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ON IC-COLORINGS FOR COMPLETE PARTITE GRAPHS

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Abstract. Some systematic techniques for proving IC-colorings and finding IC-indices of complete d-partite graphs are established. A complete solution of the maximal colorings for $K_{1,1,n}$ is obtained.

1. Introduction

For a given graph G with the vertex set $V(G)$, a coloring $f : V(G) \to \mathbb{N}$ can *produce* α if $\alpha = \sum_{u \in V(H)} f(u)$ for some connected subgraph H of G ($\sum_{u \in V(H)} f(u) = 0$ if $V(H) = \varnothing$). The coloring f is an IC-*coloring* of G if f can produce each $\alpha \in \{0, 1, \cdots, S(f)\}$, where $S(f)$ is the maximum number that can be produced by f. The IC-*index* $M(G)$ of the graph G is the number $\max\{S(g) \,|\, g \text{ is an IC-coloring of } G\}$. A coloring f of G is *maximal* if it is an IC-coloring of G such that $S(f) = M(G)$. The problem of finding IC-indices and IC-colorings of finite graphs was introduced by Salehi et al. in 2005 [7], and it can be considered as a derived problem of the postage stamp problem in number theory, which has been extensively studied [1, 2, 3, 4, 5]. A graph G is *complete d-partite*, $1 < d < \infty$, if $V(G)$ can be partitioned into d disjoint partite sets such that two distinct vertices of G are adjacent if and only if they are in different partite sets. The IC-index of a complete bipartite graph (i.e., $d = 2$) was obtained by Shiue and Fu [8]. In this paper, we shall study the theory of IC-colorings for complete d-partite graphs by means of partite sets and establish some techniques which may be used to prove IC-colorings and find maximal colorings.

2. Definitions and notations

A *complete d-partite graph* G is a family $\{V_r\}_{r=1}^d$ of disjoint finite sets, where $1 < d < \infty$. The class of all complete d-partite graphs $G = \{V_r\}_{r=1}^d$ with $|V_r| = \alpha_r$ $(1 \le r \le d)$ is denoted by $K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$. We shall assume $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_d$. Let $G = \{V_r\}_{r=1}^d \in K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$ be given. We call $V(G) := \bigcup_{r=1}^d V_r$ the *vertex set* of G and each V_r a *partite set* of G. We write $u \sim v$ if $u, v \in V_r$ for some $1 \le r \le d$, that is, u and v are *in the same partite set*. A subset H of $V(G)$ is an *IC-subgraph* (*induced connected subgraph*) of G if $H \not\subset V_r$ for all $1 \le r \le d$ whenever $|H| > 1$. The collection of all IC-subgraphs of G is denoted by $\mathcal{B}_G$. The graph G is *connected* if $V(G) \in \mathcal{B}_G$. For each coloring $f : V(G) \to \mathbb{N}$ of G there is a corresponding *sum operator* $S_f : \mathcal{B}_G \to \mathbb{Z}$ which is given by

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$S_f(H) := \sum_{u\in H} f(u)$ ($S_f(H) := 0$ if $H = \varnothing$). We call $S_f(H)$ the *sum* of f on H and $S(f) := \max S_f(\mathcal{B}_G)$ the *IC-sum* of f, where $S_f(\mathcal{B}_G) := \{S_f(H) \,|\, H \in \mathcal{B}_G\}$ is the range of S_f. We see that $S(f) = S_f(G)$ if G is connected. We say that $\alpha \in \mathbb{Z}$ can be *produced* by f if $\alpha \in S_f(\mathcal{B}_G)$ and that f is an *IC-coloring* of G if $S_f(\mathcal{B}_G) = \{\alpha \in \mathbb{Z} \,|\, 0 \le \alpha \le S(f)\}$, that is, f can produce each of the integers $0, 1, \cdots, S(f)$. The *IC-index* of the class $K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$ is $M(K_{\alpha_1,\alpha_2,\cdots,\alpha_d}) := \max\{S(g) \,|\, g$ is an IC-coloring of some member in $K_{\alpha_1,\alpha_2,\cdots,\alpha_d}\}$ and any maximizer is a *maximal coloring* (or a *maximal IC-coloring*) for $K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$.When $f : V(G) \to \mathbb{N}$ is one-to-one, that is, $f(u) \ne f(v)$ whenever $u \ne v$, $V(G)$ and $f(V(G))$ can be put in a one-to-one correspondence, and we may identify u with $f(u)$ for all $u \in f(G)$. We shall use the notation $f = \langle V_1, V_2, \cdots, V_d\rangle$ to denote the one-to-one coloring of the graph $G = \{V_r\}_{r=1}^d$ in which f is the identity function on the vertex set $V(G)$ (thus G is a family of disjoint subsets of $\mathbb{N}$ in this case) and we shall write $f = \langle V_1, V_2, \cdots, V_d\rangle \simeq K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$ if $G = \{V_r\}_{r=1}^d \in K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$. Two one-to-one colorings $f_i : V(G_i) \to \mathbb{N}$, where $G_i = \{V_r^{(i)}\}_{r=1}^d \in K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$ for $i = 1, 2$, are *IC-equivalent* if $\{f_1(V_r^{(1)}) \,|\, 1 \le r \le d\} = \{f_2(V_r^{(2)}) \,|\, 1 \le r \le d\}$. We see that $h := f_2^{-1} \circ f_1$ is a one-to-one correspondence from $V(G_1)$ to $V(G_2)$ satisfying $f_1(u) = f_2(h(u))$ for all $u \in V(G_1)$ and $\{h(V_r^{(1)}) \,|\, 1 \le r \le d\} = \{V_r^{(2)} \,|\, 1 \le r \le d\}$, so that $h(H) \in \mathcal{B}_{G_2}$ if and only if $H \in \mathcal{B}_{G_1}$ for all $H \subset V(G_1)$, it follows that f_1 can produce some $\alpha \in \mathbb{Z}$ if and only if f_2 can, thus f_1 and f_2 share the same property of being an IC-coloring or being a maximal coloring. We also see that every one-to-one coloring of some complete d-partite graph is IC-equivalent to a coloring of the form $f = \langle V_1, V_2, \cdots, V_d\rangle$. For a given one-to-one coloring $f = \langle V_1, V_2, \cdots, V_d\rangle$ with $V(G) \ne \varnothing$, we shall write $V(G) = \{x_1, x_2, \cdots, x_k\}$, $s_i = x_1+x_2+\cdots+x_i$ $(1 \le i \le k)$, where $x_1 < x_2 < \cdots < x_k$ and $k = |V(G)|$, and define $s_0 = 0$, $x_{k+1} = s_{k+1} = \infty$ and $f^+ = \{x_i \,|\, x_i = s_{i-1} + 1\}$. If $G = \{V_r\}_{r=1}^d \in K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$ and $u \ne v$ where u, $v \in V(G)$, then a subset H of $V(G)$ is *uv-preconnected* provided $H \cap \{u, v\} = \varnothing$, $H \cup \{u\} \in \mathcal{B}_G$ and $H \cup \{v\} \in \mathcal{B}_G$ and the collection of all uv-preconnected subsets of $V(G)$ is denoted by $\mathcal{B}_G(u, v)$. For a coloring $f : V(G) \to \mathbb{N}$, if $f(u) = f(v)$ for some $u \ne v$ then $S_f(H \cup \{u\}) = S_f(H \cup \{v\})$ for all $H \in \mathcal{B}_G(u, v)$ so that $|S_f(\mathcal{B}_G)| \le |\mathcal{B}_G| - |\mathcal{B}_G(u, v)|$, it follows that if $|S_f(\mathcal{B}_G)| > |\mathcal{B}_G| - \min\limits_{u\ne v} |\mathcal{B}_G(u, v)|$ then f is one-to-one, here $|V(G)| \ge 2$.

