Journal of Nonlinear and Convex Analysis
Volume 11, Number 3, 2010, 547–566
Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

STRONG CONVERGENCE THEOREMS AND NONLINEAR ANALYTIC METHODS FOR LINEAR CONTRACTIVE MAPPINGS IN BANACH SPACES

WATARU TAKAHASHI, JEN-CHIH YAO*, AND TAKASHI HONDA

Dedicated to the memory of Professor Ky Fan

Abstract. In this paper, we study nonlinear analytic methods for linear contractive mappings in Banach spaces. Using these results, we obtain some new strong convergence theorems for linear contractive operators in Banach spaces. In theorems, the limit points are characterized by suny generalized nonexpansive retractions.

1. Introduction

Let E be a real Banach space and let C be a closed convex subset of E. For a mapping $T : C \to C$, we denoted by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. In particular, a nonexpansive mapping $T : E \to E$ is called contractive if it is linear, that is, a linear contactive mapping $T : E \to E$ is a linear operator satisfying $\|T\| \leq 1$. From [38] and [55] we know a weak convergence theorem by Mann's iteration for nonexpansive mappings in a Hilbert space: Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ such that

$$\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty.$$

Then, $\{x_n\}$ converges weakly to an element z of $F(T)$, where $z = \lim_{n\to\infty} Px_n$ and P is the metric projection of H onto $F(T)$. By Reich [44], such a theorem was proved in a uniformly convex Banach space with a Fréchet differentiable norm.

2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 47H10, 60G05.

Key words and phrases. Banach space, linear contactive mapping, generalized nonexpansive mapping, generalized projection, sunny generalized nonexpansive retraction, fixed point, homogeneous mapping.

The first author and the second author were partially supported by Grant-in-Aid for Scientific Research No. 19540167 from Japan Society for the Promotion of Science and by the grant NSC 98-2115-M-110-001, respectively.

*Corresponding author.

However, we have not known whether the fixed point z is characterized under any projections in a Banach space. Recently, using nonlinear analytic methods obtained by [27], [28] and [21], Takahashi and Yao [56] solved such a problem for positively homogeneous nonexpansive mappings in a Banach space. We also know that there are many papers which discuss Reich's theorem for another nonlinear mappings in a Banach space; see, for instance, Kohsaka and Takahashi [35], Matsushita and Takahashi [39] and Ibaraki and Takahashi [25]. In 1938, Yosida [59] also proved the following mean ergodic theorem for linear bounded operators: Let E be a real Banach space and let T be a linear operator of E into itself such that there exists a constant C with $\|T^n\| \leq C$ for $n \in \mathbb{N}$, and T is weakly completely continuous, i.e., T maps the closed unit ball of E into a weakly compact subset of E. Then, for each $x \in E$, the Cesàro means

$$S_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x$$

converge strongly as $n \to \infty$ to a fixed point of T; see also Kido and Takahashi [34].

In this paper, motivated by these theorems, we study nonlinear analytic methods for linear contractive mappings in a Banach space and obtain some new strong convergence theorems for linear contractive operators in a Banach space. One of them extends Bauschk, Deutsch, Hundal and Park's theorem [7] from a Hilbert space to a Banach space.

2. Preliminaries

Throughout this paper, we assume that a Banach space E with the dual space E^* is real. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of all positive integers and all real numbers, respectively. We also denote by $\langle x, x^* \rangle$ the dual pair of $x \in E$ and $x^* \in E^*$. A Banach space E is said to be strictly convex if $\|x + y\| < 2$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x \neq y$. A Banach space E is said to be smooth provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. Let E be a Banach space. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The multivalued operator $J : E \to E^*$ is called the normalized duality mapping of E. From the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$. We know that E is smooth if and only if J is single-valued. If E is strictly convex, then J is one-to-one, i.e., $x \neq y \Rightarrow J(x) \cap J(y) = \emptyset$. If E is reflexive, then J is a mapping of E onto E^*. So, if E is reflexive, strictly convex and smooth, then J is single-valued, one-to-one and onto. In this case, the normalized duality mapping J_* from E^* into E is the inverse of J, that is, $J_* = J^{-1}$; see [50] and [51] for more details. Let E be a smooth Banach space and let J be the normalized duality mapping of E. We define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. We also define the function $\phi_* : E^* \times E^* \to \mathbb{R}$ by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle x^*, J^{-1}y^*\rangle + \|y^*\|^2$$

