Journal of Nonlinear and Convex Analysis
Volume 12, Number 1, 2011, 1–4

ISSN 1880-5221 ONLINE JOURNAL

A PROOF OF FRITZ JOHN'S ELLIPSOID THEOREM

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In Memory of Professor Ky Fan

Abstract. We present a simple proof of Fritz John's ellipsoid theorem using a projection theorem proved by the Hahn-Banach theorem.

Fritz John showed in 1948 that any symmetric convex body in $\mathbb{R}^n$ lies between two concentric homothetic ellipsoids of ratio $1/\sqrt{n}$, and the parameter $1/\sqrt{n}$ is optimal [5]. This basic result in convex geometry is of great impact on geometry of Banach spaces [6, 7], complexity of algorithms [2], and a classification of the existence of Liapunov quadratic form for switched linear system [1].

John's proof was based on Lagrange's multiplier rule where the subsidiary conditions are inequalities [5]. A proof based on differential equations is seen in [4]. Functional-theoretic proofs of the complex version of John's ellipsoid theorem may be found in [1, 6, 7]. Here we present a simple proof of this result based on a projection theorem proved by the Hahn-Banach theorem.

Let $\mathbb{C}^n$ be the n-dimensional Hilbert space equipped with the inner product

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k \overline{y_k}$$

and the associated norm

$$\|x\| = \langle x, x\rangle^{1/2}.$$

For a Hermitian matrix A, $A > 0$ denotes that A is positive definite.

We shall establish the following:

Theorem 1. *Let* $|||\cdot|||$ *be a norm on* $\mathbb{C}^n$ *and* $x_0 \in \mathbb{C}^n$ *such that* $|||x_0||| = 1$ *and* $\langle x_0, x_0\rangle = \min_{|||x|||=1} \langle x, x\rangle$. *Then*

$$|\langle x, x_0\rangle| \leq \langle x_0, x_0\rangle \text{ for all } |||x||| \leq 1.$$

Proof. By the Hahn-Banach theorem, there is a linear functional f on $\mathbb{C}^n$ such that

$$f(x_0) = |||x_0||| \text{ and } |f(x)| \leq |||x||| \text{ for all } x \in \mathbb{C}^n.$$

By the Riesz representation theorem, there is a $y \in \mathbb{C}^n$ such that

$$f(x) = \langle x, y\rangle \text{ for all } x \in \mathbb{C}^n.$$

2010 *Mathematics Subject Classification.* 15A60, 15A63.
Key words and phrases. Fritz John ellipsoid theorem, projection.
*Corresponding author. This work was supported by the National Science Council of the Republic of China.

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Since $x_0 \neq 0$, we have $y \neq 0$. Denote by W the subspace generated by x_0. According to the orthogonal decomposition

$$\mathbb{C}^n = W \oplus W^{\perp},$$

y can be written as $\alpha x_0 + z$ for some $\alpha \in \mathbb{C}$ and $z \in W^{\perp}$. Then

$$1 = |||x_0||| = f(x_0) = \bar{\alpha}\langle x_0, x_0\rangle,$$

so α is real and

$$\langle x_0, x_0\rangle = \frac{1}{\alpha}.$$

It follows from $\langle y, y\rangle = f(y) \leq |||y|||$ and $\langle x_0, x_0\rangle \leq \langle x, x\rangle$ for all $|||x||| = 1$ that

$$\frac{1}{\alpha} \leq \langle \frac{y}{|||y|||}, \frac{y}{|||y|||}\rangle \leq \frac{1}{|||y|||} \leq \frac{1}{\langle y, y\rangle} = \frac{1}{\alpha + \langle z, z\rangle} \leq \frac{1}{\alpha}.$$

From these we conclude that $z = 0$. Thus

$$y = \frac{x_0}{\langle x_0, x_0\rangle},$$

and the assertion follows from

$$\left|\frac{\langle x, x_0\rangle}{\langle x_0, x_0\rangle}\right| = |f(x)| \leq |||x||| \text{ for all } x \in \mathbb{C}^n.$$

□

Theorem 2 (Fritz John's Ellipsoid Theorem). *Let $|||\cdot|||$ be a norm on $\mathbb{C}^n$ and*

$$\mathscr{A} = \{A > 0\,;\ \langle Ax, x\rangle \leq |||x|||^2 \text{ for all } x \in \mathbb{C}^n\}.$$

Then there exists a unique $A_0 \in \mathscr{A}$ such that

$$\det A_0 = \max_{A \in \mathscr{A}} \det A.$$

Moreover,

$$\langle nA_0 x, x\rangle \geq |||x|||^2 \text{ for all } x \in \mathbb{C}^n.$$

Proof. For each $A \in \mathscr{A}$ and $\|x\| \leq 1$, we have

$$\|A^{1/2}x\| = \langle Ax, x\rangle^{1/2} \leq |||x|||,$$

so

$$\|A\| \leq \|A^{1/2}\|^2 \leq \max_{\|x\| \leq 1} |||x|||^2 \text{ for all } A \in \mathscr{A}.$$