3. Complete d-partite graphs and IC-colorings

Proposition 3.1. *Let $G = \{V_r\}_{r=1}^d \in K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$. Then:*

(a) $|\mathcal{B}_G| = 2^{\sum_{r=1}^d \alpha_r} - \sum_{r=1}^d 2^{\alpha_r} + \sum_{r=1}^d \alpha_r + d$.

(b) $|\mathcal{B}_G(u, v)| = 2^{\left(\sum_{r=1}^d \alpha_r\right)-2} - 2^{\alpha_s-2} + 1$ *if $\{u, v\} \subset V_s$ and $u \ne v$ for some $1 \le s \le d$.*

(c) $|\mathcal{B}_G(u, v)| = 2^{\left(\sum_{r=1}^d \alpha_r\right)-2} - 2^{\alpha_s-1} - 2^{\alpha_t-1} + 2$ *if $u \in V_s$ and $v \in V_t$ for some $1 \le s < t \le d$.*

(d) $\min\limits_{u\ne v} |\mathcal{B}_G(u, v)| = 2^{\left(\sum_{r=1}^d \alpha_r\right)-2} - 2^{\alpha_{d-1}-1} - 2^{\alpha_d-1} + 2$ *if $\alpha_{d-1} > 0$.*

Proof. We denote by 2^T the collection of all subsets of a given set T. We have $\mathcal{B}_G = 2^{V(G)} \setminus \bigcup_{r=1}^{d}(2^{V_r} \setminus (\{\varnothing\} \cup (\bigcup_{u \in V_r}\{\{u\}\})))$, so that $|\mathcal{B}_G| = 2^{|V(G)|} - \sum_{r=1}^{d}(2^{|V_r|} - (1 + |V_r|))$ and (a) follows. (b) follows from $\mathcal{B}_G(u,v) = 2^{V(G)\setminus\{u,v\}} \setminus (2^{V_s \setminus \{u,v\}} \setminus \{\varnothing\})$ if $\{u,v\} \subset V_s$ and $u \neq v$. (c) follows from $\mathcal{B}_G(u,v) = 2^{V(G)\setminus\{u,v\}} \setminus ((2^{V_s \setminus \{u\}} \setminus \{\varnothing\}) \cup (2^{V_t \setminus \{v\}} \setminus \{\varnothing\}))$ if $u \in V_s$ and $v \in V_t$ with $s \neq t$. To see (d), we recall that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d$. If the right side of the equality in (d) is subtracted from that in (b), we obtain $2^{\alpha_d - 1} - 2^{\alpha_s - 2} + 2^{\alpha_{d-1} - 1} - 1 > 0$, and the right side of the equality in (c) minus that in (d) is $2^{\alpha_{d-1}-1} - 2^{\alpha_s - 1} + 2^{\alpha_d - 1} - 2^{\alpha_t - 1} \geq 0$. These two inequalities imply (d). □

Proposition 3.2. *Let $G = \{V_r\}_{r=1}^{d} \in K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$, $f : V(G) \to \mathbb{N}$, $V(G) = \{u_1, u_2, \cdots, u_k\}$, $x_i = f(u_i)$ $(1 \leq i \leq k)$ such that $x_1 \leq x_2 \leq \cdots \leq x_k$ and let $s_0 = 0$, $s_i = x_1 + x_2 + \cdots + x_i$ $(1 \leq i \leq k)$ and $s_{k+1} = \infty$. Then:*

(a) *If $s_{j-1} < \alpha < x_{j+1}$ $(\alpha \in \mathbb{N})$ for some $1 \leq j \leq k$, then u_j must be used in producing α in the sense that if $\alpha = \sum_{u \in H} f(u)$ then $u_j \in H$.*
(b) *Let α, p, $q \in \mathbb{Z}$ be such that $1 \leq p < q \leq k$. If $x_i \leq s_{i-1} + \alpha$ for all $p < i \leq q$ then $s_q + \alpha \leq 2^{q-p}(s_p + \alpha)$.*

