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TWO POSITIVE SOLUTIONS FOR AN INHOMOGENEOUS SCALAR FIELD EQUATION

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Abstract. We consider the following nonlinear elliptic equation:

$$-\Delta u + u = g(u) + f(x),\ x \in \mathbb{R}^N,$$

where $N \geq 3$, $f(x) \not\equiv 0$. For a wide class of nonlinearities, we show the existence of two positive solutions to this problem when $\|f\|_{L^2(\mathbb{R}^N)}$ is small.

1. Introduction

In this paper, we consider the following nonlinear elliptic equation:

$$-\Delta u + u = g(u) + f(x),\ x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$. When $f(x) \equiv 0$, it is known that there is a nontrivial solution of (1.1) for a wide class of nonlinearities (see [5]). Even though $f(x) \not\equiv 0$, we can expect the existence of a nontrivial solution if $f(x)$ is small in a suitable sense. Our purpose is to show the existence of positive solutions of (1.1) via the variational approach when $\|f\|_{L^2(\mathbb{R}^N)}$ is small.

For the nonlinearity g, we assume
(g1) $g \in C^1(\mathbb{R},\mathbb{R})$, $g(s) \equiv 0$ for all $s \leq 0$.
(g2) There exists $s_0 > 0$ such that $\int_0^{s_0}(g(s) - s)ds > 0$.
(g3) $\displaystyle\lim_{s\to\infty} \frac{g(s)}{s^{\frac{N+2}{N-2}}} = 0$.
(g4) There exists $\eta > 0$ such that $\lim_{s\to 0} \frac{g(s)}{s^{1+\eta}} = 0$.
(g5) There exists $\delta_0 > 0$ such that $\frac{g(s)}{s}$ is non-decreasing on $(0, \delta_0]$.

Compared with assumptions in [1], we only require stronger assumptions on the behavior of the nonlinearity g near zero.

For the inhomogeneous term f, we assume
(f1) $f(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $f(x) \not\equiv 0$.
(f2) There exist $c > 0$ and $a > 0$ such that

$$|D^\alpha f(x)| \leq ce^{-(1+a)|x|} \text{ for all } x \in \mathbb{R}^N \text{ and } |\alpha| \leq 1.$$

(f3) $\dfrac{\partial f}{\partial x_i} x_i \leq 0$ and $f(x_1,\dots,x_i,\dots,x_N) = f(x_1,\dots,-x_i,\dots,x_N)$ for $i = 1,2,\dots,N$.

We obtain the following result.

Theorem 1.1. *There exists $C^* > 0$ such that if $\|f\|_{L^2} \leq C^*$, then problem (1.1) has at least two positive solutions.*

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This kind of inhomogeneous problems has been studied widely (see e.g. [1], [2], [4], [6], [7], [8], [9], [10], [12], [14], [19], [21], [23] and references therein). Compared with the previous works, the class of the inhomogeneous term f is rather restricted. However we can treat a wide class of the nonlinearity g. Especially we do not require neither the convexity of g nor so-called global Ambrosetti-Rabinowitz condition, namely

$$\text{there exists } \mu > 2 \text{ such that } 0 < \mu \int_0^u g(s)ds \leq g(u)u \text{ for all } u > 0. \tag{1.2}$$

Moreover since we do not impose any condition on the behavior of g at infinity except for (g3), we can treat the case that $g(s)$ is asymptotically linear at infinity or $g(s)$ is negative at infinity. Especially we can apply Theorem 1.1 for the case g is Fitz-Hugh Nagomo type, that is, $g(s) - s = s(s-b)(c-s)$ for $b < c$ with $bc = 1$.

The main idea to find two solutions is the followings. We define an energy functional $I : H^1(\mathbb{R}^N) \mapsto \mathbb{R}$ by

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx - \int_{\mathbb{R}^N} G(u)dx - \int_{\mathbb{R}^N} f(x)udx,$$

where $G(u) = \int_0^u g(s)ds$. Firstly we will show that if $\|f\|_{L^2}$ is small, then there exist $\rho_0 > 0$ and $v \in H^1(\mathbb{R}^N)$ with $\|v\|_{H^1(\mathbb{R}^N)} < \rho_0$ such that $I(u) > 0$ on $\|u\|_{H^1(\mathbb{R}^N)} = \rho_0$ and $I(v) < 0$. Then by Ekeland's Variational Principle, we can see that there is a local minimizer u_0 which satisfies $I(u_0) < 0$ and $\|u_0\|_{H^1(\mathbb{R}^N)} < \rho_0$.

The second solution will be obtained by the Mountain Pass Method. We consider a Mountain Pass value whose paths connect from the local minimizer to a point where the energy is negative and the norm is greater than ρ_0. We can see that this Mountain Pass value is positive if $\|f\|_{L^2}$ is small. Since we do not impose any global condition like (1.2), we can not obtain the boundedness of a Palais-Smale sequence directly. To this aim, we use the Monotonicity Trick due to [15] and [20]. Then we can obtain a bounded Palais-Smale sequence. As a consequence, we can prove the existence of a second nontrivial critical point. However this critical point might lose its energy, that is, the energy might be strictly less that the Mountain Pass value. Hence we can not distinguish from the local minimizer readily.

To this aim, we construct a suitable path and give a precise interaction estimate to obtain certain energy estimate. This estimate will enable us to establish the existence of the second critical point The most hardest part of this paper is to obtain the interaction estimate. If we assume the convexity of $g(s)$, we can easily prove the desired estimate. However since we don't assume neither the convexity nor the positivity of $g(s)$, we need to estimate the interaction term more carefully. To this aim, we restrict the class of the inhomogeneous term $f(x)$ to obtain sharp informations of solutions of (1.1). This kind of interaction estimates appears when we study elliptic problems with group symmetries (see [3], [13]).

This paper is organized as follows. In section 2, we correct basic properties of solutions of (1.1) and a corresponding homogeneous scalar field equation (i.e. problem (1.1) with $f(x) \equiv 0$). In section 3, we prove the existence of a local minimizer via Ekeland's Variational Principle. In section 4, we show one energy estimate which plays an important role to find a second solution. Finally in section

5, we prove the existence of the second solution by using the Mountain Pass Method and the Monotonicity Trick.

2. Preliminaries

2.1. **Properties of solutions for the homogeneous scalar field equation.** We consider the following scalar field equation:

$$-\Delta u + u = g(u),\ x \in \mathbb{R}^N. \tag{2.1}$$

By (f2), the inhomogeneous term $f(x)$ decays to zero at infinity. Thus problem (2.1) can be regarded as the problem at infinity. We define $I_\infty : H^1(\mathbb{R}^N) \mapsto \mathbb{R}$ by

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx - \int_{\mathbb{R}^N} G(u)dx.$$

We denote c_∞ by the least energy level for I_∞, that is,

$$c_\infty := \inf\{I_\infty(u); u \in H^1(\mathbb{R}^N),\ I'_\infty(u) = 0\}.$$

Proposition 2.1 ([5]). *Assume (g1)-(g4). Then (2.1) has a positive least energy solution $w(x) \in C^2(\mathbb{R}^N)$ (namely $I_\infty(w) = c_\infty$, $I'_\infty(w) = 0$) and it satisfies*

(i) $w(x) = w(|x|) = \tilde{w}(r)$ *and* $\frac{\partial \tilde{w}}{\partial r} < 0$ *for all* $r > 0$.

(ii) $w(x)$ *satisfies the following Pohozaev type identity:*

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla w|^2 dx = N\int_{\mathbb{R}^N} G(u) - \frac{u^2}{2}dx.$$

(iii) *There exist positive constants* c_1, c'_1, c_2, c'_2 *and* R_0 *such that*

$$c_1|x|^{-\frac{N-1}{2}}e^{-|x|} \le w(x) \le c_2|x|^{-\frac{N-1}{2}}e^{-|x|},$$

$$-c'_1 r^{-\frac{N-1}{2}}e^{-r} \le \tilde{w}'(r) \le -c'_2 r^{-\frac{N-1}{2}}e^{-r} \text{ for all } r = |x| \ge R_0.$$

Although results above are obtained under weaker assumptions on the nonlinearity, we do not provide precise statements here. By Proposition 2.1 (ii), it follows

$$c_\infty = I_\infty = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla w|^2 dx > 0.$$

Especially $I_\infty(u) > 0$ for any nontrivial critical point u of I_∞.

2.2. **Properties of solutions of (1.1).** In this subsection, we correct basic properties of solutions of (1.1) and the energy functional which corresponds to (1.1).

Lemma 2.2. *Assume (f1)-(f3) and (g1)-(g4). Let $u(x)$ be a positive solution of (1.1). Then*

(i) $u(x) \in L^\infty(\mathbb{R}^N)$ *and* $u(x) \to 0$ *as* $|x| \to \infty$.
(ii) $(\nabla u \cdot x)(x) < 0$ *and* $u(x_1, \dots, x_i, \dots, x_N) = u(x_1, \dots, -x_i, \dots, x_N)$.
(iii) *There exist positive constants* c_3, c'_3, c_4, c'_4 *and* R_1 *such that*

$$c_3|x|^{-\frac{N-1}{2}}e^{-|x|} \le u(x), |\nabla u(x)| \le c_4|x|^{-\frac{N-1}{2}}e^{-|x|}, \text{for all } |x| \ge R_1.$$

Proof. (i) By (f2), it follows $f(x) \in L^q(\mathbb{R}^N)$ for all $q \in [1, \infty]$. Then we can show that $u \in L^\infty(\mathbb{R}^N)$ and $u(x) \to 0$ as $|x| \to \infty$ (see [7], [23] for the proof).

