Journal of Nonlinear and Convex Analysis
Volume 11, Number 3, 2010, 587–595

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

FINITE DIMENSIONAL INVARIANT SUBSPACE PROPERTIES AND AMENABILITY

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In memory of Professor Ky Fan with greatest respect and admiration.

Abstract. The purpose of this paper is to report on some results related to the Ky Fan Finite Dimensional Invariant Subspace Theorem and related problems.

1. Introduction

In [7] (see also [5] and [6]), Ky Fan established the following remarkable "Invariant Subspace Theorem":

Theorem 1.1. *Let S be a left amenable semigroup. Then S satisfies the following property:*

If $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as continuous linear operators from a separated locally convex space E and $X \subseteq E$ containing an n-dimensional subspace such that $T_s(L)$ is an n-dimensional subspace contained in X whenever L is one and $s \in S$, and there exists a closed $\mathcal{S}$-invariant subspace H in E of codimension n with the property that $(x + H) \cap X$ is compact convex for each $x \in E$, then there exists an n-dimensional subspace L_0 contained in X such that $T_s(L_0) = L_0$ for all $s \in S$.

The purpose of this paper is to report on some results related to Ky Fan's Invariant Subspace Theorem and open problems.

The paper is organized as follows: In Section 2, we present some preliminaries and notations we shall need; in Section 3, we present results on the relation between Ky Fan's finite invariant subspace property and amenability for semigroup of linear operators; in Section 4, we discuss a similar property in the setting of an F-algebra i.e. a Banach algebra A which is the predual space of a von Neumann algebra such that set of positive functionals in A forms a semigroup. This class of Banach algebras includes the predual algebra of a Holf-von Neumann algebra, and in particular the group algebra $L^1(\mathbb{G})$ of a quantum group $\mathbb{G}$. Two classical examples of $L^1(\mathbb{G})$ are the group algebra $L^1(G)$ and the Fourier algebra $A(G)$ of a locally compact group G.

2. Some preliminaries and notations

Let S be a *semitopological semigroup*, i.e. S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \to as$ and $s \to sa$ from S into S are continuous. Let $\ell^\infty(S)$ denote the C^*-algebra of bounded complex-valued

2010 *Mathematics Subject Classification.* 43A07, 47H10 and 47H20.
Key words and phrases. Amenable semigroups, invariant means, finite dimensional invariant subspace, F-algebras, Fourier algebra, group algebra, quantum group algebra.
This research is supported by NSERC Grant MS100.

functions on S with the supremum norm and pointwise multiplication. For each $a \in S$ and $f \in \ell^\infty(S)$, let $\ell_a f$ and $r_a f$ denote the left and right translate of f by a respectively, i.e $(\ell_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa)$, $\quad s \in S$. Let X be a closed subspace of $\ell^\infty(C)$ containing constants and invariant under translations. Then a linear functional $m \in X^*$ is called a *mean* if $\|m\| = m(1) = 1$; m is called a *left invariant mean*, denoted by LIM, if $m(\ell_a f) = m(f)$ for all $a \in S,\ f \in X$.

Let $C(S)$ denote the space of all bounded continuous complex-valued functions on S. Let $LUC(S)$ be the space of *left uniformly continuous functions* on S, i.e., all $f \in C(S)$ such that the mappings $a \to \ell_a f$ from S into $C(S)$ are continuous when $C(S)$ has the sup norm topology. Then $LUC(S)$ is a C^*-subalgebra of $C(S)$ invariant under translations and contains the constant functions. S is called *left amenable* if $LUC(S)$ has a LIM. Left amenable semitopological semigroups include all commutative semigroups, all compact groups and all solvable groups. But the free group (or semigroup) on two generators is not left amenable. Interested readers should consult the fundamental paper of Day [1], the classic of Greenleaf [8], Paterson [30] or our survey article [23].