Proof. If $s_{j-1} < \sum_{u \in H} f(u) < x_{j+1}$ then $H \not\subset \{u_1, u_2, \cdots, u_{j-1}\}$ and $H \subset \{u_1, u_2, \cdots, u_j\}$. This proves (a). If $x_i \leq s_{i-1} + \alpha$ then $s_i + \alpha = s_{i-1} + x_i + \alpha \leq 2(s_{i-1} + \alpha)$, thus $s_q + \alpha \leq 2(s_{q-p} + \alpha) \leq \cdots \leq 2^{q-1}(s_p + \alpha)$ and (b) follows. □

Proposition 3.3. *Let $f = \langle V_1, V_2, \cdots, V_d \rangle \simeq K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$ be an IC-coloring and $\alpha_{d-1} \neq 0$. Then:*

(a) *$x_i \leq s_{i-1} + 1$ for all $1 \leq i \leq k$.*
(b) *If $x_i \in f^+$ and $s_{i-1} < x_i + x_j < x_{j+1}$ for some $1 \leq i < j \leq k$ then $\{x_i, x_j\} \in \mathcal{B}_G$ and $S_f(H) = x_i + x_j$ only if $H = \{x_i, x_j\}$.*

Proof. As $\alpha_{d-1} \neq 0$, $G = \{V_r\}_{r=1}^{d}$ is connected. As f is an IC-coloring of G and G is connected, f can produce 0, 1, $\cdots$, $s_k = S(f)$. If $s_{j-1} + 1 < x_j$ for some $1 \leq j \leq k$, then $s_{j-1} < s_{j-1} + 1 < x_{j+1}$, so that, by Proposition 3.2 (a) with $\alpha = s_{j-1} + 1$, x_j should be used in producing $s_{j-1} + 1$, which contradicts $s_{j-1} + 1 < x_j$. This proves (a). To see (b), let $x_i + x_j = S_f(H)$. By Proposition 2(a) with $\alpha = x_i + x_j$, we obtain $x_j \in H$. As $x_i \in f^+$, we have $s_{i-1} < s_{i-1} + 1 = x_i < x_{i+1}$, so that, by Proposition 3.2 (a) again, x_i must be used in producing x_i. Hence $H = \{x_i, x_j\} \in \mathcal{B}_G$. □

Proposition 3.4. *Let α, x_1, x_2, $\cdots$, $x_k \in \mathbb{N}$, $s_0 = 0$, $s_i = x_1 + x_2 + \cdots + x_i (1 \leq i \leq k)$ and $\alpha \leq s_k$. Assume that $x_i \leq s_{i-1} + 1$ for all $1 \leq i \leq k$, then $\alpha = x_{j_1} + x_{j_2} + \cdots + x_{j_p}$ for some $1 \leq j_p < \cdots < j_2 < i_1 \leq k$.*

Proof. As $0 = s_0 < s_1 < \cdots < s_k$ and $0 < \alpha \leq s_k$, there is a unique $1 \leq j_1 \leq k$ such that $s_{j_1 - 1} < \alpha \leq s_{j_1}$, so that $s_{j_1-1} - x_{j_1} < \alpha - x_{j_1} \leq s_{j_1} - x_{j_1}$, that is, $(s_{j_1-1} + 1) - x_{j_1} \leq \alpha - x_{j_1} \leq s_{j_1-1}$. By assumption with $i = j_1$, we have $0 \leq \alpha - x_{j_1} \leq s_{j_1} - 1$. If $\alpha - x_{j_1} \neq 0$ then we have $0 = s_0 < s_1 < \cdots < s_{j_1-1}$ and $0 < \alpha - x_{j_1} \leq s_{j_1-1}$. A similar argument shows that $0 \leq \alpha - x_{j_1} - x_{j_2} \leq s_{j_2-1}$ for some unique $1 \leq j_2 \leq j_1 - 1$. Continuing in this manner if necessary, we will eventually obtain $\alpha - x_{j_1} - x_{j_2} - \cdots - x_{j_p} = 0$ for some $1 \leq j_p < \cdots < j_2 < j_1 \leq k$. □

The following statements (I1)-(I7) will be used frequently in finding maximal colorings and they hold for any IC-coloring $f = \langle V_1, V_2, \cdots, V_d\rangle \simeq K_{\alpha_1,\alpha_2,\cdots,\alpha_d}$ with $\alpha_{d-1} \neq 0$.

(I1) If $b_i \leq s_i$ ($b_i \in \mathbb{R}$) for some $1 \leq i \leq k$ then $b_i - s_{i-1} \leq x_i \leq s_{i-1} + 1$.
(I2) If $1 \leq p < q \leq k$ ($p, q \in \mathbb{N}$) then $s_q < 2^{q-p}(s_p + 1)$.
(I3) If $1 \leq p < q \leq k$ ($p, q \in \mathbb{N}$) and $x_i \notin f^+$ for all $p < i \leq q$ then $s_q \leq 2^{p-q} s_p$.
(I4) If $x_i \in f^+$ and $s_{j-1} < x_i + x_j < x_{j+1}$ for some $1 \leq i < j \leq k$ then $\{x_i, x_j\} \in \mathcal{B}_G$.
(I5) If $x_i \in f^+$ and $x_i \sim x_j$ for some $1 \leq i < j \leq k$ then $x_j \leq s_{j-1} - x_i$ or $x_{j+1} \leq x_i + x_j$.
(I6) If $s_j \leq s_{j-1} - x_i$ for some $i, j \in \{1, 2, \cdots, k\}$ then $s_j \leq 2s_{j-1} - x_i$.
(I7) If $x_{j+1} \leq x_i + x_j$ for some $i, j \in \{1, 2, \cdots, k\}$ then $s_{j+1} \leq 3s_{j-1} + 2 + x_i$.