(ii) By (f3), the claim follows from the moving plane method due to [11], [17], [18].

(iii) By (f2), we can prove that for any $\delta > 0$, there exists $c_\delta > 0$ such that

$$c_\delta e^{-(1+\delta)|x|} \le u(x) \le c_\delta e^{-(1-\delta)|x|} \tag{2.2}$$

for all $x \in \mathbb{R}^N$ (see [23] for the proof). Using assumption (g4) and (f2), we have

$$g(u(x)) + f(x) = o(|x|^{-\frac{N-1}{2}} e^{-|x|}) \text{ as } |x| \to \infty.$$

Then by the asymptotic result by Gidas, Ni and Nirenberg [11], we obtain

$$u(x) = O(|x|^{-\frac{N-1}{2}} e^{-|x|}) \text{ as } |x| \to \infty.$$

Moreover using (g4) and (f2) again, we also have

$$|\nabla u(x)| = O(|x|^{-\frac{N-1}{2}} e^{-|x|}) \text{ as } |x| \to \infty.$$

Choosing c_3, c_3', c_4, c_4' suitably, we can prove the claims. □

Next we prepare the Pohozaev type identity which we will use in section 5. The proof is similar to that of Proposition 2.1 (ii).

Lemma 2.3. *Let $u(x)$ be a nontrivial solution of (1.1). Then $u(x)$ satisfies the following Pohozaev type identity:*

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx = N \int_{\mathbb{R}^N} G(u) - \frac{u^2}{2} dx + N \int_{\mathbb{R}^N} f(x) u dx + \int_{\mathbb{R}^N} \nabla f \cdot x u dx.$$

Next we recall the energy functional:

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx - \int_{\mathbb{R}^N} G(u) dx - \int_{\mathbb{R}^N} f(x) u dx.$$

Hereafter in this paper, we denote $\|u\|^2 := \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx$.

Next we need some modification because we don't impose any condition on $g(s)$ at infinity except for (g3). We claim that there exists $M > 0$ such that $g(s) + Ms > 0$ for all $s > 0$ without loss of generality.

Let $\overline{s} > 0$ be a constant which satisfies $\overline{s} > \max\{s_0, \|f\|_{L^\infty}\}$, where s_0 is a constant defined in (g2). We consider the case that there exists $\tilde{s} \ge \overline{s}$ such that

$$g(\tilde{s}) - \tilde{s} + \|f\|_{L^\infty} = 0. \tag{2.3}$$

Then we define

$$\tilde{g}(s) = \begin{cases} g(s), & 0 \le s \le \tilde{s} \\ g(\tilde{s}), & s \ge \tilde{s}. \end{cases}$$

We claim that if $u(x)$ is a solution of the problem:

$$-\Delta u + u = \tilde{g}(u) + f(x), \ x \in \mathbb{R}^N,$$

then u is a solution of the original problem (1.1). In fact if $\Omega := \{x \in \mathbb{R}^N; u(x) > \tilde{s}\} \neq \emptyset$, then we have

$$-\Delta u = \tilde{g}(u) - u + f \le \tilde{g}(\tilde{s}) - \tilde{s} + \|f\|_{L^\infty} = 0 \text{ on } \Omega.$$

By Maximum Principle, this leads a contradiction.

Now we take $M > 0$ so that

$$M > \max\Big\{0, -\min_{0\le s\le \tilde{s}} \frac{g(s)}{s}\Big\}.$$

Then we can see that $\tilde{g}(s) + Ms > 0$ for all $s > 0$. Thus replacing $g(s)$ by $\tilde{g}(s)$, we can set

$$g(s) + Ms > 0 \text{ for all } s > 0. \tag{2.4}$$

Remark 2.4. (i) Suppose $g(s) - s + \|f\|_{L^\infty} \ge 0$ for all $s \ge \bar{s}$. Then it follows $g(s) \ge \bar{s} - \|f\|_{L^\infty} > 0$ for all $s \ge \bar{s}$. Choosing $M > \max\{0, -\min_{0\le s\le \bar{s}} \frac{g(s)}{s}\}$, we have $g(s) + Ms > 0$ for all $s \ge 0$.

(ii) (2.3) implies $g(\tilde{s}) - \tilde{s} < 0$ because $f(x) \ge 0$. Then we can see that $\|u\|_{L^\infty} \le \tilde{s}$ for any solution of (2.1). Thus we can replace $g(s)$ by $\tilde{g}(s)$ in (2.1) without loss of generality.

Lemma 2.5. *Let u be a nontrivial critical point of $I(u)$. Then $u(x) > 0$ for all $x \in \mathbb{R}^N$.*

Proof. Let $u_-(x) = \min\{0, u(x)\}$. Since $I'(u) = 0$, it follows

$$\|u_-\|^2 = \int_{\mathbb{R}^N} g(u)u_- dx + \int_{\mathbb{R}^N} f(x)u_- dx.$$

By (f1) and (g1), we have $\|u_-\| = 0$ and hence $u(x) \ge 0$ for all $x \in \mathbb{R}^N$. Then by Maximum Principle, we obtain $u(x) > 0$. □

Finally we introduce a global compactness type result for I.

Lemma 2.6. *Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a sequence such that*

$$I(u_n) \to c \in \mathbb{R},\ I'(u_n) \to 0,\ \|u_n\| \text{ is bounded.}$$

Then there exist $u_0 \in H^1(\mathbb{R}^N)$, $k \in \mathbb{N}\cup\{0\}$, $\{y_n^i\} \subset \mathbb{R}^N$, $w^i \in H^1(\mathbb{R}^N)$, $i = 1, \dots, k$ such that

(i) $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$, $I'(u_0) = 0$.

(ii) $|y_n^i| \to \infty,\ |y_n^i - y_n^{i'}| \to \infty,\ i \ne i'$.

(iii) $w^i \not\equiv 0,\ I'_\infty(w^i) = 0,\ i = 1, \dots, k$.

(iv) $\|u_n - u_0 - \sum_{i=1}^k w^i(\cdot - y_n^i)\| \to 0$.

(v) $I(u_n) \to I(u_0) + \sum_{i=1}^k I_\infty(w^i)$.

3. Existence of a local minimizer

In this section, we show that there is a local minimizer of I if $\|f\|_{L^2}$ is small.

Lemma 3.1. *There exist $C_0 >$ and $\rho_0 > 0$ such that if $\|f\|_{L^2} \le C_0$, then $I(u) > 0$ for all $u \in H^1(\mathbb{R}^N)$ with $\|u\| = \rho_0$.*

Proof. We choose $0 < \epsilon < 1$ arbitrary. By (g3) and (g4), there exists $c_\epsilon > 0$ such that

$$|G(s)| \leq \frac{\epsilon}{2}s^2 + c_\epsilon s^{\frac{2N}{N-2}} \text{ for all } s \geq 0. \tag{3.1}$$

Then we obtain

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} |G(u)|dx - \int_{\mathbb{R}^N} fudx \\ &\geq \frac{1}{2}(1-\epsilon)\|u\|^2 - c'\|u\|^{\frac{2N}{N-2}} - \|f\|_{L^2}\|u\| \end{aligned} \tag{3.2}$$

for some $c' > 0$. Choosing $\rho_0 > 0$ so that $\frac{1}{2}(1-\epsilon) - c'\rho_0{}^{\frac{4}{N-2}} > 0$, we have

$$I(u) \geq \rho_0^2\Big(\frac{1}{2}(1-\epsilon) - c'\rho_0^{\frac{4}{N-2}}\Big) - \rho_0\|f\|_{L^2}$$

for all $u \in H^1(\mathbb{R}^N)$ with $\|u\| = \rho_0$. Then there exists $C_0 > 0$ such that if $\|f\|_{L^2} \leq C_0$, it follows $I(u) > 0$ for all $u \in H^1(\mathbb{R}^N)$ with $\|u\| = \rho_0$. □

Lemma 3.2. *There exists $\phi \in H^1(\mathbb{R}^N)$ such that $I(\phi) < 0$ and $\|\phi\| \leq \rho_0$.*

Proof. We fix $u \in H^1(\mathbb{R}^N)$ so that $\int_{\mathbb{R}^N} f(x)udx > 0$. For $t > 0$, we have

$$\frac{I(tu)}{t} = \frac{t}{2}\|u\|^2 - \int_{\mathbb{R}^N} \frac{G(tu)}{t}dx - \int_{\mathbb{R}^N} fudx.$$

From (3.1), we have $\lim_{t\to 0}\int_{\mathbb{R}^N} \frac{|G(tu)|}{t}dt = 0$. Choosing t sufficiently small, $\phi := tu$ satisfies $I(\phi) < 0$ and $\|\phi\| \leq \rho_0$. □

Lemma 3.3. *There exists $u_0 \in H^1(\mathbb{R}^N)$ such that*

$$I(u_0) = \inf_{u\in B_{\rho_0}} I(u) < 0,$$

where $B_{\rho_0} = \{u \in H^1(\mathbb{R}^N); \|u\| \leq \rho_0\}$. Moreover it follows $\|u_0\| < \rho_0$.

Proof. First we observe that there exists $0 < \rho' < \rho_0$ such that

$$I(u) \geq \frac{1}{2}\inf_{u\in B_{\rho_0}} I(u) \tag{3.3}$$

for any $u \in H^1(\mathbb{R}^N)$ with $\rho' \leq \|u\| \leq \rho_0$.