Let $AP(S)$ denote the space of all $f \in C(S)$ such that $\mathcal{LO}(f) = \{\ell_s f;\, s \in S\}$ is relatively compact in the norm topology of $C(S)$ and $WAP(S)$ denote the space of all $f \in C(S)$ such that $\mathcal{LO}(f)$ is relatively compact in the weak topology of $C(S)$. Functions in $AP(S)$ (resp. $WAP(S)$) are called *almost periodic* (resp. *weakly almost periodic*). In general the following inclusions hold:

$$AP(S) \subseteq LUC(S) \subseteq C(S) \quad \text{and} \quad AP(S) \subseteq WAP(S) \subseteq C(S).$$

If S is discrete, then:

$$AP(S) \subseteq WAP(S) \subseteq LUC(S) = \ell^\infty(S).$$

If S is compact then:

$$AP(S) = LUC(S) \subseteq WAP(S) = C(S).$$

Let G be a locally compact group with fixed left Haar measure λ. Let $C(G)$ be the space of bounded continuous complex-valued functions on G, and let $C_0(G)$ be the subspace of $C(G)$ consisting of all those functions that vanish at infinity.

The dual space $C_0(G)^*$ may be identified with $M(G)$, the Banach space of regular Borel measures of G with total variation norm. $M(G)$ is a Banach algebra with product

$$\langle \mu * \nu, f\rangle = \iint f(xy)\,d\nu(y)\,d\mu(x)$$

for $\mu, \nu \in M(G),\ f \in C_0(G)$. $\big(M(G), *\big)$ is called the *measure algebra* of G. Furthermore, $L^1(G)$, the space of λ-integrable functions on G, may be identified with the closed ideal in $M(G)$ consisting of all μ which is absolutely continuous with respect to λ. Then $\big(L^1(G), *\big)$ is called the *group algebra* of G.

We define $C^*(G)$, the group C^*-algebra of G, to be the completion of $L^1(G)$ with respect to the norm

$$\|f\|_* = \sup \|\pi_f\|,$$

where the supremum is taken over all nondegenerate $*$-representations π of $L^1(G)$ as an $*$-algebra of bounded operators on a Hilbert space.

A function $\phi \in C(G)$ is called *positive definite* if for any $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in G$, $\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\lambda}_i \lambda_j \phi(x_i^{-1} x_j) \geq 0$.

Denote the set of continuous positive definite functions on G by $P(G)$, and the set of continuous functions on G with compact support by $C_{00}(G)$. Define the *Fourier-Stieltjes algebra* of G, denoted by $B(G)$, to be the linear span of $P(G)$. Then $B(G)$ may be identified with the dual Banach space of $C^*(G)$ with

$$\langle \phi, f \rangle = \int \phi(x) f(x) d\lambda(x)$$

for $\phi \in B(G)$ and $f \in L^1(G)$. $B(G)$ is then a commutative Banach algebra. Let $A(G)$ denote the closed subalgebra of $B(G)$ generated by functions with compact support. $A(G)$ is called the *Fourier algebra* of G. When G is abelian, then $B(G) \cong M(\widehat{G})$ and $A(G) \cong L^1(\widehat{G})$, where $\widehat{G}$ is the dual group of G. (See [4], [9] and [10] for details.)

If $f \in L^\infty(G)$ and $\mathcal{L}O(f) = \{\ell_g f;\, g \in G\}$ is relatively weakly compact, then $f \in C(G)$. In particular $f \in WAP(G)$ (see [20]). In this case $B(G) \subseteq WAP(G) \subseteq LUC(G)$.