(I1) follows from $x_i = s_i - s_{i-1}$ and Proposition 3.3(a). (I2) and (I3) follow from (I1) and Proposition 3.2(b). (I4) follows from Proposition 3.3(b). (I5) is a contrapositive of (I4). (I6) is an immediate consequence of $s_j = s_{j-1} + x_j$. Finally, the inequalities $x_j \leq s_{j-1} + 1$ and $x_{j+1} \leq x_i + x_j$ imply $s_{j+1} = s_{j-1} + x_j + x_{j+1} \leq s_{j-1} + 2x_j + x_i \leq 3s_{j-1} + 2 + x_i$ and (I7) follows.

4. Maximal colorings for $K_{1,1,n}$

The aim of this section is to illustrate some techniques in proving IC-colorings and finding maximal colorings by exploring the class $K_{1,1,n}$. Numbers in f^+ are usually printed in boldface.

Theorem 4.1. *Up to IC-equivalence, we have:*

(a) *The maximal coloring for* $K_{1,1,1}$ *is* $\langle\{\mathbf{1}\}, \{\mathbf{2}\}, \{\mathbf{4}\}\rangle$.
(b) *The maximal colorings for* $K_{1,1,2}$ *are*
  (1) $\langle\{3\}, \{\mathbf{7}\}, \{\mathbf{1}, \mathbf{2}\}\rangle$,
  (2) $\langle\{\mathbf{2}\}, \{\mathbf{4}\}, \{\mathbf{1}, 6\}\rangle$,
  (3) $\langle\{\mathbf{1}\}, \{6\}, \{\mathbf{2}, \mathbf{4}\}\rangle$,
  (4) $\langle\{\mathbf{1}\}, \{\mathbf{2}\}, \{3, \mathbf{7}\}\rangle$.
(c) *The maximal colorings for* $K_{1,1,n}$, $n \geq 3$, *are*
  (1) $\langle\{\mathbf{2}\}, \{\mathbf{4}\}, \{\mathbf{1}, 6, \cdots, 3 \cdot 2^{n-1}\}\rangle$,
  (2) $\langle\{\mathbf{1}\}, \{\mathbf{2}\}, \{3, 6, \cdots, 3 \cdot 2^{n-2}, \mathbf{3 \cdot 2^{n-1} + 1}\}\rangle$.
  Consequently, the IC-index of $K_{1,1,n}$ *is* $M(K_{1,1,n}) = 3 \cdot 2^n + 1$.

Theorem 4.1 will follow from the following Lemmas 4.2–4.9.

Lemma 4.2. *The colorings in Theorem 4.1 are IC-colorings.*

Proof. We shall see that each $f = \langle V_1, V_2, V_3\rangle$ in Theorem 4.1 satisfies $x_i \leq s_{i-1} + 1$ for all $1 \leq i \leq n + 2$. Since $G = \{V_1, V_2, V_3\} \in K_{1,1,n}$ is connected, $S(f) = s_{n+2}$. From Proposition 3.4 with $k = n + 2$, it follows that if $\alpha \in \mathbb{N}$ and $\alpha \leq S(f)$ then $\alpha = \sum_{x\in H} x$ for some $H \subset V(G)$. Thus α can be produced by f if $H \in \mathcal{B}_G$. When $H \notin \mathcal{B}_G$, by modifying H, we shall obtain some $K \in \mathcal{B}_G$ such that $\alpha = S_f(K)$.

(a) The coloring $f = \langle\{\mathbf{1}\}, \{\mathbf{2}\}, \{\mathbf{4}\}\rangle \simeq K_{1,1,1}$ satisfies $x_i = s_{i-1} + 1$ for $i \in \{1, 2, 3\}$ and every subset H of $V(G)$ is in $\mathcal{B}_G$. Hence f is an IC-coloring.

(*b*) (1) $f = \langle\{3\}, \{\mathbf{7}\}, \{\mathbf{1}, \mathbf{2}\}\rangle$ satisfies $x_i = s_{i-1} + 1$ for $i \in \{1, 2, 4\}$ and $x_i = s_{i-1}$ for $i = 3$. The only subset of $V(G)$ not an IC-subgraph is $\{x_1, x_2\}$ and $x_1 + x_2 = x_3 = S_f(\{x_3\})$. Hence f is an IC-coloring.

(2) $f = \langle\{\mathbf{2}\}, \{\mathbf{4}\}, \{\mathbf{1}, 6\}\rangle$ satisfies $x_i = s_{i-1} + 1$ for $i \in \{1, 2, 3\}$ and $x_i = s_{i-1} - 1$ for $i = 4$. We see that if $H \subset V(G)$ and $H \notin \mathcal{B}_G$ then $H = \{x_1, x_4\}$. As $x_1 + x_4 = S_f(\{x_1, x_2, x_3\})$, f is an IC-coloring.

(3) $f = \langle\{\mathbf{1}\}, \{6\}, \{\mathbf{2}, \mathbf{4}\}\rangle$ satisfies $x_i \le s_{i-1} + 1$ for all $1 \le i \le 4$ as in the previous one. As $x_2 + x_3 = S_f(\{x_4\})$, f is an IC-coloring.

(4) $f = \langle\{\mathbf{1}\}, \{\mathbf{2}\}, \{3, \mathbf{7}\}\rangle$ is an IC-coloring because $x_i \le s_{i-1} + 1$ for all $1 \le i \le 4$ and $x_3 + x_4 = S_f(\{x_1, x_2, x_4\})$.