Now let $\{u_n\}$ be a sequence such that $I(u_n) \to \inf_{u\in B_{\rho_0}} I(u)$. From (3.3), we may assume that $\|u_n\| \leq \rho'$. Let B_{ρ_0} be endowed with the metric dist $(u_1, u_2) = \|u_1 - u_2\|$, u_1, $u_2 \in B_{\rho_0}$. By Ekeland's Variational Principle, there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$I(u_n) = \inf_{u\in B_{\rho_0}} I(u) + \epsilon_n\|u_n - u\|.$$

Then we can see that $\{u_n\}$ is a Palais-Smale sequence for I. Since $\|u_n\| \leq \rho'$, we may assume that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$ for some $u_0 \in H^1(\mathbb{R}^N)$. By the weakly lower semi-continuity, we have $\|u_0\| \leq \rho' < \rho_0$. By Lemma 2.6, we have

$$I'(u_0) = 0 \text{ and } \inf_{u\in B_{\rho_0}} I(u) = I(u_0) + \sum_{i=1}^k I_\infty(w^i)$$

for some $k \in \mathbb{N} \cup \{0\}$. Suppose $k \geq 1$. Then it follows

$$\inf_{u \in B_{\rho_0}} I(u) \geq I(u_0) + kc_\infty > I(u_0).$$

This is a contradiction. Thus we have $k = 0$ and hence $I(u_0) = \inf_{u \in B_{\rho_0}} I(u)$. □

Next we prove that $I(u_0) + c_\infty > 0$ if $\|f\|_{L^2}$ is small. This estimate will be used to find a second nontrivial critical point of I.

Lemma 3.4. *There exists $C_1 > 0$ such that if $\|f\|_{L^2} \leq C_1$, then $I(u_0) + c_\infty > 0$.*

Proof. From (3.2), we have

$$I(u) \geq \|u\|^2\Big(\frac{1}{2}(1-\epsilon) - c'\rho_0^{\frac{4}{N-2}}\Big) - \|f\|_{L^2}\|u\| =: c_{\rho_0}\|u\|^2 - \|f\|_{L^2}\|u\|$$

for all $u \in B_{\rho_0}$.

First we claim that there exists $0 < \rho_1 < \rho_0$ such that if $\|u\| \leq \rho_1$, then $I(u) > -c_\infty$. Now we choose $\epsilon_1 > 0$ so that $\epsilon_1\|f\|_{L^2}^2 < \frac{c_\infty}{2}$. Then by Young's inequality, we have

$$I(u) \geq \|u\|^2\Big(c_{\rho_0} - \frac{1}{\epsilon_1}\Big) - \epsilon_1\|f\|_{L^2}^2 \geq -\|u\|^2\Big|c_{\rho_0} - \frac{1}{\epsilon_1}\Big| - \frac{c_\infty}{2}.$$

Next we choose $\rho_1 > 0$ so that $|c_{\rho_0} - \frac{1}{\epsilon_1}|\rho_1^2 < \frac{c_\infty}{2}$. Then we obtain $I(u) > -c_\infty$ for all $u \in H^1(\mathbb{R}^N)$ with $\|u\| \leq \rho_1$.

Next we claim that there exists $C_1 = C_1(\rho_0, \rho_1) > 0$ such that if $\|f\|_{L^2} \leq C_1$, then $I(u) > -c_\infty$ for all $u \in B_{\rho_0}$. We take $\epsilon_2 > 0$ so that $c_{\rho_0}\rho_1^2 - \epsilon_2\rho_0^2 > 0$. Using Young's inequality again, we have

$$I(u) \geq c_{\rho_0}^2\rho_1^2 - \epsilon_2\rho_0^2 - \frac{1}{\epsilon_2}\|f\|_{L^2}^2.$$

Choosing $\|f\|_{L^2(\mathbb{R}^N)}^2 < \epsilon_2(c_\infty + c_{\rho_0}^2\rho_1^2 - \epsilon_2\rho_0^2)$, we obtain $I(u) > -c_\infty$ for all $u \in B_{\rho_0}$. Thus it follows

$$I(u_0) = \inf_{u \in B_{\rho_0}} I(u) > -c_\infty.$$

□

Hereafter in this paper, we put $C^* = \max\{C_0, C_1\}$ and assume $\|f\|_{L^2} \leq C^*$.

4. An interaction estimate

In this section, we prove one energy estimate which plays an important role to find a second solution. We consider a path $\gamma_l(t) : [0,1] \mapsto H^1(\mathbb{R}^N)$ by:

$$\gamma_l(t) = \begin{cases} u_0 + w(\frac{\cdot}{t} - le), & t \geq 0 \\ u_0, & t = 0, \end{cases}$$

where $l \geq 0$, $e = (1, 0, \ldots, 0) \in \mathbb{R}^N$ and $w(x)$ is a least energy solution of (2.1). We notice that $\|\gamma_l(0)\| < \rho_0$ and $\|\gamma_l(t)\| > \rho_0$ for large t where ρ_0 is a constant which appears in Lemma 3.1. Then we have $\sup_{t \geq 0} I(\gamma_l(t)) > 0$ by Lemma 3.1. On the other hand by Lemma 3.4, we know that $I(u_0) + c_\infty > 0$. Our purpose is to show that

$$0 < \sup_{t \geq 0} I(\gamma_l(t)) < I(u_0) + c_\infty \tag{4.1}$$

for sufficiently large l.

Now we consider a function $h(t) := I_\infty(w(\frac{x}{t}))$ for $t > 0$. Then by Proposition 2.1 (ii), we have

$$h(t) = \Big(\frac{t^{N-2}}{2} - \frac{N-2}{2N}t^N\Big)\|\nabla w\|_{L^2}^2. \tag{4.2}$$

Since $N \geq 3$, it follows $h(t) < 0$ for large t. We choose $T > 1$ so that $I_\infty(w(\frac{x}{L})) + I(u_0) < 0$ and $\|u_0 + w(\frac{\cdot}{T} - le)\| > \rho_0$. We note that T is independent of l because $\|u_0 + w(\frac{\cdot}{t} - le)\| \geq \|w(\frac{\cdot}{t} - le)\| - \|u_0\| \geq t^{N-2}\|w\| - \|u_0\|$ for $t \geq 1$.

To show (4.1), we use the following lemma.

Lemma 4.1. *Assume (g1)-(g5). Let $K > 0$ be arbitrary given.*

(i) *There exists $c_K > 0$ such that if $u_1 \in (0, \frac{K}{2})$ and $u_2 \in (0, \frac{K}{2})$, then*

$$\Big|G(u_1+u_2) - G(u_1) - G(u_2) - \frac{1}{2}g(u_1)u_2 - \frac{1}{2}g(u_2)u_1\Big| \leq c_K(u_1^{1+\eta}u_2 + u_2^{1+\eta}u_1).$$

Here η is a positive constant defined in (g4).

(ii) *For any $\epsilon > 0$, there exists $\delta_{\epsilon,K} > 0$ such that if $u_1 \in (0, \frac{K}{2})$ and $u_2 \in (0, \delta_{\epsilon,K}]$, then*

$$\begin{aligned} &G(u_1+u_2) - G(u_1) - G(u_2) - \frac{1}{2}g(u_1)u_2 - \frac{1}{2}g(u_2)u_1 \\ &\geq \frac{1}{2}g(u_1)u_2 - \epsilon\Big(u_2 + \frac{1}{2}u_2^2 + \frac{1}{2}u_1u_2\Big). \end{aligned}$$

Conversely if $u_1 \in (0, \delta_{\epsilon,K}]$ and $u_2 \in (0, \frac{K}{2})$, then

$$\begin{aligned} &G(u_1+u_2) - G(u_1) - G(u_2) - \frac{1}{2}g(u_1)u_2 - \frac{1}{2}g(u_2)u_1 \\ &\geq \frac{1}{2}g(u_2)u_1 - \epsilon\Big(u_1 + \frac{1}{2}u_1^2 + \frac{1}{2}u_1u_2\Big). \end{aligned}$$

(iii) *If u_1, $u_2 \in (0, \frac{\delta_0}{2})$, then*

$$G(u_1+u_2) - G(u_1) - G(u_2) - \frac{1}{2}g(u_1)u_2 - \frac{1}{2}g(u_2)u_1 \geq 0.$$

Here δ_0 is a positive constant defined in (g5).

Proof. (i) Now we have

$$\begin{aligned} &G(u_1+u_2) - G(u_1) - G(u_2) - \frac{1}{2}g(u_1)u_2 - \frac{1}{2}g(u_2)u_1 \\ &= \int_0^{u_2} g(u_1+\tau) - g(\tau)d\tau - \frac{1}{2}g(u_1)u_2 - \frac{1}{2}g(u_2)u_1 \\ &= \int_0^{u_2}\int_0^{u_1} g'(s+\tau)dsd\tau - \frac{1}{2}g(u_1)u_2 - \frac{1}{2}g(u_2)u_1. \end{aligned}$$

From (g4), there exists $c > 0$ such that $g'(s) \leq cs^\eta$ for all $s \in [0, \frac{k}{2}]$. Thus we obtain

$$\begin{aligned} &G(u_1+u_2) - G(u_1) - G(u_2) - \frac{1}{2}g(u_1)u_2 - \frac{1}{2}g(u_2)u_1 \\ &\leq c(u_1+u_2)^\eta u_1u_2 + cu_1^{1+\eta}u_2 + cu_2^{1+\eta}u_1 \leq cu_1^{1+\eta}u_2 + cu_2^{1+\eta}u_1. \end{aligned}$$

(ii) Now for any $\epsilon > 0$, there exists $\delta_{\epsilon,K} > 0$ such that

$$G(u_1 + u_2) - G(u_1) - G(u_2) \geq g(u_1)u_2 - \epsilon u_2$$

if $u_2 \in [0, \epsilon_k]$. Moreover choosing $\delta_{\epsilon,K}$ smaller if necessary, we have

$$g(s) \leq \epsilon s, \; G(s) \leq \frac{\epsilon}{2}s^2 \text{ for all } s \in [0, \delta_{\epsilon,K}].$$

Thus for $u_1 \in (0, \frac{K}{2})$ and $0 < u_2 \leq \delta_{\epsilon,K}$, we obtain

$$\begin{aligned}G(u_1 + u_2) - G(u_1) - G(u_2) - \frac{1}{2}g(u_1)u_2 - \frac{1}{2}g(u_2)u_1 \\ \geq \frac{1}{2}g(u_1)u_2 - \epsilon\Big(u_2 + \frac{1}{2}u_2^2 + \frac{1}{2}u_1u_2\Big).\end{aligned}$$

(iii) For the proof, we refer to [3]. □

We will apply Lemma 4.1 provided $K = 2\max\{\|u_0\|_{L^\infty}, \|w\|_{L^\infty}\}$, $u_1 = u_0(tx)$ and $u_2 = w(x - le)$. As we will see later, the terms $g(u_1)u_2$ and $g(u_2)u_1$ will be key terms to obtain strict inequality in (4.1). However we can not see the sign of them readily because $g(s)$ may change its sign.