for all $x^*, y^* \in E^*$. It is easy to see that $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$. Thus, in particular, $\phi(x, y) \geq 0$ for all $x, y \in E$. We also know the following:

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.1}$$

for all $x, y, z \in E$. Further, we have

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.2}$$

for all $x, y, z, w \in E$. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{2.3}$$

for all $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Leftrightarrow x = y. \tag{2.4}$$

The following lemma due to Kamimura and Takahashi [33] is well-known.

Lemma 2.1 ([33]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \phi(z, x) = \min_{y\in C} \phi(y, x)\}$$

is always a singleton. Let us define the mapping Π_C of E onto C by $z = \Pi_C x$ for every $x \in E$, i.e.,

$$\phi(\Pi_C x, x) = \min_{y\in C} \phi(y, x)$$

for every $x \in E$. Such Π_C is called the generalized projection of E onto C; see Alber [1]. The following lemma is due to Alber [1] and Kamimura and Takahashi [33].

Lemma 2.2 ([1, 33]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, the following hold:*

(a) $z = \Pi_C x$ *if and only if* $\langle y - z, Jx - Jz\rangle \leq 0$ *for all* $y \in C$;
(b) $\phi(z, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(z, x)$.

Let D be a nonempty closed subset of a smooth Banach space E, let T be a mapping from D into itself and let $F(T)$ be the set of fixed points of T. Then, T is said to be generalized nonexpansive [24] if $F(T)$ is nonempty and $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in D$ and $u \in F(T)$. Let C be a nonempty subset of E and let R be a mapping from E onto C. Then R is said to be a retraction, or a projection if $Rx = x$ for all $x \in C$. It is known that if a mapping P of E into E satisfies $P^2 = P$, then P is a projection of E onto $\{Px : x \in E\}$. A mapping $T : E \to E$ with $F(T) \neq \emptyset$ is a retraction if and only if $F(T) = R(T)$, where $R(T)$ is the range of T. The mapping R is also said to be sunny if $R(Rx + t(x - Rx)) = Rx$ whenever $x \in E$ and

$t \geq 0$. A nonempty subset C of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto C. The following lemmas were proved by Ibaraki and Takahashi [24].

Lemma 2.3 ([24]). *Let C be a nonempty closed subset of a smooth, strictly convex and reflexisve Banach space E and let R be a retraction from E onto C. Then, the following are equivalent:*

(a) *R is sunny and generalized nonexpansive;*
(b) *$\langle x - Rx, Jy - JRx\rangle \leq 0$ for all $(x, y) \in E \times C$.*

Lemma 2.4 ([24]). *Let C be a nonempty closed sunny and generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then, the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([24]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then, the following hold:*

(a) *$z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$;*
(b) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

The following theorems were proved by Kohsaka and Takahashi [37].

Theorem 2.6 ([37]). *Let E be a smooth, strictly convex and reflexive Banach space, let C^* be a nonempty closed convex subset of E^* and let Π_{C^*} be the generalized projection of E^* onto C^*. Then the mapping R defined by $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$.*

Theorem 2.7 ([37]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty subset of E. Then, the following are equivalent.*

(1) *D is a sunny generalized nonexpansive retract of E;*
(2) *D is a generalized nonexpansive retract of E;*
(3) *JD is closed and convex.*

In this case, D is closed.