(*c*) (1) $f = \langle V_1, V_2, V_3\rangle \simeq K_{1,1,n}$ $(n \ge 3)$ is the coloring satisfying $V_1 = \{x_2\}$, $V_2 = \{x_3\}$, $V_3 = \{x_1\} \cup \{x_4, x_5, \cdots, x_{n+2}\}$, $x_i = s_{i-1}+1$ for $i \in \{1, 2, 3\}$, and $x_i = s_{i-1} - 1$ for $i \in \{4, 5, \cdots, n + 2\}$. Let $H \subset V(G)$, $H \notin \mathcal{B}_G$. Then $|H| > 1$ and $H \subset V_3$. If $x_j = \min(V_3 \backslash \{x_1\})$ then $j \ge 4$ and $x_j = s_{j-1} - 1 = (x_1 + x_2 + \cdots + x_{j-1}) - x_1$, so that $\sum_{x\in H} x = S_f(K)$, where $K = (H\backslash\{x_j\}) \cup \{x_2, x_3, \cdots, x_{j-1}\}$. Hence f is an IC-coloring.

(2) We have $V_1 = \{x_1\}$, $V_2 = \{x_2\}$, $V_3 = \{x_3, x_4, \cdots, x_{n+2}\}$, $x_i = s_{i-1} + 1$ for $i \in \{1, 2, n + 2\}$, and $x_i = s_{i-1}$ for $i \in \{3, 4, \cdots, n + 1\}$. If $|H| > 1$, $H \subset V_3$ and $x_j = \min(H\backslash\{x_{n+2}\})$, then $3 \le j \le n + 1$ and $x_j = s_{j-1} = x_1 + x_2 + \cdots + x_{j-1}$, so that $\sum_{x\in H} x = S_f(K)$, where $K = (H\backslash\{x_j\}) \cup \{x_1, x_2, \cdots, x_{j-1}\}$. Hence $f = \langle V_1, V_2, V_3\rangle \simeq K_{1,1,n}$ $(n \ge 3)$ is an IC-coloring.

□

Lemma 4.3. *Maximal colorings for $K_{1,1,n}$ are one-to-one.*

Proof. By Lemma 4.2, we have $M(K_{1,1,n}) \ge 3\cdot 2^n + 1$. If $G \in K_{1,1,n}$ and $f : V(G) \to \mathbb{N}$ is maximal for $K_{1,1,n}$ then $|S_f(\mathcal{B}_G)| = |\{0, 1, 2, \cdots, M(K_{1,1,n})\}| = M(K_{1,1,n}) + 1$, and, by Proposition 3.1, $|\mathcal{B}_G| - \min_{u\neq v} |\mathcal{B}_G(u, v)| = 3 \cdot 2^n - 2^{n-1} + n$. It follows that $|S_f(\mathcal{B}_G)| - (|\mathcal{B}_G| - \min_{u\neq v} |\mathcal{B}_G(u, v)|) \ge 2^{n-1} - n + 2 > 0$. Hence f is one-to-one as mentioned at the end of Section 2. □

For convenience sake, we define the *$b(1, 1, n)$-sequence* b_1, b_2, $\cdots$, b_{n+3} to be $b_1 = 1$, $b_i = 3 \cdot 2^{i-2}$ for $2 \le i \le n + 2$, and $b_{n+3} = \infty$. In the following, b_1, b_2, $\cdots$, b_{n+3} will be the $b(1, 1, n)$-sequence when we deal with a given coloring $f = \langle V_1, V_2, V_3\rangle \simeq K_{1,1,n}$ which is maximal.

Lemma 4.4. *Let $f = \langle V_1, V_2, V_3\rangle \simeq K_{1,1,n}$ be maximal. Then:*

(*a*) $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, 3)$ *or* $(\mathbf{1}, \mathbf{2}, \mathbf{4})$.

(*b*) $b_i \le s_i$ *for all* $1 \le i \le n + 3$.

(*c*) $|f^+| \ge 3$.

Proof. As $x_1 < x_2 < x_3$ and $x_i \le s_{i-1} + 1$ $(1 \le i \le 3)$, (*a*) follows. As $s_{n+2} = M(K_{1,1,n}) \ge 3 \cdot 2^n + 1$ and $s_{n+3} = \infty$, we see that $b_i \le s_i$ for $i \in \{1, 2, 3\} \cup \{n + 2, n + 3\}$. If $s_p < b_p$ for some $4 \le p < n + 2$, by (I2) with $q = n + 2$, we would have $S_{n+2} \le 2^{n+2-p} b_p = 3 \cdot 2^n < s_{n+2}$. Hence (*b*) is true. Similarly, if $|f^+| < 3$ then $f^+ = \{\mathbf{1}, \mathbf{2}\}$, by (I3) with $(p, q) = (2, n + 2)$, we would have $s_{n+2} \le 3 \cdot 2^n < s_{n+2}$. This prove (*c*). □

Lemma 4.5. *Let $f = \langle V_1, V_2, V_3\rangle \simeq K_{1,1,n}$ ($n \geq 2$) be maximal. Then:*

(*a*) *If $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, 3)$ then $6 \leq x_4 \leq 7$ and $\{x_1, x_3\}$, $\{x_2, x_3\} \in \mathcal{B}_G$.*

(*b*) *If $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, \mathbf{4})$ then $6 \leq x_4 \leq 8$ and $\{x_1, x_2\}$, $\{x_1, x_3\} \in \mathcal{B}_G$.*

