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A NOTE ON EXPLICITLY QUASICONVEX SET-VALUED MAPS

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Abstract. This note concerns explicitly quasiconvex set-valued maps, defined on a nonempty convex subset of a real linear space with values in a real linear space, partially ordered by a solid vectorially closed convex cone. It is shown that these generalized convex set-valued maps can be characterized in terms of classical explicit quasiconvexity of certain scalar functions.

1. Introduction

Real-valued explicitly quasiconvex functions (also known under different terminology in the literature) constitute a special class of quasiconvex functions. Enjoying certain properties similar to that of convex functions, they play an important role in scalar optimization. For instance, as already mentioned in the classical monograph of Stoer and Witzgall [15], every local minimum of such a function is indeed a global one. In what concerns multicriteria optimization problems, several interesting questions have been solved assuming that the vector-valued objective function is componentwise explicitly quasiconvex: the connectedness and contractibility of efficient sets in [5], [3] and [2], the parametric methods for scalarizing bicriteria optimization problems in [14] and [10], and the Pareto reducibility of multicriteria optimization problems in [11] and [12].

The notion of explicit quasiconvexity has been extended in [13] for set-valued maps and vector-valued functions in a more general framework, where the componentwise setting doesn't have any sense, since these functions take values in a real linear space partially ordered by an arbitrary relatively solid convex cone. The principal aim of our work is to prove that, under some mild assumptions, explicitly cone-quasiconvex set-valued maps can be characterized in terms of classical explicit quasiconvexity of certain scalar functions, in the same manner as set-valued cone-quasiconvex maps have been characterized in [4].

The paper is organized as follows. In Section 2 we state some properties of solid convex cones, which allow us to extend the Gerstewitz's scalarization function introduced in [6] to our general (algebraic, but not necessarily topological) setting. In Section 3 we state our main result (Theorem 3.7) concerning the characterization of explicitly cone-quasiconvex set-valued maps and we derive from it a characterization of explicitly cone-quasiconvex vector-valued functions.

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2. Gerstewitz's scalarization functions in algebraic framework

Throughout this paper Y will be a linear space over the field $\mathbb{R}$ of real numbers. For convenience, let us denote $\mathbb{R}_+ := [0,+\infty[$, $\mathbb{R}_+^* :=]0,+\infty[$, and $\overline{\mathbb{R}} := \mathbb{R}\cup\{-\infty,+\infty\}$. Recall that the algebraic interior and the relative algebraic interior of any set $M \subset Y$ are given by:

$$\begin{aligned}\operatorname{cor} M &:= \{x \in M \mid \forall\, y \in Y,\, \exists\, \lambda \in \mathbb{R}_+^*,\, x + [0,\lambda]\cdot y \subset M\};\\ \operatorname{icr} M &:= \{x \in M \mid \forall\, y \in \operatorname{span}(M-M),\, \exists\, \lambda \in \mathbb{R}_+^*,\, x + [0,\lambda]\cdot y \subset M\}.\end{aligned}$$

The set M is called solid (relatively solid) if $\operatorname{cor} M \neq \emptyset$ (resp. $\operatorname{icr} M \neq \emptyset$). Following Adán and Novo [1], we say that M is vectorially closed if $M = \operatorname{vcl} M$, where the so-called vector closure of M is given by

$$\begin{aligned}\operatorname{vcl} M &:= \{y \in Y \mid \exists\, y' \in Y,\, \forall\, \lambda \in \mathbb{R}_+^*,\, \exists\, \lambda' \in]0,\lambda],\, y + \lambda' y' \in M\}\\ &= \{y \in Y \mid \exists\, \tilde{y} \in Y,\, \exists\, \{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R},\, \lambda_n \to 0,\, y + \lambda_n \tilde{y} \in M,\, \forall n \in \mathbb{N}\}.\end{aligned}$$

Notice that, if M is convex, then $\operatorname{vcl} M = M \cup \{y \in Y \mid \exists\, x \in M,\, \forall\, t \in [0,1[,\, (1-t)x + ty \in M\} =: \operatorname{lin} M$, the latter set representing the so-called algebraic closure of M. When a convex set M is relatively solid, we also have $\operatorname{vcl} M = \operatorname{vcl}(\operatorname{icr} M)$ and $(1-t)\cdot \operatorname{icr} M + t\cdot \operatorname{vcl} M \subset \operatorname{icr} M$ for all $t \in [0,1[$. In the particular case when Y is endowed with a linear topology, the vector closure of any solid convex set coincides with its topological closure (see [1] and [7]).