3. Finite dimensional invariant subspace properties and amenability

Let E be a separated locally convex space. Let $\mathcal{S} = \{T_s;\, s \in S\}$ be a representation of a semigroup S as linear mappings from E into E. We assume (throughout this section) that the map $\psi : S \times E \to E,\ (s,x) \to T_s(x),\ s \in S$ and $x \in E$ is separately continuous, i.e., ψ is continuous in each of the two variables when the other is kept fixed. Then $\mathcal{S}$ is *jointly continuous* on a subset $K \subseteq E$ if the map ψ is continuous on $S \times K$ when $S \times K$ has the product topology; $\mathcal{S}$ is *quasi-equicontinuous* on K if the closure of $\mathcal{S}$ in the product space E^K consists of continuous mappings from K to E; and $\mathcal{S}$ is *equicontinuous* on K if for each $y \in K$, U a neighbourhood of $0 \in E$, there exist a neighbourhood V of 0 such that whenever $x \in K,\ x - y \in V$, then $T_s x - T_s y \in U$ for all $s \in S$.

Let S be a semigroup and $\mathcal{S} = \{T_s;\, s \in S\}$ be a representation of S as continuous linear mappings on a separated locally convex space E. Let X be a subset of E, and $\mathcal{L}_n(X)$ denote all n-dimensional subspaces of E. We say that X is *n-consistent* if:

(a) $\mathcal{L}_n(X)$ is non-empty and S-invariant (i.e. $T_s(L) \subseteq L$ for all $s \in S$); and
(b) there exists a closed S-invariant subspace H of E with codimension n and $(x + H) \cap X$ is compact convex for each $x \in E$.

Theorem 3.1. *Let S be a semitopological semigroup.*

(a) *If $LUC(S)$ has a LIM, then S satisfies $P(n)$ for each n :*
$P(n)$: Let E be a separated locally convex space and $\mathcal{S} = \{T_s;\, s \in S\}$ is a continuous representation of S as linear operators from E into E and jointly continuous on compact convex subsets of E. Let X be an n-consistent subset of E. Then there exists $L_0 \in \mathcal{L}_n(X)$ such that $T_s(L_0) = L_0$ for each $s \in S$.

(b) *If $P(n)$ holds for some $n = 1, 2, \dots$ then $LUC(S)$ has a LIM. In particular $P(m)$ and $P(n)$ are equivalent for all $m, n = 1, 2, \dots$.*

Remark 3.2. Theorem 3.1(a) was proved by Ky Fan in [5] and by Lau in [17] for semitopological semigroups (see [22, p.555] for correction). Theorem3.1(b) was proved in [17].

Consider on S the following finite dimensional invariant subspace property:

$P'(n)$: Let E be a separated locally convex space and $\mathcal{S} = \{T_s;\ s \in S\}$ be a continuous representation of S as linear operators from E into E. Let X be an n-consistent subset of E. Then there exists $L_0 \in \mathcal{L}_n(X)$ such that $T_s(L_0) = L_0$ for each $s \in S$.

Then clearly $P'(n) \Longrightarrow P(n)$.

Open problem 1: Does $P(n) \Longrightarrow P'(n)$?

Let $P_A(n)$ denote the same property as $P(n)$ with joint continuity replaced by equicontinuity on compact convex subsets and $P_W(n)$ denotes $P(n)$ with joint continuity replaced by quasi-equicontinuity on compact convex subsets K of E (see [25]).

Proposition 3.3. *Let S be a semitopological semigroup. Then for any positive integer $n = 1, 2, 3, \dots$*

(a) *$P_A(n)$ implies $AP(S)$ has a LIM*
(b) *$P_W(n)$ implies $WAP(S)$ has a LIM.*