Proof. Let $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, 3)$. By (I1) with $i = 4$, we obtain $6 \leq x_4 \leq 7$. By (I4) with $i \in \{1, 2\}$ and $j = 3$, we have $\{x_i, x_3\} \in \mathcal{B}_G$ ($1 \leq i \leq 2$). This prove (a). To see (b), let $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, \mathbf{4})$. By (I4) with $(i, j) = (1, 2)$, we see that $\{x_1, x_2\} \in \mathcal{B}_G$. We prove $\{x_1, x_3\} \in \mathcal{B}_G$ by a contradiction. If $x_1 \sim x_3$, by (I5) with $(i, j) = (1, 3)$, we would have $(x_1, x_2, x_3, x_4) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, 5)$, and, by (I1) with $i = 5$, we would have $x_5 \geq 12$, it follows from these and (I4) with $(i, j) = (3, 4)$ that $\{x_3, x_4\} \in \mathcal{B}_G$. Now, $x_1 \sim x_3$, $\{x_1, x_2\} \in \mathcal{B}_G$ and $\{x_3, x_4\} \in \mathcal{B}_G$ would imply $V_1 \cup V_2 = \{x_2, x_4\}$ and $\{x_1, x_3\} \subset V_3$, so that $x_5 \in V(G)$ ($s_4 < M(K_{1,1,2})$) and, by (I5)-(I7) with $(i, j) = (1, 5)$, $s_5 \leq 2s_4 - x_1 = 23 < b_5$ or $s_6 \leq 3s_4 + 2 + x_1 = 39 < b_6$, which contradicts Lemma 4.4 (b). Hence $\{x_1, x_3\} \in \mathcal{B}_G$ is proved. By (I1) with $i = 4$, we have $5 \leq x_4 \leq 8$. Let us check the case $x_4 = 5$. We have $(x_1, x_2, x_3, x_4) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, 5)$ and $x_5 \geq 12$ by (I1), so that $x_5 > s_4 - x_i$ for all $1 \leq i \leq 3$. If $x_i \sim x_5$ for some $1 \leq i \leq 3$, by (I5) and (I7) with $j = 5$, we would have $s_6 \leq 3s_4 + 2 + x_i \leq 42 < b_6$ (we have proved that $x_5 \leq s_4 - x_i$ is false). This contradiction proves that $\{x_i, x_5\} \in \mathcal{B}_G$ for all $1 \leq i \leq 3$. It follows from this and $\{x_1, x_2\}$, $\{x_1, x_3\} \in \mathcal{B}_G$ that $x_2 \sim x_3$ (we have only three partite sets), so that $\{x_1, x_5\} = V_1 \cup V_2$ and $\{x_2, x_3, x_4\} \subset V_3$. By $x_4 > s_3 - x_3$ and (I5) with $(i, j) = (3, 4)$, we have $x_5 \leq x_1 + x_4 = 9$, which would imply $s_5 \leq 21 < b_5$. Thus $x_4 = 5$ is impossible and we have $6 \leq x_4 \leq 8$. This completes the proof. □

Lemma 4.6. *Let $f = \langle V_1, V_2, V_3\rangle \simeq K_{1,1,n}$ ($n \geq 2$) be maximal. If $x_1 \in V_3$ and $x_2 \in V_3$ then, up to IC-equivalence, $f = \langle\{3\}, \{\mathbf{7}\}, \{\mathbf{1}, \mathbf{2}\}\rangle$.*

Proof. By (I5) with $(i, j) = (1, 2)$, we obtain $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, 3)$. By Lemma 4.5 ($a$), we have $6 \leq x_4 \leq 7$. Thus $x_5 = s_5 - s_4 \geq b_4 - s_4 \geq 24 - 13 = 11$ (note $x_5 = \infty$ if $n = 2$), so that, by (I4) with $i = 1$ and $j \in \{3, 4\}$, $\{x_1, x_3\} \in \mathcal{B}_G$ and $\{x_1, x_4\} \in \mathcal{B}_G$. This proves that $\{x_3, x_4\} = V_1 \cup V_2$ and $\{x_1, x_2\} \subset V_3$. If $x_4 = 6$ then $x_5 \in V(G)$ ($s_4 < M(K_{1,1,2})$) and, by (I5)-(I7) with $(i, j) = (3, 5)$, $s_5 \leq 2s_4 - x_1 = 11 < b_5$ or $s_6 \leq 3s_4 + 2 + x_1 = 39 < b_6$, which contradicts Lemma 4.4 (b). Hence $x_4 \neq 6$ and, by $6 \leq x_4 \leq 7$, we have $(x_1, x_2, x_3, x_4) = (\mathbf{1}, \mathbf{2}, 3, \mathbf{7})$. We claim that $n = 2$. If not, by (I5)-(I7) with $(i, j) = (2, 5)$, we would have $s_5 \leq 24 \leq b_5$ or $s_6 \leq 43 < b_6$, so that $s_5 = 24 < M(K_{1,1,3})$, and, by (I5)-(I8) with $(i, j) = (2, 6)$, we obtain the contradiction $s_6 \leq 46 < b_6$ or $s_7 \leq 76 < b_7$. □

Lemma 4.7. *Let $f = \langle V_1, V_2, V_3\rangle \simeq K_{1,1,n}$ ($n \geq 2$) be maximal. If $x_1 \notin V_3$ and $x_2 \in V_3$ then, up to IC-equivalence, $f = \langle\{\mathbf{1}\}, \{6\}, \{\mathbf{2}, \mathbf{4}\}\rangle$.*