Now let $\tau \in (0, T)$ be arbitrary given. We choose $R > \frac{1}{\tau}\max\{R_0, R_1, 1\}$ so that $u_0(Tx) < \frac{\delta_0}{2}$, $w(x) < \frac{\delta_0}{2}$ for all $x \in \mathbb{R}^N \setminus B_R(0)$. Here $B_R(0) := \{x \in \mathbb{R}^N; |x| \leq R\}$, R_0 and R_1 are positive constants which appear in Proposition 2.1 (iii) and Lemma 2.2 (iii) respectively. We note that if $|x| \geq R$, then $u_0(tx) < \frac{\delta_0}{2}$ for all $t \in (0, T]$. Then we can obtain the following estimates.

Lemma 4.2 ([22]). *Assume (f1)-(f3) and (g1)-(g4). Then there exist $c_R > 0$ and $l_R > 0$ which are independent of $t \in [\tau, T]$ such that*

(i) $$\int_{B_R(0)} (g(u_0(tx)) + f(tx))w(x - le)dx \geq c_R \int_{B_R(0)} w(x - le)dx,$$

(ii) $$\int_{B_R(le)} g(w(x - le))u_0(tx)dx \geq c_R \int_{B_R(le)} u_0(tx)dx$$

for all $l \geq l_R$ and $t \in [\tau, T]$.

Proof. First we prove (i). Integrating by parts, we obtain

$$\begin{aligned}
&\int_{B_R(0)} (g(u_0(tx)) + f(tx))w(x - le)dx = \int_{B_R(0)} (-\Delta u_0(tx) + u_0(tx))w(x - le)dx \\
&= \int_{B_R(0)} \nabla u_0(tx)\nabla w(x - le) + u_0(tx)w(x - le)dx - \int_{\partial B_R(0)} \frac{\partial u_0}{\partial n}(tx)w(x - le)dS \\
&= \int_{B_R(0)} u_0(tx)(-\Delta w(x - le) + w(x - le))dx \\
&\quad + \int_{\partial B_R(0)} u_0(tx)\frac{\partial w}{\partial n}(x - le)dS - \int_{\partial B_R(0)} \frac{\partial u_0}{\partial n}(tx)w(x - le)dS \\
&= \int_{B_R(0)} u_0(tx)g(w(x - le))dx + \int_{\partial B_R(0)} u_0(tx)\frac{\partial w}{\partial n}(x - le)dS \\
&\quad - \int_{\partial B_R(0)} \frac{\partial u_0}{\partial n}(tx)w(x - le)dS.
\end{aligned}$$

Here we denote n by the unit normal vector on $\partial B_R(0)$. We recall here that $w(|x|) = \tilde{w}(r)$. Writing $\frac{\partial \tilde{w}}{\partial r} = \tilde{w}'$, we have

$$\frac{\partial w}{\partial n}(x - le) = \tilde{w}'(|x - le|)\frac{x \cdot (x - le)}{|x - le||x|}.$$

Thus we obtain

$$\begin{aligned}\int_{B_R(0)} (g(u_0(tx)) + f(tx))w(x - le)dx &= \int_{B_R(0)} u_0(tx)g(w(x - le))dx \\ &\quad - \int_{\partial B_R(0)} \frac{\partial u_0}{\partial n}(tx)w(x - le)dS \\ &\quad + \int_{\partial B_R(0)} u_0(tx)\tilde{w}'(|x - le|)\frac{x \cdot (x - le)}{|x - le||x|}dS \\ &=: (I) + (II) + (III).\end{aligned}$$

First we claim that there exists $c_R > 0$ such that $(I) \geq -c_R w(-le)^{1+\eta}$ for large l. In fact by (g4), we have

$$(I) \geq -\int_{B_R(0)} u_0(tx)|g(w(x - le))|dx \geq -c\int_{B_R(0)} u_0(tx)w(x - le)^{1+\eta}dx.$$

Now we recall that if l is large, then for $x \in B_R(0)$,

$$c' \leq \frac{w(x - le)}{|x - le|^{-\frac{N-1}{2}}e^{-|x-le|}} \leq c$$

by Proposition 2.1 (iii). We also observe that $\lim_{l\to\infty} \frac{|x-le|^{-\frac{N-1}{2}}}{l^{-\frac{N-1}{2}}} = 1$,

$$\lim_{l\to\infty}(-|x - le| + | - le|) = \lim_{l\to\infty}\Big(\frac{-|x|^2}{|x - le| + l} + \frac{2x_1 l}{|x - le| + l}\Big) = x_1$$

for $x \in B_R(0)$. Thus there exist c, c' and $l_0 > 0$ such that if $l \geq l_0$, then

$$c'e^{x_1} \leq \frac{w(x - le)}{w(-le)} \leq ce^{x_1} \text{ for } x \in B_R(0). \tag{4.3}$$

Using (4.3), we obtain

$$(I) \geq -c\|u_0\|_{L^\infty}\int_{B_R(0)} e^{(1+\eta)x_1}dx \times w(-le)^{1+\eta} =: -c_R w(-le)^{1+\eta}.$$

Next we prove that there exists $c'_R > 0$ such that $(II) \geq c'_R w(-le)$ for large l. Now by Lemma 2.2 (ii), it follows $(\nabla u_0 \cdot n) < 0$. Then from (4.3), we obtain

$$(II) \geq -c \sup_{t\in[\tau,T]}\sup_{x\in\partial B_R(0)} (\nabla u_0 \cdot n)(tx)\int_{\partial B_R(0)} e^{x_1}dS \times w(-le) =: c'_R w(-le).$$

Finally we claim that $(III) \geq c''_R w(-le)$ for large l. Now by Proposition 2.1 (iii), it follows if l is large, then for $x \in \partial B_R(0)$,

$$\frac{\tilde{w}'(|x - le|)}{w(x - le)} \geq -c.$$

We also have

$$\lim_{l\to\infty}\frac{x\cdot(x-le)}{|x-le||x|}=-\frac{x_1}{|x|}.$$

Thus there exists $l_1>0$ such that if $l\geq l_1$, then

$$\tilde{w}'(|x-le|)\frac{x\cdot(x-le)}{|x-le||x|}=\frac{\tilde{w}'(|x-le|)}{w(x-le)}\frac{w(x-le)}{w(-le)}w(-le)\frac{x\cdot(x-le)}{|x-le||x|}$$

$$\geq\frac{c}{R}x_1e^{x_1}w(-le)\text{ for }x\in\partial B_R(0).$$

Hence we obtain

$$(III)\geq\frac{c}{R}\int_{\partial B_R(0)}u_0(tx)x_1e^{x_1}dS\times w(-le).$$

By Lemma 2.2 (ii), we observe that

$$\begin{aligned}\int_{\partial B_R(0)}u_0(tx)x_1e^{x_1}dS=&\int_{\partial B_R(0)\cup\{x_1>0\}}u_0(tx)x_1e^{x_1}dS\\&+\int_{\partial B_R(0)\cup\{x_1<0\}}u_0(tx)x_1e^{x_1}dS\\=&\int_{\partial B_R(0)\cup\{x_1>0\}}u_0(tx)x_1e^{x_1}dS\\&-\int_{\partial B_R(0)\cup\{x_1>0\}}u_0(tx)x_1e^{-x_1}dS\\\geq&\inf_{t\in[\tau,T]}\inf_{x\in\partial B_R(0)}u_0(tx)\int_{\partial B_R(0)\cup\{x_1>0\}}x_1(e^{x_1}-e^{-x_1})dS\\>&0.\end{aligned}$$

Thus we have $(III)\geq c''_Rw(-le)$ for some $c''_R>0$. Consequently, we have

$$\int_{B_R(0)}(g(u_0(tx))+f(tx))w(x-le)dx\geq cw(-le)(1-w(-le)^\eta)$$

for $l\geq\max\{l_0,l_1\}$. Since $w(-le)$ decays exponentially as $l\to\infty$, we may assume that $w(-le)^\eta\leq\frac{1}{2}$ for all $l\geq l_R$. We also observe that from (4.3),

$$\int_{B_R(0)}w(x-le)dx\leq c\int_{B_R(0)}e^{x_1}dx\times w(-le)=:\tilde{c}_Rw(-le).$$