In the sequel we will assume that the real linear space Y is partially ordered by a solid convex cone C, i.e., $\emptyset \neq \operatorname{cor} C \subset C = \mathbb{R}_+\cdot C = C + C \subset Y$. It is easy to check that for every $e \in \operatorname{cor} C$ we have:

$$\mathbb{R}_+^* \cdot e - C = Y; \tag{2.1}$$

$$\mathbb{R}_+^* \cdot e + C = \operatorname{cor} C. \tag{2.2}$$

Lemma 2.1. *Assume that C is vectorially closed and let $e \in \operatorname{cor} C$. Then, for every $x \in Y$, the set*

$$A_e(x) := \{\alpha \in \mathbb{R} \mid x \in \alpha e - C\}$$

is nonempty and closed. Moreover, if $C \neq Y$, then $A_e(x)$ is bounded from below.

Proof. Consider an arbitrary point $x \in Y$. The nonemptiness of $A_e(x)$ follows by (2.1).

Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $A_e(x)$, which converges to some $\alpha \in \mathbb{R}$. For all $n \in \mathbb{N}$ we have $\alpha_n e - x \in C$ and hence $\alpha e - x = (\alpha - \alpha_n)e + (\alpha_n e - x) \in \operatorname{vcl} C = C$, which yields $\alpha \in A_e(x)$. Thus $A_e(x)$ is closed.

Assume now that $C \neq Y$ and suppose on the contrary that $A_e(x)$ is not bounded from below. Then there exists $n_0 \in \mathbb{N}$ such that $-n \in A_e(x)$, i.e., $-e \in \frac{1}{n}x + C$, for all $n \in \mathbb{N} \cap [n_0,+\infty[$, showing that $-e \in \operatorname{vcl} C = C$. Hence $0_Y = e + (-e) \in (\operatorname{cor} C) + C = \operatorname{cor} C$, which yields $C = Y$, a contradiction. □

Lemma 2.1 shows that, for each couple $(e,v) \in (\operatorname{cor} C) \times Y$, one can define a function $h_{e,v} : Y \to \mathbb{R}$ by

$$h_{e,v}(y) := \min A_e(y-v) = \min\{\alpha \in \mathbb{R} \mid y \in v + \alpha e - C\},\ \forall\, y \in Y. \tag{2.3}$$

Notice that, in the particular case when Y is a real topological linear space, partially ordered by a proper closed convex cone with nonempty interior, the function $h_{e,v}$ actually represents the Gerstewitz's scalarization function, introduced in [6] (the so-called smallest strictly monotonic function in [9]).

3. Explicitly quasiconvex functions

In what follows S will be a nonempty convex subset of a real linear space X. For all points $x_1, x_2 \in X$ we denote $]x_1, x_2[:= \{(1-t)x_1 + tx_2 \mid t \in]0,1[\}$. Notice that $]x_1, x_2[$ becomes a singleton whenever $x_1 = x_2$.

Definition 3.1. An extended real-valued function $\varphi : S \to \overline{\mathbb{R}}$ is said to be *explicitly quasiconvex* if for all $x_1, x_2 \in S$ and $x \in]x_1, x_2[$ one has

$$\varphi(x) \leq \max\{\varphi(x_1), \varphi(x_2)\},$$

where the strict inequality holds whenever $\varphi(x_1) \neq \varphi(x_2)$.