Proof. (a) Let $H = AP(S)^*$ with the weak*-topology and $E = H \times \mathbb{R}^n$ with the product topology. Let M be the set of means on $AP(S)$ i.e. all $m \in AP(S)^*$ such that $\|m\| = m(1) = 1$. For each $m \in M$, let $V_m = \text{span } \{(m, b);\ b \in B\}$ where $B = \{e_i\} \subseteq \mathbb{R}^n$, $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (with "1" in the i^{th} place), and identify H with $H \times \{0\}$, $\mathbb{R}^n$ with $\{0\} \times \mathbb{R}^n$. Then $E = H \oplus V_m$ for each $m \in M$. Let $X = \cup\{V_m;\ m \in M\}$. Consider the continuous representation $\mathcal{S} = \{T_s;\ s \in S\}$ of S on E defined by $T_s(\phi, t) = (\ell_s^* \phi, t)$, $\phi \in H$, $t \in \mathbb{R}^n$. Then $\mathcal{S}$ is equicontinuous on compact convex subsets of E. Note that $T_s V_m = V_{\ell_s^* m}$ for each $s \in S$, $m \in M$. Also for each $x = (\phi, t) \in E$, $\phi \in AP(S)^*$, $t \in \mathbb{R}^n$, and $t = \sum_{i=1}^n \lambda_i e_i$, then

$$(x + H) \cap X = (t + H) \cap X = t + \Big(\sum_{i=1}^n \lambda_i\Big) M$$

which is compact and convex. By $P_A(n)$, we can find $L_0 \in \mathcal{L}_n(X)$ such that $T_s(L_0) = L_0$ for each $s \in S$. Necessarily $L_0 = V_m$ for some $m \in M$. Clearly m is a LIM on $AP(S)$.

(b) Can be proved by similar arguments. □

Open problem 2: (a) Does $AP(S)$ have a LIM imply $P_A(n)$ for all n?

(b) Does $WAP(S)$ have a LIM imply $P_W(n)$ for all n?

It follows from [11] (see [27]) if S is a left reversible semigroup i.e. $aS \cap bS \neq \emptyset$ for all $a, b \in S$, then $WAP(S)$ has a left invariant mean. But the converse is not true (see [27]).

Open problem 3: Let S be a left reversible semigroup. Does S have the finite invariant subspace property $P_A(n)$ or $P_W(n)$ for each positive integer n? This problem is not known even when S is a group.

Remark 3.4. We now consider the bicyclic semigroups S_2 and S_3: S_2 is the semigroup generated by $\{e, a, b, c\}$ such that e is the unit element and a, b and c satisfies the relation $ab = ac = e$; S_3 is the semigroup generated by $\{e, a, b, c, d\}$ such that e is the unit element in S_3 and $ac = e$, $bd = e$. Then as shown in [27]:

(a) Both S_2 and S_3 are *not* left amenable;
(b) S_2 is right amenable, but S_3 is not right amenable;
(c) S_2 is not left reversible but $WAP(S_2)$ has a LIM;
(d) $WAP(S_3)$ does not have a LIM.

Open problem 4:

(a) Does S_2 have property $P_A(n)$ or $P_W(n)$ for each n?
(b) Does the bicyclic semigroup S_3 have the property $P_A(n)$ for each n?

Note that S_3 cannot have property $P_W(n)$ for any n by Proposition 3.3.

In a conference on Analysis and Semigroups held in Richmond, Virginia 1984, Theodore Mitchell [28] (see also [23] and [27]) showed that the bicyclic semigroups S_2 and S_3 are not left reversible, but $AP(S_2)$ and $AP(S_3)$ have a LIM.

Let S be a semitopological semigroup. The existence of LIM on $LUC(S)$, $WAP(S)$ and $AP(S)$ can be characterized by the following Markov-Kakutani type fixed point properties.