Thus we obtain

$$\int_{B_R(0)}(g(u_0(tx))+f(tx))w(x-le)dx\geq\frac{c}{\tilde{c}_R}\int_{B_R(0)}w(x-le)dx\text{ for all }l\geq l_R.$$

Next we prove (ii). In a similar calculation, we obtain

$$\begin{aligned}\int_{B_R(le)} g(w(x-le))u_0(tx)dx &= \int_{B_R(0)} g(w(x))u_0(t(x+le))dx \\ &= \int_{B_R(0)} w(x)g(u_0(t(x+le)))dx \\ &\quad + \int_{B_R(0)} w(x)f(t(x+le))dx \\ &\quad - \int_{\partial B_R(0)} \frac{\partial w}{\partial n}(x)u_0(t(x+le))dS \\ &\quad + \int_{\partial B_R(0)} w(x)\frac{\partial u_0}{\partial n}(t(x+le))dS \\ &\geq \int_{B_R(0)} w(x)g(u_0(t(x+le)))dx \\ &\quad - \tilde{w}'(R)\int_{\partial B_R(0)} u_0(t(x+le))dS \\ &\quad + \tilde{w}(R)\int_{\partial B_R(0)} \frac{\partial u_0}{\partial n}(t(x+le))dS.\end{aligned}$$

Here we used $w(x) = \tilde{w}(r)$ and $w(x)f(t(x+le)) \geq 0$. We claim that

$$\int_{\partial B_R(0)} \frac{\partial u_0}{\partial n}(t(x+le))dS \geq \tilde{c}_R u_0(tle)$$

for large l and some $\tilde{c}_R > 0$. Now we have

$$\int_{\partial B_R(0)} \frac{\partial u_0}{\partial n}(t(x+le))dS = t\int_{\partial B_R(0)} \frac{(\nabla u_0 \cdot x)(t(x+le))}{|t(x+le)|}\frac{t(x+le)\cdot x}{|t(x+le)||x|}dS.$$

By Lemma 2.2 (iii), there exists $l_2 > 0$ such that for $l \geq l_2$,

$$c'e^{-x_1} \leq \frac{u_0(t(x+le))}{u_0(tle)} \leq ce^{-x_1} \text{ for } x \in B_R(0),$$

$$-c' \leq \frac{(\nabla u_0 \cdot x)(t(x+le))}{u_0(t(x+le))|t(x+le)|} \leq c \text{ for } x \in \partial B_R(0).$$

Moreover it follows

$$\frac{t(x+le)\cdot x}{|t(x+le)||x|} \to \frac{x_1}{|x|} \text{ for } x \in \partial B_R(0).$$

Thus we have

$$\int_{\partial B_R(0)} \frac{\partial u_0}{\partial n}(t(x+le))dS \geq c\int_{\partial B_R(0)} -x_1e^{-x_1}dS \times u_0(tle) =: \tilde{c}_R u_0(tle).$$

Here we used

$$\int_{\partial B_R(0)} -x_1e^{-x_1}dS = \int_{\partial B_R(0)\cup\{x_1<0\}} -x_1(e^{-x_1}-e^{x_1})dS > 0.$$

Finally we can see that there exists $l_3 > 0$ such that for $l \geq l_3$,

$$\int_{B_R(le)} u_0(tx)dx \leq c \int_{B_R(0)} e^{-x_1} dx \times u_0(tle).$$

Thus for $l \geq \max\{l_2, l_3\}$, it follows

$$\int_{\partial B_R(0)} \frac{\partial u_0}{\partial n}(t(x+le))dS \geq c \int_{B_R(le)} u_0(tx)dx.$$

Arguing similarly as (i), we obtain

$$\int_{B_R(le)} g(w(x-le))u_0(tx)dx \geq c \int_{B_R(le)} u_0(tx)dx.$$

Putting $l_R := \max\{l_0, l_1, l_2, l_3\}$, the claims follow. □

Now we state the main result of this section.

Proposition 4.3. *There exist* $l^* > l_R$ *and* $0 < t_0 < 1$ *such that for all* $l \geq l^*$,

(i) $\sup_{t \in (0, t_0]} I\Big(u_0(x) + w\Big(\frac{x}{t} - le\Big)\Big) < 0,$

(ii) $I\Big(u_0(x) + w\Big(\frac{x}{T} - le\Big)\Big) < 0,$

(iii) $\sup_{t \in [t_0, T]} I\Big(u_0(x) + w\Big(\frac{x}{t} - le\Big)\Big) < I(u_0) + c_\infty.$

Proof. **Step 1**: [Decomposition of the energy]. Now we have

$$\begin{aligned}
I\Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big) =& \frac{1}{2}\int_{\mathbb{R}^N} \Big|\nabla\Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big)\Big|^2 dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^N} \Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big)^2 dx \\
&- \int_{\mathbb{R}^N} G\Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big) dx \\
&- \int_{\mathbb{R}^N} f(x)\Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big) dx \\
=& I(u_0) + I_\infty\Big(w\Big(\frac{x}{t}\Big)\Big) + \frac{1}{t}\int_{\mathbb{R}^N} \nabla u_0 \nabla w\Big(\frac{x}{t} - le\Big) dx \\
&+ \int_{\mathbb{R}^N} u_0 w\Big(\frac{x}{t} - le\Big) dx - \int_{\mathbb{R}^N} f(x) w\Big(\frac{x}{t} - le\Big) dx \\
&+ \int_{\mathbb{R}^N} G(u_0) dx + \int_{\mathbb{R}^N} G\Big(w\Big(\frac{x}{t} - le\Big)\Big) dx \\
&- \int_{\mathbb{R}^N} G\Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big) dx.
\end{aligned}$$

Since $I'(u_0)=0$, we have

$$\begin{aligned}\frac{1}{2t}\int_{\mathbb{R}^N}\nabla u_0\nabla w\Big(\frac{x}{t}-le\Big)dx=&-\frac{1}{2}\int_{\mathbb{R}^N}u_0w\Big(\frac{x}{t}-le\Big)dx\\&+\frac{1}{2}\int_{\mathbb{R}^N}g(u_0)w\Big(\frac{x}{t}-le\Big)dx\\&+\frac{1}{2}\int_{\mathbb{R}^N}fw\Big(\frac{x}{t}-le\Big)dx.\end{aligned}$$

Since $I'_\infty(w)=0$, we also have

$$\frac{1}{2t}\int_{\mathbb{R}^N}\nabla u_0\nabla w\Big(\frac{x}{t}-le\Big)dx=-\frac{t}{2}\int_{\mathbb{R}^N}u_0w\Big(\frac{x}{t}-le\Big)dx+\frac{t}{2}\int_{\mathbb{R}^N}g\Big(w\Big(\frac{x}{t}-le\Big)\Big)u_0dx.$$

Thus we obtain

$$\begin{aligned}I\Big(u_0+w\Big(\frac{x}{t}-le\Big)\Big)&-I(u_0)-I_\infty(w)\le I_\infty\Big(w\Big(\frac{x}{t}\Big)\Big)-I_\infty(w)\\&-\frac{t^N}{2}\int_{\mathbb{R}^N}f(tx)w(x-le)dx+\frac{t^N}{2}(1-t)\int_{\mathbb{R}^N}u_0(tx)w(x-le)dx\\&+\frac{t^N}{2}(1-t)\int_{\mathbb{R}^N}g(w(x-le))u_0(tx)dx+t^N\int_{\mathbb{R}^N}G(u_0(tx))\\&+G(w(x-le))-G(u_0(tx)+w(x-le))dx\\&+\frac{t^N}{2}\int_{\mathbb{R}^N}g(u_0(tx))w(x-le)+g(w(x-le))u_0(tx)dx\\&\le I_\infty\Big(w\Big(\frac{x}{t}\Big)\Big)-I_\infty(w)-\frac{t^N}{2}\int_{\mathbb{R}^N}f(tx)w(x-le)dx\\&+ct^N|t-1|\int_{\mathbb{R}^N}u_0(tx)w(x-le)dx-t^N\int_{\mathbb{R}^N}J(x)dx.\end{aligned}\tag{4.4}$$

Here we put

$$\begin{aligned}J(x):=&\,G(u_0(tx)+w(x-le))-G(u_0(tx))-G(w(x-le))\\&-\frac{1}{2}g(u_0(tx))w(x-le)-\frac{1}{2}g(w(x-le))u_0(tx).\end{aligned}$$

□

Step 2: [Proof of (i)]. From (4.4) and by Lemma 4.1 (i), it follows

$$\begin{aligned}I\Big(u_0+w\Big(\frac{x}{t}-le\Big)\Big)&\le I(u_0)+I_\infty\Big(w\Big(\frac{x}{t}\Big)\Big)+\frac{t^N}{2}|1-t|\int_{\mathbb{R}^N}u_0(tx)w(x-le)dx\\&\quad+\frac{t^N}{2}\int_{\mathbb{R}^N}f(tx)w(x-le)dx+ct^N\int_{\mathbb{R}^N}u_0(tx)^{1+\eta}w(x-le)\\&\quad+u_0(tx)w(x-le)^{1+\eta}dx\\&\le c(t^{N+1}+t^N)\int_{\mathbb{R}^N}u_0(tx)w(x-le)dx\\&\quad+\frac{t^N}{2}\int_{\mathbb{R}^N}f(tx)w(x-le)dx+I(u_0)+I_\infty\Big(w\Big(\frac{x}{t}\Big)\Big).\end{aligned}$$

Now we have

$$\int_{\mathbb{R}^N} u_0(tx)w(x-le)dx \le t^{-\frac{N}{2}}\|w\|_{L^2}\|u_0\|_{L^2}.$$

Combining with (4.2), we obtain

$$I\Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big) \le I(u_0) + c\Big(t^{\frac{N}{2}} + t^{\frac{N}{2}+1} + t^N + t^{N-2}\Big).$$

Since $N > 2$, we can take small $t_0 > 0$ so that

$$I(u_0) + c\Big(t^{\frac{N}{2}} + t^{\frac{N}{2}+1} + t^N + t^{N-2}\Big) < \frac{1}{2}I(u_0) < 0$$

for all $0 \le t \le t_0$. Thus we obtain (i). □

Now We decompose

$$\mathbb{R}^N = \Omega \cup B_R(0) \cup B_R(le),\ \Omega := \mathbb{R}^N \setminus (B_R(0) \cup B_R(le)).$$

As we mentioned earlier, it follows $u_0(tx) < \frac{\delta_0}{2}$, $w(x) < \frac{\delta_0}{2}$ on Ω.