Let E be a smooth, strictly convex and reflexive Banach space, let J be the normalized duality mapping from E onto E^* and let C be a closed subset of E such that JC is closed and convex. Then, we can define a unique sunny generalized nonexpansive retraction R_C of E onto C as follows:

$$R_C = J^{-1}\Pi_{JC}J,$$

where Π_{JC} is the generalized projection from E^* onto JC.

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \|z - x\| = \min_{y\in C} \|y - x\|\}$$

is always a singleton. Let us define the mapping P_C of E onto C by $z = P_C x$ for every $x \in E$, i.e.,

$$\|P_C x - x\| = \min_{y\in C} \|y - x\|$$

for every $x \in E$. Such P_C is called the metric projection of E onto C; see [50]. The following lemma is in [50].

Lemma 2.8 ([50])**.** *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, $z = P_C x$ if and only if $\langle y - z, J(x - z)\rangle \leq 0$ for all $y \in C$.*

An operator $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x - y, x^* - y^*\rangle \geq 0$ for any $(x, x^*), (y, y^*) \in A$. An operator A is said to be strictly monotone if $\langle x-y, x^*-y^*\rangle > 0$ for any $(x, x^*), (y, y^*) \in A$ $(x \neq y)$. Let J be the normalized duality mapping from E into E^*. Then, J is monotone. If E is strictly convex, then J is one to one and strictly monotone; for instance, see [50].

Let E be a Banach space and let

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| = \epsilon\right\}.$$

We call the function $\delta : [0, 2] \to [0, 1]$ the modulus of convexity. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. In a uniformly convex Banach space, we know the following lemma.

Lemma 2.9 ([50])**.** *Let E be a uniformly convex Banach space and let δ be the modulus of convexity in E. Let ϵ and r be real numbers with $0 < \epsilon \leq 2r$. Then, $\delta\left(\frac{\epsilon}{r}\right) > 0$ and*

$$\|\lambda x + (1 - \lambda)y\| \leq r\left\{1 - 2\lambda(1 - \lambda)\delta\left(\frac{\epsilon}{r}\right)\right\}$$

for all $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x - y\| \geq \epsilon > 0$ and $\lambda \in [0, 1]$.

3. Homogeneous mappings in Banach spaces

In this section, we discuss some properties for homogeneous generalized nonexpansive mappings in a Banach space. Let E be a Banach space and let K be a closed convex cone of E. Then, $T : K \to K$ is called a positively homogeneous mapping if $T(\alpha x) = \alpha Tx$ for all $\alpha \geq 0$ and $x \in K$. Let M be a closed linear subspace of E. Then, $S : M \to M$ is called a homogeneous mapping if $T(\beta x) = \beta Tx$ for all $\beta \in \mathbb{R}$ and $x \in M$.

Remark 3.1. In L^p spaces, $1 \leq p \leq \infty$, we know examples of nonexpansive and positively homogeneous mappings; see, for instance, Wittmann [58].

We start with the following theorem.

Theorem 3.2. *Let E be a smooth Banach space and let K be a closed convex cone of E. Then, a positively homogeneous mapping $T : K \to K$ is generalized nonexpansive if and only if for any $x \in K$ and $u \in F(T)$,*

$$\|Tx\| \leq \|x\| \text{ and } \langle x - Tx, Ju\rangle \leq 0.$$

Furthermore, let M be a closed linear subspace of E. Then, a homogeneous mapping $S : M \to M$ is generalized nonexpansive if and only if for any $x \in M$ and $v \in F(T)$,

$$\|Sx\| \leq \|x\| \text{ and } \langle x - Sx, Jv\rangle = 0.$$

Proof. Since T is positively homogeneous, $F(T)$ must contain the origin. Further, we have that for any $x \in K$, $u \in F(T)$ and $\alpha > 0$,

