$. Using the proof of [6, Theorem 3.1] and Proposition 3.1, we obtain

$$|x_r - x_{r+p}|^2 = |x_r - x_{r+2k}|^2 \le M(|x_r|^2 - |x_{r+2k}|^2) = M(|x_r|^2 - |x_{r+p}|^2),$$

where $M = C/2$. Assume now that $p = 2k + 1$ for some $k \ge 1$. Then

$$x_{r+p} = x_{r+2k+1} = P_1(P_2P_1)^k x_r = P_X(P_Y P_X)^k x_r,$$

which again leads to (4.1). □

Note that Proposition 4.1 does *not* follow from [6, Corollary 3.2] because the constant there *does* depend on the dimensions of $X$ and $Y$. It is Proposition 3.1 which has enabled us to overcome this crucial obstacle.

*Proof of Theorem 1.1.* Since the numerical sequence $\{|x_i| : i \in \mathbb{N}\}$ decreases to its limit as $i \to \infty$, Proposition 4.1 shows that $\{x_i : i \in \mathbb{N}\}$ is a Cauchy sequence which converges in norm as $i \to \infty$ to $P_S x_0$ by part (c) of [7, Lemma 2.1]. □

Alternatively [8], once we know that

$$|x_r - x_{r+2k}|^2 \leq M(|x_r|^2 - |x_{r+2k}|^2),$$

we have

$$|x_{2j} - x_{2(j+k)}|^2 \leq M(|x_{2j}|^2 - |x_{2(j+k)}|^2), \tag{4.4}$$

so $x_{2j} = (P_2P_1)^j x_0 \to z$, a fixed point of $P_2P_1$, as $j \to \infty$. This limit $z$ clearly belongs to $S_2$. If $z$ were not in $S_1$, then we would obtain $|P_2P_1z| \leq |P_1z| < |z|$, a contradiction. Thus $z \in S_1$, $x_{2j+1} = P_1x_{2j} \to P_1z = z$ as $j \to \infty$, and the whole sequence $\{x_i : i \in \mathbb{N}\}$ converges in norm as $i \to \infty$ to $z = P_S x_0$, as asserted.

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