Step 3: [Estimate of J]. By Lemma 4.1 (ii), it follows

$$\begin{aligned}-\int_{B_R(0)} Jdx \le &-\frac{1}{2}\int_{B_R(0)} g(u_0(tx))w(x-le)dx\\ &+\epsilon\int_{B_R(0)} w(x-le) + \frac{1}{2}w(x-le)^2\\ &+\frac{1}{2}w(x-le)u_0(tx)dx.\end{aligned}$$

Now we can apply Lemma 4.2 provided $\tau = t_0$ because t_0 is independent of l. Then for $l \ge l_R$, we obtain

$$\begin{aligned}-\int_{B_R(0)} Jdx \le &-\frac{c_R}{2}\int_{B_R(0)} w(x-le)dx + \frac{1}{2}\int_{B_R(0)} f(tx)w(x-le)dx\\ &+\epsilon\int_{B_R(0)} w(x-le) + \frac{1}{2}w(x-le)^2 + \frac{1}{2}w(x-le)u_0(tx)dx.\end{aligned}$$

Now we choose $\epsilon > 0$ so that

$$\epsilon\Big(1 + \frac{1}{2}\|w\|_{L^\infty} + \frac{1}{2}\|u_0\|_{L^\infty}\Big) \le \frac{c_R}{4}.$$

Then we have

$$-\int_{B_R(0)} Jdx \le -\frac{c_R}{4}\int_{B_R(0)} w(x-le)dx + \frac{1}{2}\int_{B_R(0)} f(tx)w(x-le)dx. \tag{4.5}$$

Similarly we have

$$-\int_{B_R(le)} Jdx \le -\frac{c_R}{4}\int_{B_R(le)} u_0(tx)dx. \tag{4.6}$$

Finally by Lemma 4.1 (iii), it follows $J(x) \geq 0$ for $x \in \Omega$. Thus from (4.4), (4.5) and (4.6), we obtain

$$
\begin{aligned}
I\Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big) - I(u_0) - I_\infty(w) &\leq I_\infty\Big(w\Big(\frac{x}{t}\Big)\Big) - I_\infty(w) \\
&+ct^N|t-1| \int_{\mathbb{R}^N} u_0(tx)w(x-le)dx \\
&-\frac{c_R}{4}t^N\Big(\int_{B_R(0)} w(x-le)dx + \int_{B_R(le)} u_0(tx)dx\Big). \qquad (4.7)
\end{aligned}
$$

Here we dropped the term $f(tx)w(x-le)$ because $f(x) \geq 0$ and $w(x) > 0$. □

Step 4: [Exponentially decay estimate]. Here we use the exponential decays of u_0 and w. We choose $0 < \delta$ arbitrary. By Proposition 2.1 (iii) and (2.2), we have

$$
\int_{B_R(0)} w(x-le)dx \geq c\int_{B_R(0)} e^{-(1+\delta)|x-le|}dx \geq ce^{-(1+\delta)l},
$$

$$
\int_{B_R(le)} u_0(tx)dx \geq c\int_{B_R(le)} e^{-t(1+\delta)|x|}dx \geq ce^{-t(1+\delta)l}.
$$

Moreover it follows

$$
\int_{\mathbb{R}^N} u_0(tx)w(x-le)dx \leq c\int_{\mathbb{R}^N} e^{-t(1-\delta)|x|}e^{-(1-\delta)|x-le|}dx \qquad (4.8)
$$

If $t \geq 1$, then

$$
\begin{aligned}
\text{r.h.s. of (4.8)} &\leq \int_{\mathbb{R}^N} e^{-(1-\delta)(|x|+|x-le|)}dx \\
&\leq ce^{-(1-2\delta)l}\int_{\mathbb{R}^N} e^{-\delta(|x|+|x-le|)}dx \\
&\leq ce^{-(1-2\delta)l}.
\end{aligned}
$$

If $t_0 \leq t < 1$, then

$$
\begin{aligned}
\text{r.h.s. of (4.8)} &\leq \int_{\mathbb{R}^N} e^{-t(1-\delta)(|x|+|x-le|)}dx \\
&\leq ce^{-t(1-2\delta)l}\int_{\mathbb{R}^N} e^{-t_0\delta(|x|+|x-le|)}dx \\
&\leq ce^{-t(1-2\delta)l}.
\end{aligned}
$$

Thus we obtain

$$
\int_{\mathbb{R}^N} u_0(tx)w(x-le)dx \leq ce^{-(1-2\delta)l} + ce^{-t(1-2\delta)l}.
$$

□

From (4.7) and by step 4, we obtain

$$
\begin{aligned}
I\Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big) - I(u_0) - I_\infty(w) &\leq ct^{N-1}|t-1|(e^{-t(1-2\delta)l} + e^{-(1-2\delta)l}) \\
&- ct^N(e^{-(1+\delta)l} + e^{-t(1+\delta)l}) + I_\infty\Big(w(\frac{x}{t})\Big) - I_\infty(w).
\end{aligned}
$$

Now we choose $0 < \delta$ so that $\delta < \frac{1}{5}$.

Step 5: [Proof of (ii)]. From (4.9), we obtain

$$\begin{aligned} I\Big(u_0 + w\Big(\frac{x}{T} - le\Big)\Big) \le\ & I(u_0) + I_\infty\Big(w\Big(\frac{x}{T}\Big)\Big) - cT^N(e^{-(1+\delta)l} + e^{-T(1+\delta)l}) \\ & + cT^{N-1}|T-1|(e^{-T(1-2\delta)l} + e^{-(1-2\delta)l}). \end{aligned}$$

By Young's inequality, we have

$$\begin{aligned} I\Big(u_0 + w\Big(\frac{x}{T} - le\Big)\Big) \le\ & I(u_0) + I_\infty\Big(w\Big(\frac{x}{T}\Big)\Big) + \epsilon(cT^{N-1}|T-1|)^2 \\ & + c_\epsilon(e^{-2T(1-2\delta)l} + e^{-2(1-2\delta)l}) \\ & - cT^N(e^{-(1+\delta)l} + e^{-T(1+\delta)l}). \end{aligned}$$

First we take $\epsilon > 0$ so that $\epsilon(cT^{N-1}|T-1|)^2 + I(u_0) + I_\infty(w(\frac{x}{T})) < 0$. Next we observe that $\delta < \frac{1}{5}$ implies $2(1-2\delta) > 1+\delta$. Thus we can choose $l_2 \ge l_R$ so that $I(u_0 + w(\frac{x}{T} - le)) < 0$ for all $l \ge l_2$. □

Step 6: We claim that there exists $A > 0$ such that

$$I_\infty\Big(w\Big(\frac{x}{t}\Big)\Big) - I_\infty(w) \le -A(t-1)^2 \text{ for all } t \in (0, T]. \tag{4.9}$$

In fact by Proposition 2.1 (ii), we have

$$\begin{aligned} I_\infty\Big(w\Big(\frac{x}{t}\Big)\Big) - I_\infty(w) =\ & \frac{1}{2}(t^{N-1} - 1)\int_{\mathbb{R}^N} |\nabla w|^2 dx \\ & + \frac{1}{2}(t^N - 1)\int_{\mathbb{R}^N} w^2 dx - (t^N - 1)\int_{\mathbb{R}^N} G(w) dx \\ =& - \frac{1}{2N}((N-2)t^N - Nt^{N-2} + 2)\int_{\mathbb{R}^N} |\nabla w|^2 dx \\ =& - \frac{1}{2N}(t-1)^2 a(t)\int_{\mathbb{R}^N} |\nabla w|^2 dx, \end{aligned}$$

where $a(t) = (N-2)t^{N-2} + \sum_{k=0}^{N-3} 2(k+1)t^k$. We can see that $a(t) > 0$ for all $t \in (0, T]$. Thus we obtain

$$I_\infty\Big(w\Big(\frac{x}{t}\Big)\Big) - I_\infty(w) \le -(t-1)^2 \times \Big(\frac{1}{2N}\inf_{t\in(0,L]} a(t)\int_{\mathbb{R}^N} |\nabla w|^2 dx\Big).$$

□

Step 7: [Proof of (iii)]. From (4.9) and (4.10), we obtain

$$\begin{aligned} I\Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big) - I(u_0) - I_\infty(w) \le\ & - A(t-1)^2 \\ & + ct^{N-1}|t-1|(e^{-(1-2\delta)l} + e^{-t(1-2\delta)l}) \\ & - ct^N(e^{-(1+\delta)l} + e^{-t(1+\delta)l}). \end{aligned}$$