$$\begin{aligned}\phi(T(\alpha x), u) \leq \phi(\alpha x, u) \\ &\Leftrightarrow \phi(\alpha Tx, u) \leq \phi(\alpha x, u) \\ &\Leftrightarrow \|\alpha Tx\|^2 - 2\langle \alpha Tx, Ju\rangle \leq \|\alpha x\|^2 - 2\langle \alpha x, Ju\rangle \\ &\Leftrightarrow \big(\|x\|^2 - \|Tx\|^2\big)\alpha^2 - 2\alpha\langle x - Tx, Ju\rangle \geq 0 \\ &\Leftrightarrow \big(\|x\|^2 - \|Tx\|^2\big)\alpha - 2\langle x - Tx, Ju\rangle \geq 0.\end{aligned}$$

Letting $\alpha \to 0$, we obtain $\langle x - Tx, Ju\rangle \leq 0$. From $0 \in F(T)$, we have also $\|x\|^2 - \|Tx\|^2 \geq 0$ and hence $\|Tx\| \leq \|x\|$. Conversely, if a positively homogeneous mapping $T : K \to K$ satisfies that for any $x \in K$ and $u \in F(T)$,

$$\|Tx\| \leq \|x\| \text{ and } \langle x - Tx, Ju\rangle \leq 0,$$

then we have

$$\|Tx\| \leq \|x\| \text{ and } \langle x, Ju\rangle \leq \langle Tx, Ju\rangle.$$

So, we have

$$\begin{aligned}\phi(Tx, u) &= \|Tx\|^2 - 2\langle Tx, Ju\rangle + \|u\|^2 \\ &\leq \|x\|^2 - 2\langle x, Ju\rangle + \|u\|^2 \\ &= \phi(x, u).\end{aligned}$$

Then, T is generalized nonexpansive.

Similarly, since S is a homogeneous mapping of M into itself, $F(S)$ must contain the origin. Further, we have that for any $x \in M$, $v \in F(S)$ and $\beta < 0$, we have

$$\big(\|x\|^2 - \|Sx\|^2\big)\beta - 2\langle x - Sx, Jv\rangle \leq 0.$$

Letting $\beta \to 0$, we obtain $\langle x - Sx, Jv\rangle \geq 0$. Since $\langle x - Sx, Jv\rangle \leq 0$ for $\beta > 0$, we obtain $\langle x - Sx, Jv\rangle = 0$. $\|Sx\| \leq \|x\|$ is obvious. The reverse is obvious. □

We also know the follwing theorem from Takahashi and Yao [56]; see also Honda, Takahashi and Yao [21].

Theorem 3.3. *Let E be a smooth Banach space and let K be a closed convex cone in E If $T : K \to K$ is a positively homogeneous nonexpansive mapping, then T is generalized nonexpansive. In particular, if $T : E \to E$ is a linear contractive mapping, then T is generalized nonexpansive.*

From Theorems 3.3 and 3.2, we have the following corollary.

Corollary 3.4. *Let E be a smooth Banach space and let K be a closed convex cone of E. If a mapping $T : K \to K$ is positively homogeneous nonexpansive, then for any $x \in K$ and $u \in F(T)$,*

$$\|Tx\| \leq \|x\| \textit{ and } \langle x - Tx, Ju\rangle \leq 0.$$

Furthermore, let M be a closed linear subspace of E. If a mapping $S : M \to M$ is homogeneous nonexpansive, then for any $x \in M$ and $v \in F(T)$,

$$\|Sx\| \leq \|x\| \textit{ and } \langle x - Sx, Jv\rangle = 0.$$

Let E be a smooth Banach space and let C be a closed subset in E. We call a mapping $T : C \to C$ a firmly generalized nonexpansive type [27] and [28] or firmly skew-nonspreading [15] if it satisfies

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(x, Tx) + \phi(y, Ty) \leq \phi(x, Ty) + \phi(y, Tx)$$

for all $x, y \in C$. By from (2.2), this inequality is equivalent to

$$\langle x - Tx, JTx - JTy\rangle \geq \langle y - Ty, JTx - JTy\rangle$$

for all $x, y \in C$.