Proposition 3.2. *For any function $\varphi : S \to \overline{\mathbb{R}}$ the following assertions are equivalent:*

1° *φ is explicitly quasiconvex.*

2° *For all $\lambda \in \mathbb{R} \cup \{+\infty\}$ and $x_1, x_2 \in S$ with $\varphi(x_1) < \lambda$ and $\varphi(x_2) \leq \lambda$ one has $\varphi(x) < \lambda$ for every $x \in]x_1, x_2[$.*

Proof. 1° $\Rightarrow$ 2°. Assume that 1° holds and let $\lambda \in \mathbb{R} \cup \{+\infty\}$. Let $x_1, x_2 \in S$ with $\varphi(x_1) < \lambda$, $\varphi(x_2) \leq \lambda$, and let $x \in]x_1, x_2[$. If $\varphi(x_1) = \varphi(x_2)$, then 1° implies $\varphi(x) \leq \max\{\varphi(x_1), \varphi(x_2)\} = \varphi(x_1) < \lambda$. Otherwise, if $\varphi(x_1) \neq \varphi(x_2)$, then we can also deduce by 1° that $\varphi(x) < \max\{\varphi(x_1), \varphi(x_2)\} \leq \lambda$. Hence 2° holds.

2° $\Rightarrow$ 1°. Assume that 2° holds and consider some arbitrary points $x_1, x_2 \in S$ and $x \in]x_1, x_2[$. Without loss of generality we can assume that $\varphi(x_1) \leq \varphi(x_2)$.

Case 1: $\varphi(x_1) = \varphi(x_2)$.

If $\varphi(x_2) = +\infty$, then $\varphi(x) \leq +\infty = \max\{\varphi(x_1), \varphi(x_2)\}$. Otherwise, if $\varphi(x_2) \in \mathbb{R}$, then for each real number $\varepsilon > 0$, denoting $\lambda_\varepsilon := \varphi(x_2) + \varepsilon$, we have $\varphi(x_1) = \varphi(x_2) < \lambda_\varepsilon$. By 2° we can deduce that $\varphi(x) < \lambda_\varepsilon$ and then, letting $\varepsilon \searrow 0$, we get $\varphi(x) \leq \varphi(x_2) = \max\{\varphi(x_1), \varphi(x_2)\}$.

Case 2: $\varphi(x_1) \neq \varphi(x_2)$, i.e., $\varphi(x_1) < \varphi(x_2)$.

In this case, denoting $\lambda := \varphi(x_2)$, we have $\lambda \in \mathbb{R} \cup \{+\infty\}$, $\varphi(x_1) < \lambda$, and $\varphi(x_2) \leq \lambda$. By 2° it follows that $\varphi(x) < \lambda = \max\{\varphi(x_1), \varphi(x_2)\}$. □

Definition 3.3. ([13]) A set-valued map $F : S \to 2^Y$, defined on a nonempty convex subset S of X, is said to be *explicitly C-quasiconvex* if for all $x_1, x_2 \in S$ and $x \in]x_1, x_2[$ the following inclusion holds:

$$(F(x_1) + \operatorname{cor} C) \cap (F(x_2) + C) \subset F(x) + \operatorname{cor} C.$$

A function $f : S \to Y$ is called *explicitly C-quasiconvex* if the set-valued map $F : S \to 2^Y$, defined for all $x \in S$ by $F(x) = \{f(x)\}$, is explicitly C-quasiconvex.

Proposition 3.4. *Let $\Phi : S \to 2^{\mathbb{R}}$ be a set-valued map and let $\varphi : S \to \overline{\mathbb{R}}$ be its marginal function, defined for all $x \in S$ by $\varphi(x) = \inf \Phi(x)$. Then the following hold:*

a) If φ is explicitly quasiconvex, then Φ is explicitly $\mathbb{R}_+$-quasiconvex.

b) If Φ is explicitly $\mathbb{R}_+$-quasiconvex and its values are nonempty and closed, then φ is explicitly quasiconvex.

Proof. a) Assuming that φ is explicitly quasiconvex, consider some arbitrary $x_1, x_2 \in S$ and $x \in]x_1, x_2[$. Then, for any $\lambda \in (\Phi(x_1) + \mathrm{int}\mathbb{R}_+) \cap (\Phi(x_2) + \mathbb{R}_+)$, there exist $y_1 \in \Phi(x_1)$ and $y_2 \in \Phi(x_2)$ such that $\varphi(x_1) \le y_1 < \lambda$ and $\varphi(x_2) \le y_2 \le \lambda$. Since φ is explicitly quasiconvex, it follows by Proposition 3.2 that $\varphi(x) < \lambda$, i.e., $\inf \Phi(x) < \lambda$, which shows that $\lambda \in \Phi(x) + \mathrm{int}\mathbb{R}_+$. We infer that $(\Phi(x_1) + \mathrm{int}\mathbb{R}_+) \cap (\Phi(x_2) + \mathbb{R}_+) \subset \Phi(x) + \mathrm{int}\mathbb{R}_+$, proving that Φ is explicitly $\mathbb{R}_+$-quasiconvex.

b) Since for all $x \in S$ the nonempty set $\Phi(x)$ is closed, we actually have $\varphi(x) = \min \Phi(x)$, hence the conclusion follows from Proposition 3.2. □

Remark 3.5. The closeness assumption in Proposition 3.4 (*b*) is essential, as shown below.