By Young's inequality, we obtain

$$\begin{aligned} I\Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big) - I(u_0) - I_\infty(w) \le & - A|t-1|^2 + \epsilon(cT^{N-1})^2|t-1|^2 \\ & + c_\epsilon(e^{-2(1-2\delta)l} - e^{-(1+\delta)l}) \\ & + c_\epsilon(e^{-2t(1-2\delta)l} - e^{-t(1+\delta)l}). \end{aligned}$$

Now we choose $\epsilon > 0$ so that $\epsilon(cT^{N-1})^2 \leq \frac{A}{2}$. Next we take large $l_3 \geq l_R$ so that $e^{-(1-5\delta)t_0 l_2} \leq \frac{1}{4}$. Then we have

$$c_\epsilon(e^{-2(1-2\delta)l} - e^{-(1+\delta)l}) \leq -\frac{1}{2}e^{-(1+\delta)l}, c_\epsilon(e^{-2t(1-2\delta)l} - e^{-t(1+\delta)l}) \leq -\frac{1}{2}e^{-t(1+\delta)l}$$

for all $t \in [t_0, T]$ and $l \geq l_3$. Thus for $l \geq l_3$, we obtain

$$\sup_{t\in[t_0,T]} I(u_0 + w(\frac{x}{t} - le)) \leq -\frac{A}{2}|t-1|^2 - \frac{c_\epsilon}{2}(e^{-t(1+\delta)l} + e^{-(1+\delta)l}) < 0.$$

Putting $l^* = \max\{l_2, l_3\}$, the proof is complete. □

Hereafter in this paper, we fix $l \geq l^*$.

5. Existence of the second solution

In this section, we prove the existence of another nontrivial critical point of I. To this aim, we use the Monotonicity Trick.

Now by Proposition 4.3, we know that

$$\sup_{t\geq 0} I\Big(u_0 + w\Big(\frac{x}{t} - le\Big)\Big) < I(u_0) + c_\infty,\ I\Big(u_0 + w\Big(\frac{x}{T} - le\Big)\Big) < 0. \tag{5.1}$$

For $\lambda_0 \in (0,1)$, we put

$$I_{\lambda_0}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + (1+M)u^2 dx - \lambda_0 \int_{\mathbb{R}^N} G(u) + \frac{M}{2}u^2 dx - \int_{\mathbb{R}^N} fu dx.$$

Here M is a positive constant defined in (2.4). We choose $\lambda_0 < 1$ sufficiently close to 1. Then from (5.1), we have $I_{\lambda_0}(u_0) < 0$,

$$I_{\lambda_0}\Big(u_0(x) + w\Big(\frac{x}{T} - le\Big)\Big) < 0,\ \sup_{t\in[0,T]} I_{\lambda_0}\Big(u_0(x) + w\Big(\frac{x}{t} - le\Big)\Big) < I(u_0) + c_\infty. \tag{5.2}$$

For $\lambda \in [\lambda_0, 1]$, we define

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + (1+M)u^2 dx - \lambda \int_{\mathbb{R}^N} G(u) + \frac{M}{2}u^2 dx - \int_{\mathbb{R}^N} fu dx.$$

We define

$$\Gamma := \Big\{\gamma \in C([0,1], H^1(\mathbb{R}^N)); \gamma(0) = u_0, \gamma(1) = u_0 + w\Big(\frac{\cdot}{T} - le\Big)\Big\},$$

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)).$$

Since $G(s) + \frac{M}{2}s^2 \geq 0$, it follows

$$I(u) = I_1(u) \leq I_\lambda(u) \leq I_{\lambda_0}(u) \text{ for all } u \in H^1(\mathbb{R}^N),$$

and hence $c_1 \leq c_\lambda \leq c_{\lambda_0}$. Moreover we have $\|u_0\| < \rho_0 < \|u_0 + w(\frac{x}{T} - le)\|$. Then for all $\gamma \in \Gamma$, we have $\gamma([0,1]) \cap \{u \in H^1(\mathbb{R}^N); \|u\| = \rho_0\} \neq \emptyset$. Thus by Lemma 3.1 and from (5.2), we obtain

$$0 < c_1 \leq c_\lambda \leq c_{\lambda_0} < I(u_0) + c_\infty. \tag{5.3}$$

Next we introduce the Monotonicity Trick due to [15] and [20].

Proposition 5.1 ([15], [20]). *Let X be a Banach space with the norm $\|\cdot\|_X$ and $\Lambda \subset (0,\infty)$ be an interval. We consider a family $\{L_\lambda\}_{\lambda\in\Lambda}$ of C^1-functional on X of the form:*

$$L_\lambda(u) = A(u) - \lambda B(u),\ \ \lambda \in \Lambda,$$

where $B(u) \geq 0$ for all $u \in X$ and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\|_X \to \infty$. We assume there exist v_0, $v_1 \in X$ such that

$$m_\lambda := \inf_{\gamma\in\Gamma_X} \max_{t\in[0,1]} L_\lambda(\gamma(t)) > \max\{L_\lambda(v_0), L_\lambda(v_1)\} \text{ for all } \lambda \in \Lambda,$$

where

$$\Gamma_X = \{\gamma(t) \in C([0,1], X); \gamma(0) = v_0, \gamma(1) = v_1\}.$$

Then for almost every $\lambda \in \Lambda$, there exists a sequence $\{u_n^\lambda\} \subset X$ such that

$$(i)\ L_\lambda(u_n^\lambda) \to m_\lambda,\ \ (ii)\ L'_\lambda(u_n^\lambda) \to 0,\ \ (iii)\ \|u_n^\lambda\|_X \text{ is bounded.}$$

Moreover a map $\lambda \mapsto m_\lambda$ is left-continuous with respect to λ.

Since $G(s) + \frac{M}{2}s^2 \geq 0$ for all s ≥ 0 and

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + (1+M)u^2 dx - \int_{\mathbb{R}^N} fudx \geq \Big(\frac{1}{2} - \epsilon\Big)\|u\|^2 - \frac{1}{\epsilon}\|f\|_{L^2}^2$$

for any $\epsilon > 0$, we can apply Proposition 5.1. Then for a.e. $\lambda \in [\lambda_0, 1]$, there exists $\{u_n\} \subset H^1(\mathbb{R}^N)$ such that

$$I_\lambda(u_n) \to c_\lambda, I'_\lambda(u_n) \to 0, \{u_n\} \text{ is bounded in } H^1(\mathbb{R}^N).$$

Now we fix $\lambda \in [\lambda_0, 1]$. Then we may assume that $u_n \rightharpoonup u_\lambda$ in $H^1(\mathbb{R}^N)$ for some $u_\lambda \in H^1(\mathbb{R}^N)$.

Lemma 5.2. *u_λ satisfies $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) \leq c_\lambda$. Moreover it follows either $I_\lambda(u_\lambda) = c_\lambda$ or $\|u_\lambda\| \geq \rho_0$.*

Proof. The proof of the first part is rather standard. Applying Concentration Compactness Principle to I_λ, we have

$$I_\lambda(u_n) \to c_\lambda = I_\lambda(u_\lambda) + \sum_{i=1}^k I_\lambda^\infty(w_\lambda^i),$$

$$\Big\|u_n - u_\lambda - \sum_{i=1}^k w_\lambda^i(\cdot - y_{n,\lambda}^i)\Big\| \to 0 \text{ as } n \to \infty,$$

$$I_\lambda^{\infty\prime}(w_\lambda^i) = 0,\ |y_{n,\lambda}^i| \to \infty,\ |y_{n,\lambda}^i - y_{n,\lambda}^{i'}| \to \infty \text{ for } i \neq i'$$

for $k \in \mathbb{N} \cup \{0\}$, $\{y_{n,\lambda}^i\} \subset \mathbb{R}^N$, $w_\lambda^i \in H^1(\mathbb{R}^N) \setminus \{0\}$. Here

$$I_\lambda^\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + (1+M)u^2 dx - \lambda\int_{\mathbb{R}^N} G(u) + \frac{M}{2}u^2 dx.$$

We can see that for any nontrivial critical point u of I_λ^∞, it follows $I_\lambda^\infty(u) > 0$. Thus we have $c_\lambda \geq I_\lambda(u_\lambda)$.

Now we have to distinguish into two cases. (i) The case $k = 0$. In this case, we have $c_\lambda = I_\lambda(u_\lambda)$.

(ii) The case $k \geq 1$. In this case, it follows $\|u_\lambda\| \geq \rho_0$. Now we have

$$c_\lambda = I_\lambda(u_\lambda) + \sum_{i=1}^{k} I_\lambda^\infty(w_\lambda^i) \geq I_\lambda(u_\lambda) + kc_\lambda^\infty \geq I(u_\lambda) + kc_\lambda^\infty, \tag{5.4}$$

where c_λ^∞ is the least energy level for I_λ^∞. We can see that $c_\lambda^\infty \geq c_\infty$ for $\lambda \leq 1$. Thus from (5.4), we obtain $c_\lambda \geq I(u_\lambda) + c_\infty$.

On the other hand from (5.3), it follows $c_\lambda \leq c_{\lambda_0} < I(u_0) + c_\infty$. Thus we obtain

$$I(u_0) > I(u_\lambda) + (k-1)c_\infty \geq I(u_\lambda). \tag{5.5}$$

If $\|u_\lambda\| < \rho_0$, then we have

$$I(u_0) > I(u_\lambda) \geq \inf_{u \in B_{\rho_0}} I(u) = I(u_0).$$

This is a contradiction. Thus it follows $\|u_\lambda\| \geq \rho_0$. □

Now for a non-decreasing sequence $\lambda_j \to 1$ as $j \to \infty$, we apply Proposition 5.1. Then by Lemma 5.2, there exists $\{(\lambda_j, u_j)\} \subset [\lambda_0, 1] \times H^1(\mathbb{R}^N)$ with $\lambda_j \nearrow 1$ as $j \to \infty$ such that

$$I'_{\lambda_j}(u_j) = 0, I_{\lambda_j}(u_j) \leq c_{\lambda_j} \text{ and either } \|u_j\| \geq \rho_0 \text{ or } I_{\lambda_j}(u_j) = c_{\lambda_j}. \tag{5.6}$$

Next we prove the boundedness of $\{u_j\}$.