Theorem 3.5. *$LUC(S)$ has a LIM if and only if S has the following fixed point property:*

(F_1) *Whenever $\mathcal{S}$ is a jointly continuous representation of S as affine mappings on a non-empty compact convex subset X of a separated locally convex space, then X contains a common fixed point for $\mathcal{S}$.*

Theorem 3.6. *$WAP(S)$ has a LIM if and only if S has the following fixed point property:*

(F_2) *Whenever $\mathcal{S}$ is weakly continuous representation of S as τ-equicontinuous affine self-maps on a non-empty weakly compact convex subset X of a separated locally convex space E with topology τ, then X has a common fixed point for $\mathcal{S}$.*

Theorem 3.7. *$AP(S)$ has LIM $\Longleftrightarrow$ S has the following fixed point property:*

(F_3) *Whenever $\mathcal{S} = \{T_s : s \in S\}$ is a continuous representation of S as affine equicontinuous mappings from a non-empty compact convex subset X of a separated locally convex space into X, then X contains a common fixed point for $\mathcal{S}$.*

Remark 3.8. Theorem 3.5 was due to Day [2] (see also [3]) for discrete semigroup, and Mitchell [29] for semitopological semigroup. Theorem 3.6 is proved in [15] and Theorem 3.7 is proved in [14]. See [27] for related fixed point properties for semigroup of nonexpansive mappings.

4. Algebras of linear operators

By an F-*algebra* we shall mean a complex Banach algebra A such that A^* is a C^*-algebra and the identity of A^*, denoted by I (which always exists), is a multiplicative linear functional on A. Examples of F-algebras include the predual algebra of a Holf-von Neumann algebra, the group algebra $L^1(G)$, Fourier algebra $A(G)$ and the Fourier Stieltjes algebra $B(G)$ of a locally compact group G. It also includes the measure algebra $M(S)$ of a locally compact semigroup S.

Let S_A denote the set of all positive functionals in $A \subset A^{**}$ with norm one. Then $S_A = \{\mu \in A; \|\mu\| = I(\mu) = 1\}$. Hence, as readily checked, $(S_A, *)$, where $*$ denotes the multiplication of A, is a semigroup. A is called *left amenable* if $A^* = M$ has a topological left invariant mean (abbreviated as $TLIM$), i.e. an $m \in M^*$ such that $\|m\| = 1$, $m \geq 0$ and $m(F \cdot \mu) = m(F)$ for each $\mu \in S_A$ and $F \in M$, where $F \cdot \mu \in M$ is defined by $(F \cdot \mu, \nu) = \langle F, \mu * \nu \rangle$ for all $\nu \in A$ (see [16] and [18] for details).

A representation of an F-algebra A as operators in a locally convex space E is a map $T : A \times E \to E$ denoted by $(\mu, x) \to T_\mu(x)$ such that (1) $T_\mu : E \to E$ is continuous and linear, (2) $\mu \to T_\mu(x)$ is continuous and linear with respect to the norm topology in A for each $x \in E$ and (3) $T_{\mu * \nu} = T_\mu \circ T_\nu \,\forall\, \mu, \nu \in A$, where $*$ denotes multiplication in A. Also, let X be a subset of E containing an n-dimensional subspace. We say that $\mathcal{L}_n(X)$ is S_A-invariant under T if $T_\mu(L) \in \mathcal{L}_n(X)$ for each $L \in \mathcal{L}_n(X)$ and $\mu \in S_A$. A closed subspace H in E is called S_A-invariant under T if $T_\mu(H) \subset H \,\forall\, \mu \in S_A$ (and hence $\forall\, \mu \in A$ as well). Denote by $q : E \to E/H$ the natural map such that $q(x) = \widetilde{x}$, $x \in E$.

The action of S_A on E is called *inversely equicontinuous modulo H* if given any neighborhood U in E, there is some neighborhood V in E such that $V \subset T_\mu(U) + H$ for any $\mu \in S_A$.