Example 3.6. Let $\Phi : \mathbb{R} \to 2^{\mathbb{R}}$ be the set-valued map defined by

$$\Phi(x) := \begin{cases}]1, +\infty[& \text{if} \quad x \neq 0 \\ [0, +\infty[& \text{if} \quad x = 0. \end{cases}$$

It is easily seen that for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in (\Phi(x_1) + \mathrm{int}\mathbb{R}_+) \cap (\Phi(x_2) + \mathbb{R}_+)$, we actually have $\lambda > 1$, hence $\lambda \in \Phi(x) + \mathrm{int}\mathbb{R}_+$ for all $x \in]x_1, x_2[$. Thus Φ is explicitly $\mathbb{R}_+$-quasiconvex. However, its marginal function, given by

$$\varphi(x) := \inf \Phi(x) = \begin{cases} 1 & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0, \end{cases}$$

is not explicitly quasiconvex.

Theorem 3.7. *Consider an arbitrary point $e \in \mathrm{cor}\, C$ and let $F : S \to 2^Y$ be a set-valued map such that $F(x) + C$ is nonempty and vectorially closed for all $x \in S$. For any $v \in Y$, define the set-valued map $\Phi_v : S \to 2^{\mathbb{R}}$ by*

$$\Phi_v(x) := \{\alpha \in \mathbb{R} \mid v + \alpha e \in F(x) + C\} \quad \textit{for all} \quad x \in S,$$

and denote by $\varphi_v : S \to \overline{\mathbb{R}}$ its marginal function, i.e.

$$\varphi_v(x) = \inf \Phi_v(x) \quad \textit{for all} \quad x \in S.$$

The following assertions are equivalent:

1° *The map F is explicitly C-quasiconvex.*
2° *For every $v \in Y$ the map Φ_v is explicitly $\mathbb{R}_+$-quasiconvex.*
3° *For every $v \in Y$ the function φ_v is explicitly quasiconvex.*

Proof. In order to prove that 2° $\Leftrightarrow$ 3° one can apply Proposition 3.4 for Φ_v in the role of Φ, where $v \in Y$. We just have to check that $\Phi_v(x)$ is nonempty and closed for any $x \in S$. Indeed, $F(x) + C$ being nonempty, we can choose $y \in F(x)$. Since $e \in \mathrm{cor}\,\mathrm{C}$, it follows by (2.1) that $\emptyset \neq \{\alpha \in \mathbb{R} \mid y - v \in \alpha e - C\} \subset \Phi_v(x)$. Thus $\Phi_v(x) \neq \emptyset$. Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence in $\Phi_v(x)$, converging to $\alpha \in \mathbb{R}$. Since $v + \alpha e = v + \alpha_n e + (\alpha - \alpha_n)e$ for all $n \in \mathbb{N}$, we infer that $v + \alpha e \in \mathrm{vcl}(F(x) + C) = F(x) + C$, hence $\alpha \in \Phi_v(x)$. Thus $\Phi_v(x)$ is closed.

Assume now that 1° holds. In order to prove 2°, consider an arbitrary $v \in Y$, let $x_1, x_2 \in S$, let $x \in]x_1, x_2[$, and let $\lambda \in (\Phi_v(x_1) + \mathrm{int}\mathbb{R}_+) \cap (\Phi_v(x_2) + \mathbb{R}_+)$. Then there exist $\alpha_1 \in \Phi_v(x_1)$ and $\alpha_2 \in \Phi_v(x_2)$ such that $\alpha_1 < \lambda$ and $\alpha_2 \le \lambda$.