Lemma 5.3. *$\{u_j\}$ is bounded in $H^1(\mathbb{R}^N)$.*

Proof. We argue as in [16]. Now we have

$$\begin{aligned} c_{\lambda_j} \geq I_{\lambda_j}(u_j) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_j|^2 dx + \frac{1+(1-\lambda_j)M}{2}\int_{\mathbb{R}^N} u_j^2 dx \\ &\quad -\lambda_j \int_{\mathbb{R}^N} G(u_j)dx - \int_{\mathbb{R}^N} f u_j dx. \end{aligned}$$

On the other hand, by Lemma 2.3, it follows

$$\begin{aligned} \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u_j|^2 dx &= -\frac{N}{2}(1+(1-\lambda_j)M)\int_{\mathbb{R}^N} u_j^2 dx + N\lambda_j \int_{\mathbb{R}^N} G(u_j)dx \\ &\quad + N\int_{\mathbb{R}^N} f u_j dx + \int_{\mathbb{R}^N} \nabla f \cdot x u_j dx. \end{aligned}$$

Thus we have

$$\int_{\mathbb{R}^N} |\nabla u_j|^2 dx = Nc_{\lambda_j} - \int_{\mathbb{R}^N} \nabla f \cdot x u_j dx. \tag{5.7}$$

Now by (f2), it follows $\nabla f \cdot x \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$. Then from (5.7), we have

$$\|\nabla u_j\|_{L^2}^2 \leq Nc_{\lambda_0} + \|\nabla f \cdot x\|_{L^{\frac{2N}{N+2}}} \|u_j\|_{L^{\frac{2N}{N-2}}} \leq Nc_{\lambda_0} + c\|\nabla f \cdot x\|_{L^{\frac{2N}{N+2}}} \|\nabla u_j\|_{L^2}.$$

By Young's inequality, we have

$$\|\nabla u_j\|_{L^2} \leq c \text{ for some } c > 0. \tag{5.8}$$

Next by (g3) and (g4), we have for all $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$g(s) \leq \epsilon s + c_\epsilon s^{\frac{N+2}{N-2}} \text{ for all } s \geq 0.$$

Since $I'_{\lambda_j}(u_j)u_j = 0$, we have

$$\|\nabla u_j\|_{L^2}^2 + (1+(1-\lambda_j)M)u_j^2 dx = \lambda_j \int_{\mathbb{R}^N} g(u_j)u_j dx + \int_{\mathbb{R}^N} f u_j dx.$$

Since $\lambda_j \in [\lambda_0, 1]$ and $M \geq 0$, we have

$$\begin{aligned} \|u_j\|_{L^2}^2 &\leq \int_{\mathbb{R}^N} g(u_j)u_j dx + \int_{\mathbb{R}^N} f u_j dx \\ &\leq \epsilon \int_{\mathbb{R}^N} u_j^2 dx + c_\epsilon |u_j|^{\frac{2N}{N-2}} dx + \|f\|_{L^2}\|u_j\|_{L^2} \\ &\leq \epsilon \|u_j\|_{L^2}^2 + c'_\epsilon \|\nabla u_j\|_{L^2}^{\frac{2N}{N-2}} + \epsilon \|u_j\|_{L^2}^2 + c''_\epsilon \|f\|_{L^2}^2 \end{aligned}$$

for some c'_ϵ, $c''_\epsilon > 0$. Thus we have

$$(1-2\epsilon)\|u_j\|_{L^2}^2 \leq c\|\nabla u_j\|_{L^2}^{\frac{2N}{N-2}} + c.$$

Combining with (5.8), we obtain the boundedness of $\{u_j\}$. □

Next we show that $\{u_j\}$ is a Palais-Smale sequence for I.

Lemma 5.4. *$\{u_j\}$ is a bounded Palais-Smale sequence for I and it satisfies $\limsup_{j\to\infty} I(u_j) \leq c_1$. Moreover it follows either $I(u_j) \to c_1$ or $\|u_j\| \geq \rho_0$.*

Proof. Now by the definition of I_λ, we have

$$I(u) = I_{\lambda_j}(u_j) + \frac{(\lambda_j - 1)M}{2} \int_{\mathbb{R}^N} u_j^2 dx + (\lambda_j - 1) \int_{\mathbb{R}^N} G(u_j) dx,$$

$$I'(u_j) = I'_{\lambda_j}(u_j) + (\lambda_j - 1)M u_j + (\lambda_j - 1) g(u_j) \text{ in } H^{-1}.$$

We observe that $\int_{\mathbb{R}^N} G(u_j)dx$ and $\|g(u_j)\|_{H^{-1}}$ are bounded because of the boundedness of $\|u_j\|$. Since $\lambda_j \to 1$ and $I_{\lambda_j}(u_j) = 0$, we obtain $I'(u_j) \to 0$. Moreover we have

$$\limsup_{j\to\infty} I(u_j) = \limsup_{j\to\infty} I_{\lambda_j}(u_j) \leq \limsup_{j\to\infty} c_{\lambda_j}.$$

Since c_λ is left-continuous with respect to λ, it follows $\lim_{j\to\infty} c_{\lambda_j} = c_1$. Thus we obtain $\limsup_{j\to\infty} I(u_j) \leq c_1$.

Finally we show either $I(u_j) \to c_1$ or $\|u_j\| \geq \rho_0$. From (5.6), we have $\|u_j\| \geq \rho_0$ or $I_{\lambda_j}(u_j) = c_{\lambda_j}$. If $\|u_j\| \geq \rho_0$, we have nothing to prove. If $I_{\lambda_j}(u_j) = c_{\lambda_j}$, then it follows

$$\lim_{j\to\infty} I(u_j) = \lim_{j\to\infty} I_{\lambda_j}(u_j) = \lim_{j\to\infty} c_{\lambda_j} = c_1.$$

□

Now we may assume that $u_j \rightharpoonup u_1$ in $H^1(\mathbb{R}^N)$ for some $u_1 \in H^1(\mathbb{R}^N)$ because of the boundedness of $\{u_j\}$. Then by a standard argument, we have $I'(u_1) = 0$.

Lemma 5.5. *$u_1 \not\equiv u_0$, that is, u_1 and u_0 are different critical points of I.*

Proof. We suppose by contradiction that $u_1 \equiv u_0$. Applying Concentration Compactness Principle to I, we have

$$\lim_{j\to\infty} I(u_j) = I(u_1) + \sum_{i=0}^{k} I_\infty(w^i), \ \|u_j - u_1 - \sum_{i=0}^{k} w^i(\cdot - y_j^i)\| \to 0. \tag{5.9}$$

We distinguish into three cases: (i) $k \geq 1$. (ii) $k = 0$ and $I(u_j) \to c_1$. (iii) $k = 0$ and $\|u_j\| \geq \rho_0$.

(i) From (5.3), (5.9) and by Lemma 5.4,

$$I(u_0) + c_\infty > c_1 \geq \limsup_{j\to\infty} I(u_j) = I(u_1) + \sum_{i=0}^{k} I_\infty(w^i) \geq I(u_0) + kc_\infty.$$

Thus we have $(k-1)c_\infty < 0$. Since $k \geq 1$ and $c_\infty > 0$, this is a contradiction.

(ii) From (5.3) and (5.9), it follows

$$0 < c_1 = \lim_{j\to\infty} I(u_j) = I(u_1) = I(u_0).$$

This contradicts to the fact $I(u_0) < 0$.

(iii) From (5.9), we have

$$\rho_0 \leq \liminf_{j\to\infty} \|u_j\| = \|u_1\| = \|u_0\| < \rho_0.$$

This is a contradiction. Therefore it follows $u_1 \not\equiv u_0$. □

Remark 5.6. Suppose that $f(x) = f(|x|)$. Then we can obtain two critical points easily. Indeed if $f(x) = f(|x|)$, then all positive solution of (1.1) should be radially symmetric. Thus critical points of $I(u)$ belong to $H^1_{rad}(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N); u(x) = u(|x|)\}$.

In a same argument, we can prove the existence of a local minimizer whose energy is negative. To obtain the second critical point, we set

$$\tilde{\Gamma} := \{\gamma \in C([0,1], H^1_{rad}(\mathbb{R}^N); \gamma(0) = 0, \gamma(1) = w(\frac{\cdot}{T})\},$$

$$\tilde{c}_\lambda := \inf_{\gamma\in\tilde{\Gamma}} \max_{t\in[0,1]} I_\lambda(\gamma(t)).$$

Then we can see that $I_\lambda(\gamma(1)) < 0$ and $\tilde{c}_1 > 0$ if T is sufficiently large. Since the embedding $H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $2 < p < \frac{2N}{N-2}$, it follows $I(u_j) \to \tilde{c}_1$ where u_j is a bounded Palais-Smale sequence for I constructed by the monotonicity trick. Especially we don't need the interaction estimate. Using the compact embedding again, the weak limit u_1 of u_j satisfies $I(u_1) = \tilde{c}_1 > 0$ and $I'(u_1) = 0$. Thus we obtain two different critical points.

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