Proof. That $x_1 \notin V_3$ and $x_2 \in V_3$ imply $\{x_1, x_2\} \in \mathcal{B}_G$. If $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, 3)$, by Lemma 4.5 ($a$), we would have $\{x_i, x_j\} \in \mathcal{B}_G$ for all $1 \leq i < j \leq 3$, and by (I5) and (I7) with $(i, j) = (2, 4)$, $s_5 \leq 3s_3 + 2 + x_2 = 22 < b_5$, which contradicts $b_5 \leq s_5$. Hence $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, \mathbf{4})$. We claim that $x_3 \in V_3$. Suppose on the contrary that $\{x_1, x_3\} = V_1 \cup V_2$ and $\{x_2, x_4\} \subset V_3$. By Lemma 4.5 (b), $6 \leq x_4 \leq 8$. If $6 \leq x_4 \leq 7$, then $x_4 > s_3 - x_2$, so that, by (I5) with $(i, j) = (2, 4)$, $x_5 \leq x_2 + x_4 \leq 9$, we would have $s_5 \leq 23 < b_5$. If $x_4 = 8$ then, by (I5) with $(i, j) = (2, 4)$, $x_5 \leq x_2 + x_4 = 10 < \infty$, so that, by (I5) with $(i, j) = (4, 5)$, $x_6 \leq x_4 + x_5 \leq 18$, we would have $s_6 \le 43 < b_6$. This proves that $x_3 \in V_3$ and $x_2 \sim x_3$. By (I5) with $(i,j) = (2,3)$, $x_4 \le x_2 + x_3 = 6$. From this and Lemma 4.5 (b), it follows that $(x_1, x_2, x_3, x_4) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, 6)$. Let us now check whether $\{x_2, x_4\} \in \mathcal{B}_G$ or not. If $x_2 \sim x_4$, by (I5) with $(i,j) = (2,4)$, $x_5 \le x_2 + x_4 \le 8$, and we would have $s_5 \le 21 < b_5 \le s_5$. This proves that $\{x_2, x_4\} \in \mathcal{B}_G$ and that $\{x_1, x_4\} = V_1 \cup V_2$ and $\{x_2, x_3\} \subset V_3$. Finally, if $x_5 \in V(G)$, by (I1) with $i = 5$, we would have $x_5 \ge 11$, and by (I5) and (I7) with $(i,j) = (3,5)$, we obtain the contradiction $s_6 \le 3s_4 + 2 + x_3 = 45 < b_6$. Hence $x_5 \notin V(G)$. □

Lemma 4.8. *Let* $f = \langle V_1, V_2, V_3 \rangle \simeq K_{1,1,n}$ *($n \ge 2$) be maximal. If* $x_1 \in V_3$ *and* $x_2 \notin V_3$ *then, up to IC-equivalence,* $f = \langle \{\mathbf{2}\}, \{\mathbf{4}\}, \{\mathbf{1}, 6, \cdots, 3 \cdot 2^{n-1}\} \rangle$.

Proof. Let $x_1 \in V_3$ and $x_2 \notin V_3$. If $x_3 = 3$, by Lemma 4.5 (a), we would have $\{x_2, x_3\} = V_1 \cup V_2$, $\{x_1, x_4\} \subset V_3$ and $x_4 \ge 6 > s_3 - x_1$, so that, by (I5) and (I7) with $(i,j) = (1,4)$, we would have $s_5 \le 3s_3 + 2 + x_1 = 21 < b_5$, which contradicts $b_5 \le s_5$. Thus $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, \mathbf{4})$ and, by Lemma 4.5 (b), we have $\{x_2, x_3\} = V_1 \cup V_2$, $\{x_1, x_4\} \subset V_3$ and $6 \le x_4 \le 8$. If $x_4 \in \{7, \mathbf{8}\}$ then $x_4 > s_3 - x_1$, by (I5) and (I7) with $(i,j) = (1,4)$, we would have $s_5 \le 3s_3 + 2 + x_1 = 24 < M(K_{1,1,3})$, so that $x_6 \in V(G)$ and $b_5 \le s_5 \le 24 = b_5$. We have to discuss the following two cases for $s_5 = 24$.

Case 1 $(x_1, x_2, x_3, x_4, x_5) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, 7, 10)$. By (I4) with $(i,j) = (1,4)$, we would obtain $\{x_1, x_4\} \in \mathcal{B}_G$ which contradicts $\{x_1, x_4\} \subset V_3$.

Case 2 $(x_1, x_2, x_3, x_4, x_5) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, 9)$. By (I1) with $i = 6$, we would obtain $x_6 \ge 24$. It follows from this and (I4) with $(i,j) = (4,5)$ that $\{x_4, x_5\} \in \mathcal{B}_G$ which contradicts $\{x_4, x_5\} \subset V_3$.

Thus $(x_1, x_2, x_3, x_4) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, 6)$. Let $x_p = 3 \cdot 2^{p-3}$ and $s_p = 3 \cdot 2^{p-2} + 1$ for some $4 \le p < n + 2$. We claim that $x_{p+1} = 3 \cdot 2^{p-2}$ and $s_{p+1} = 3 \cdot 2^{p-1} + 1$. If $x_{p+1} < 3 \cdot 2^{p-2}$ then $s_{p+1} = s_p + x_{p+1} \le b_{p+1}$, so that $s_{p+1} = b_{p+1} < M(K_{1,1,p-1})$ (b_{p+1} is a lower bound for s_{p+1} by Lemma 4.4 (b)), thus $x_{p+2} \in V(G)$ and, by (I5)-(I7) with $(i,j) = (1, p+2)$, we would have $s_{p+2} \le 2s_{p+1} - x_1 < b_{p+2}$ or $s_{p+3} \le 3s_{p+1} + 2 + x_1 = 3b_{p+1} + 3 < 3b_{p+1} + b_{p+1} \le b_{p+3}$, which contradicts $b_i \le s_i$ for all $1 \le i \le n+3$. Similarly, if $x_{p+1} > 3 \cdot 2^{p-2}$ then $x_{p+1} > s_p - x_1$, so that, by (I5) and (I7) with $(i,j) = (1, p+1)$, we would have $s_{p+2} \le 3s_p + 2 + x_1 = 3b_p + 6 < 3b_p + b_p \le b_{p+2}$, a contradiction. Hence $x_{p+1} = 3 \cdot 2^{p-2}$ and our claim is proved. This completes the proof. □

Lemma 4.9. *Let* $f = \langle V_1, V_2, V_3 \rangle \simeq K_{1,1,n}$ *($n \ge 2$) be maximal. If* $x_1 \notin V_3$ *and* $x_2 \notin V_3$ *then, up to IC-equivalence,* $f = \langle \{\mathbf{1}\}, \{\mathbf{2}\}, \{3, 6, \cdots, 3 \cdot 2^{n-2}, \mathbf{3} \cdot \mathbf{2^{n-1}} + \mathbf{1}\} \rangle$.