Thus $\mathscr{A}$ is bounded. It follows that there exists an A_0 in the closure of $\mathscr{A}$ such that

$$\det A_0 = \sup_{A \in \mathscr{A}} \det A.$$

Since the right part of the above equation is positive, we conclude that A_0 is in $\mathscr{A}$. To prove the uniqueness assertion, we apply the following determinantal inequality [3]:
If A and B are $n \times n$ complex matrices with $A > 0$, $B > 0$ and $A \neq B$, then

$$\det((1-t)A + tB) > (\det A)^{1-t}(\det B)^t \text{ for all } t \in (0, 1).$$

If $B_0 \in \mathscr{A}$ is such that $\det B_0 = \max_{A \in \mathscr{A}} \det A$ and $B_0 \neq A_0$, then $\det \frac{1}{2}(A_0 + B_0) \in \mathscr{A}$ and

$$\det \frac{1}{2}(A_0 + B_0) > (\det A_0)^{1/2} (\det B_0)^{1/2} = \det A_0,$$

contradicting the maximality of $\det A_0$, proving the assertion. To prove the "moreover" part, we first consider the case of $A_0 = I$. Let $|||x_0||| = 1$ and

$$\langle x_0, x_0 \rangle = \min_{|||x|||=1} \langle x, x \rangle.$$

We have to prove that $\langle n x_0, x_0 \rangle \geq 1$. Suppose not, i.e., $\alpha = \langle x_0, x_0 \rangle < \frac{1}{n}$. Let

$$u_1 = \frac{x_0}{\langle x_0, x_0 \rangle^{1/2}}$$

and $u_2, \dots, u_n$ be such that $\{u_1, \dots, u_n\}$ forms an orthonormal basis of $\mathbb{C}^n$. Let

$$B = \operatorname{diag}\left(\frac{1}{n\alpha}, \frac{n-1}{n(1-\alpha)}, \dots, \frac{n-1}{n(1-\alpha)} \right),$$

and $B_0 = P^* B P$ where P is the matrix with row vectors $u_1^*, \dots, u_n^*$, i.e.,

$$P = \begin{pmatrix} u_1^* \\ \vdots \\ u_n^* \end{pmatrix}.$$

We now claim that $B_0 \in \mathscr{A}$, i.e., $\langle B_0 x, x \rangle \leq 1$ for all $|||x||| \leq 1$. Let $|||x||| \leq 1$. Using the basis $\{x_0, u_2, \dots, u_n\}$ of $\mathbb{C}^n$ we obtain

$$\begin{aligned} x &= c_1 x_0 + c_2 u_2 + \cdots + c_n u_n \\ &= c_1 \sqrt{\alpha} u_1 + c_2 u_2 + \cdots + c_n u_n \end{aligned}$$

for certain $c_i \in \mathbb{C}$, $i = 1, \dots, n$. According to Theorem 1, we deduce the inequality

$$\alpha |c_1| = |\langle x, x_0 \rangle| \leq \langle x_0, x_0 \rangle = \alpha,$$

so $|c_1| \leq 1$. It follows from $\langle x, x \rangle = \langle A_0 x, x \rangle \leq |||x|||^2$ that

$$|c_2|^2 + \cdots + |c_n|^2 \leq 1 - |c_1|^2 \alpha.$$

Consequently,

$$\begin{aligned} \langle B_0 x, x \rangle &= \langle B P x, P x \rangle \\ &= \frac{1}{n\alpha} |c_1|^2 \alpha + \frac{n-1}{n(1-\alpha)} (|c_2|^2 + \cdots + |c_n|^2) \\ &\leq \frac{1}{n} |c_1|^2 + \frac{n-1}{n(1-\alpha)} (1 - |c_1|^2 \alpha) \\ &= \left(\frac{1 - n\alpha}{n(1-\alpha)} \right) |c_1|^2 + \frac{n-1}{n(1-\alpha)} \\ &\leq \left(\frac{1 - n\alpha}{n(1-\alpha)} \right) + \frac{n-1}{n(1-\alpha)} = 1. \end{aligned}$$

This shows that $B_0 \in \mathscr{A}$. However, by the arithmetic-geometric means inequality, we have

$$\frac{1}{(\det B_0)^{1/n}} = \left[n\alpha \left(\frac{n(1-\alpha)}{n-1} \right)^{n-1} \right]^{1/n}$$
$$\leq \frac{1}{n} \left(n\alpha + \frac{n(1-\alpha)}{n-1} + \cdots + \frac{n(1-\alpha)}{n-1} \right) = 1.$$

Here the equality holds if and only if $\alpha = \frac{1}{n}$. Thus $\det B_0 > 1$, in contradiction to the maximality of $\det A_0$, proving the case of $A_0 = I$.

For the general case of A_0, there exist a unitary matrix Q and a diagonal matrix D such that

$$Q^* A_0 Q = D.$$

Put $U = QD^{-\frac{1}{2}}$. Consider the set

$$\mathscr{B} = \{B > 0\,;\, \langle Bx, x\rangle \leq |||Ux|||^2 \text{ for all } x \in \mathbb{C}^n\}.$$

Then $A \in \mathscr{A}$ if and only if $B = U^*AU \in \mathscr{B}$. Thus $A_0 \in \mathscr{A}$ is such that $\det A \leq \det A_0$ for all $A \in \mathscr{A}$ if and only if $U^*A_0U = I \in \mathscr{B}$ and $\det B \leq 1$ for all $B \in \mathscr{B}$. Applying the case of $A_0 = I \in \mathscr{A}$ to $U^*A_0U = I \in \mathscr{B}$, we conclude that

$$\langle nA_0Ux, Ux\rangle \leq |||Ux|||^2 \text{ for all } x \in \mathbb{C}^n,$$

and the theorem is proved. □

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Manuscript received December 28, 2010
revised February 18, 2011

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