We have $v + \lambda e = v + \alpha_1 e + (\lambda - \alpha_1)e \in F(x_1) + C + \operatorname{cor} C = F(x_1) + \operatorname{cor} C$ and $v + \lambda e = v + \alpha_2 e + (\lambda - \alpha_2)e \in F(x_2) + C + C = F(x_2) + C$. By 1° it follows that $v + \lambda e \in F(x) + \operatorname{cor} C$. Due to (2.2), we can find $\tau > 0$ such that $v + \lambda e \in \tau e + F(x) + C$, which yields $\lambda \in \Phi_v(x) + \tau \subset \Phi_v(x) + \operatorname{int}\mathbb{R}_+$. Hence $(\Phi_v(x_1) + \operatorname{int}\mathbb{R}_+) \cap (\Phi_v(x_2) + \mathbb{R}_+) \subset \Phi_v(x) + \operatorname{int}\mathbb{R}_+$. Thus 2° holds.

Finally, assume that 2° holds and suppose to the contrary that 1° is not true. Then there exist some points x_1^0, $x_2^0 \in S$, $x^0 \in]x_1^0, x_2^0[$, and $y^0 \in \big(F(x_1^0) + \operatorname{cor} C\big) \cap \big(F(x_2^0) + C\big) \setminus (F(x^0) + \operatorname{cor} C)$. By (2.2) we can find some $\tau_0 > 0$ such that $y_0 \in \tau_0 e + F(x_1^0) + C$, which yields $0 \in \Phi_{y^0}(x_1^0) + \operatorname{int}\mathbb{R}_+$. Since $y^0 \in F(x_2^0) + C$, we also have $0 \in \Phi_{y^0}(x_2^0) \subset \Phi_{y^0}(x_2^0) + \mathbb{R}_+$. Hence $0 \in \big(\Phi_{y^0}(x_1^0) + \operatorname{int}\mathbb{R}_+\big) \cap \big(\Phi_{y^0}(x_2^0) + \mathbb{R}_+\big)$. Taking into account that Φ_{y^0} is explicitly $\mathbb{R}_+$-quasiconvex (by 2°), we infer that $0 \in \Phi_{y^0}(x^0) + \operatorname{int}\mathbb{R}_+$. Thus there exists $\beta > 0$ such that $-\beta \in \Phi_{y^0}(x^0)$, i.e., $y^0 - \beta e \in F(x^0) + C$. It follows that $y^0 \in F(x^0) + C + \beta e \subset F(x^0) + C + \mathbb{R}_+^* \cdot \operatorname{cor} C = F(x^0) + \operatorname{cor} C$, which contradicts the fact that $y^0 \notin F(x^0) + \operatorname{cor} C$. □

Corollary 3.8. *Assume that $C \neq Y$ is vectorially closed and consider an arbitrary point $e \in \operatorname{cor} C$. For any vector-valued function $f : S \to Y$ the following assertions are equivalent*:

1° *f is explicitly C-quasiconvex.*

2° *The composite function $h_{e,v} \circ f : S \to \mathbb{R}$ is explicitly quasiconvex, for every $v \in Y$.*

Proof. The desired equivalence directly follows from Theorem 3.7, applied to the single-valued map defined as $F(x) := \{f(x)\}$ for all $x \in S$. Indeed, since C is vectorially closed, the set $F(x) + C = f(x) + C$ is nonempty and vectorially closed, for all $x \in S$. Moreover, for all $v \in Y$ and $x \in S$, we have $\varphi_v(x) := \inf \Phi_v(x) = \inf\{\alpha \in \mathbb{R} \mid v + \alpha e \in f(x) + C\}$. In view of (2.3), this means that $\varphi_v(x) = h_{e,v}(f(x))$ for all $x \in S$, i.e., $\varphi_v = h_{e,v} \circ f$. □

Remark 3.9. The well-known characterization of cone-quasiconvex vector-valued functions obtained by Dinh The Luc in [9] (Proposition 1.6.3), as well as the characterization of cone-quasiconvex set-valued maps established by Benoist and Popovici in [4] (Theorem 3.2) under the framework of a real topological vector space, partially ordered by a closed convex cone with nonempty interior, can be extended in the general setting of our paper. Moreover, by replacing the linear segments by continuous arcs, as in [8], further generalizations of these characterization theorems may be obtained for an appropriate definition of explicitly cone-quasiconvexity.

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