Proof. We have $\{x_1, x_2\} = V_1 \cup V_2$ and $\{x_3, x_4, \cdots, x_{n+2}\} = V_3$. Suppose on the contrary that $(x_1, x_2, x_3) = (\mathbf{1}, \mathbf{2}, \mathbf{4})$. By Lemma 4.5 (b), we would have three cases to discuss.

Case 1 $x_4 = 6$. By (I5) with $(i,j) = (3,4)$, $x_5 \le x_3 + x_4 = 10$, we would obtain the contradiction $s_5 \le 23 < b_5$.

Case 2 $x_4 = 7$. By (I5) with $(i,j) = (3,4)$, $x_5 \le x_3 + x_4 = 11$. By (I1) with $i = 5$, $x_5 \ge b_5 - s_4 = 10$. Thus $10 \le x_5 \le 11$. If $(x_1, x_2, x_3, x_4, x_5) = (\mathbf{1}, \mathbf{2}, \mathbf{4}, 7, 10)$, by (I1) with $i = 6$ ($x_6 \in V(G)$ for $s_5 = 24 < M(K_{1,1,3})$), $x_6 \ge 24$, we would have $s_4 < 21 < x_6$. By Proposition 3.2 (a) with $(j, \alpha) =$

$(5,21)$, $x_5 = 10$ should be used in producing 21, so that $\{\mathbf{4},7,10\} \in \mathcal{B}_G$ for $S_f(H) = 21$ only if $H = \{4,7,10\}$, which contradicts $\{x_3,x_4,x_5\} \subset V_3$. Similarly, if $(x_1,x_2,x_3,x_4,x_5) = (\mathbf{1},\mathbf{2},\mathbf{4},7,11)$ then $x_6 = s_6 - s_5 \geq b_6 - s_5 \geq 48 - 25 = 23$ ($x_6 = s_6 = b_6 = \infty$ if $n = 3$), we would have$s_4 < 22 < x_6$, so that, by Proposition 3.2 (a) with $(j,\alpha) = (5,22)$, $\{\mathbf{4},7,11\} \in \mathcal{B}_G$, which contradicts $\{x_3,x_4,x_5\} \subset V_3$.

Case 3 $x_4 = 8$. Then $(x_1,x_2,x_3,x_4) = (\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{8})$. By (I5) with $(i,j) = (3,4)$, $x_5 \leq x_3 + x_4 = 12$. By (I1) with $i = 5$, $x_5 \geq b_5 - s_4 = 9$. Thus $9 \leq x_5 \leq 12$. From this and (I5) with $(i,j) = (4,5)$ it would follow that $x_6 \leq x_4 + x_5 \leq 20$, so that $s_6 \leq 47 < b_6$, a contradiction.

Thus, we have proved $(x_1,x_2,x_3) = (\mathbf{1},\mathbf{2},3)$. Let $f^+ = \{x_{i_1}, x_{i_2}, x_{i_3}, \cdots\}$, where $x_{i_1} < x_{i_2} < x_{i_3} < \cdots$. Then $x_{i_1} = \mathbf{1}$, $x_{i_2} = \mathbf{2}$, and $4 \leq i_3 \leq n+2$ by Lemma 4.4 (c). By (I1) with $3 \leq i < i_3$, we see that $b_i - s_i \leq x_i \leq s_{i-1}$ (each $x_i \notin f^+$), it follows that $(x_1,x_2,x_3,\cdots,x_{i_3-1},x_{i_3}) = (\mathbf{1},\mathbf{2},b_2,\cdots,b_{i_3-2},\boldsymbol{b_{i_3-1}+1})$ and $(s_1,s_2,s_3,\cdots,s_{i_3-1},s_{i_3}) = (b_1,b_2,b_3,\cdots,b_{i_3-1},b_{i_3}+1)$. To complete the proof, we must show that $i_3 = n+2$. Suppose, to the contrary, that $i_3 < n+2$, by (I1) with $i = i_3+1$, we would obtain $b_{i_3} - 1 \leq x_{i_3+1} \leq b_{i_3}+2$ and $b_{i_3+1} \leq s_{i_3+1} \leq b_{i_3+1}+3$. By (I5) and (I8) with $(i,j) = (i_3, i_3+1)$, we would have $s_{i_3+2} \leq 3s_{i_3}+2+x_{i_3} = 2b_{i_3+1} + 6 - b_{i_3-1} \leq 2b_{i_3+1} < M(K_{1,1,i_3})$, so that $x_{i_3+3} \in V(G)$ and $x_{i_3} \sim x_{i_3+3}$. It follows from $s_{i_3+1} \leq b_{i_3+1}+3$ and (I5)-(I7) with $(i,j) = (i_3, i_3+3)$ that $s_{i_3+2} \leq 2s_{i_3+1} - x_{i_3} \leq b_{i_3+2} + 5 - b_{i_3-1} < b_{i_3+2}$ or $s_{i_3+3} \leq 3s_{i_3+1} + 2 + x_{i_3} \leq b_{i_3+3} + 12 - b_{i_3} - b_{i_3-1} < b_{i_3+3}$, which contradicts Lemma 4.4 (b). Hence $i_3 = n+2$. □

Proof of Theorem 4.1. Lemma 4.3 shows that maximal colorings for $K_{1,1,n}$ are of the form $f = \langle V_1, V_2, V_3 \rangle$ up to IC-equivalence. The desired results now follow from Lemma 4.2, Lemma 4.4 (a), and Lemmas 4.6–4.9. □

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