Theorem 4.1. *Let A be an F-algebra.*

- (a) *If A is left amenable, then A satisfies property $T(n)$ for $n = 1, 2, \ldots$ where property $T(n)$ is defined as follows. Let E be a separated locally convex space and $T : A \times E \to E$ be a representation of A as linear operators in E. Let X be a subset of E such that there exists a closed S_A-invariant subspace H of E of codimension n and $(x + H) \cap X$ is compact convex for each $x \in E$. If the action of S_A on E is inversely equicontinuous modulo H and $\mathcal{L}_n(X)$ is nonempty and S_A-invariant, then there exists $L_0 \in \mathcal{L}_n(X)$ such that $T_\mu(L_0) = L_0 \,\forall\, \mu \in S_A$.*
- (b) *If A satisfies property $T(1)$, then A is left amenable (hence A satisfies $T(n)$ for every n).*

Corollary 4.2. *Let G be a locally compact group. If G is amenable, then the group algebra $L^1(G)$ and the measure algebra $M(G)$ satisfy $T(n)$ for each $n = 1, 2, 3, \dots$. Conversely, if either $L_1(G)$ or $M(G)$ satisfies $T(1)$, then G is amenable.*

Corollary 4.3. *Let G be a locally compact group. Then both the Fourier algebra $A(G)$ and the Fourier Stieltjes algebra $B(G)$ have property $T(n)$ for each $n = 1, 2, \dots$.*

Let $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ be a von Neumann algebraic locally compact quantum group in the sense of Kustermans and Vaes ([12], [13]). By definition, (M, Γ) is a Hopf-von Neumann algebra, φ is a normal semifinite faithful left invariant weight on (M, Γ), and ψ is a normal semifinite faithful right invariant weight on (M, Γ). Since the co-multiplication Γ is a normal isometric unital $*$-homomorphism from M into $M \widetilde{\otimes} M$, it is well known that its pre-adjoint $\Gamma_* : M_* \widehat{\otimes} M_* \to M_*$ induces an associative completely contractive multiplication $*$ on M_*. Here, $\widetilde{\otimes}$ denotes the von Neumann algebra tensor product, and $\widehat{\otimes}$ denotes the operator space projective tensor product. In particular M_* is an F-algebra. Classical examples of quantum group includes the group algebra $L^1(G)$ and the Fourier algebra $A(G)$ of a locally compact group. In this case, $*$ is the usual convolution and pointwise multiplication on $L^1(G)$ and $A(G)$ respectively.

Given a quantum group $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ the von Neumann algebra M is written as $L^\infty(\mathbb{G})$ and the Banach algebra as $L^1(\mathbb{G})$. We say that $\mathbb{G}$ is *amenable* if $L^1(\mathbb{G})$ is left amenable. As a consequence of Theorem 4.1, we have the following of amenable quantum group:

Theorem 4.4. *Let $\mathbb{G}$ be a quantum group, and*

(a) *If $\mathbb{G}$ is amenable, then the Banach algebra $L^1(\mathbb{G})$ satisfies property $T(n)$ for $n = 1, 2, \dots$*
(b) *If $L^1(\mathbb{G})$ satisfies $T(1)$, then $\mathbb{G}$ is amenable, and hence $L^1(\mathbb{G})$ has properties $T(n)$ for all $n = 1, 2, \dots$.*

Theorem 4.5. *For any locally compact group G, the Fourier algebra $A(G)$ has the following fixed point property:*

(A): *For any representation of $A(G)$ on a separated locally convex space E and any compact $P_1\big(A(G)\big)$-invariant subset S of E, then S contains a common fixed point for $\big\{\tau_\phi;\ \phi \in P_1\big(A(G)\big)\big\}$.*

Remark 4.6. Theorem 4.1 is proved by Lau and Wong in [21] and Theorem 4.5 is contained in [24, Theorem 3.4].

Open problem 5: Does every quantum group $\mathbb{G}$ satisfy a fixed point property of type (A) above with an appropriate representation as in Theorem 3.6?

Note this is the case for the Fourier algebra $A(G)$, or the group algebra of an amenable group $L^1(G)$ (see [31]). Also note that for any locally compact group G, $WAP(G)$ has a LIM (see [8]) and hence has fixed point property (F_2). See also [26] for related fixed point property for amenable representations.

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Manuscript received August 26, 2010
revised December 16, 2010

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