Theorem 3.5. *Let E be a smooth Banach space, let K be a closed convex cone of E and let $T : K \to K$ be a positively homogeneous firmly generalized nonexpansive type mapping. Then, $\langle x - Tx, JTx\rangle \geq 0$ for all $x \in K$.*

Proof. From the definition of a firmly generalized nonexpansive type mapping, we have

$$\langle x - Tx, JTx - Ju\rangle \geq 0$$

for any $x \in K$ and $u \in F(T)$. Then we have

$$\langle x - Tx, JTx\rangle \geq \langle x - Tx, Ju\rangle.$$

Fix $x \in K$ and $u \in F(T)$. Let $a = \langle x - Tx, JTx\rangle$ and $b = \langle x - Tx, Ju\rangle$. From the assumption of T, we have that for any $\alpha > 0$,

$$\alpha^2 a = \langle \alpha x - T(\alpha x), JT(\alpha x)\rangle,$$
$$\alpha b = \langle \alpha x - T(\alpha x), Ju\rangle$$

and

$$\alpha^2 a \geq \alpha b.$$

From this, we have $a \geq \frac{1}{\alpha} b$. Letting $\alpha \to \infty$, we have $a \geq 0$, i.e., $a = \langle x - Tx, JTx\rangle \geq 0$. This completes the proof. □

From Theorem 3.2, we introduce the following concept.

Definition 3.6. Let E be a smooth Banch space, let $x \in E$ and let F be a nonempty subset of E. The Sizihwan region between x and F is the set

$$R(x; F) = \{z \in E : \langle x - z, Ju\rangle = 0 \text{ for all } u \in F \text{ and } \|z\| \leq \|x\|\}.$$

Lemma 3.7. *Let E be a strictly convex and smooth Banch space, let $x \in E$ and let F be a nonempty subset of E. Then $R(x; F)$ is nonempty, closed, convex and bounded, and $F \cap R(x; F)$ consists of at most one point.*

Proof. For any $x \in E$ and $F \subset E$, x is always an element of $R(x; F)$. Then $R(x; F)$ is nonempty. From the definition, it is obvious that $R(x; F)$ is convex and bounded. We show that $R(x; F)$ is closed. Let $\{z_n\}$ be a sequence in $R(x; F)$ and $z_n \to z_0$. Then, we have that for all $u \in F$,

$$0 = \langle x - z_n, Ju\rangle \to 0 = \langle x - z_0, Ju\rangle$$

and $\|z_0\| \leq \|x\|$. So, $z_0 \in R(x; F)$.

Let $z_1, z_2 \in F \cap R(x; F)$. Then $\langle x - z_1, Jz_1\rangle = 0$ and $\langle x - z_2, Jz_1\rangle = 0$. So, we have $\langle z_1 - z_2, Jz_1\rangle = 0$. Similarly, we have $\langle z_1 - z_2, Jz_2\rangle = 0$. Then we obtain $\langle z_1 - z_2, Jz_1 - Jz_2\rangle = 0$. Since E is strictly convex, we obtain $z_1 = z_2$. □

Let F be a nonempty subset of a Banach space E and $x \in E$. Then,

$$\operatorname{dist}(x, F) = \inf\{\|x - y\| : y \in F\}.$$

Lemma 3.8. *Let E be a uniformly convex and smooth Banach space, let $x \in E$ and let F be a nonempty closed subset of E. Suppose $\{x_n\}$ is a sequence in $R(x; F)$ such that $\lim_{n\to\infty} \operatorname{dist}(x_n, F) = 0$. Then $F \cap R(x; F)$ is nonempty and $\{x_n\}$ converges strongly to a unique point in $F \cap R(x; F)$.*

Proof. Choose $\{y_n\}$ in F such that $\|x_n - y_n\| \to 0$. By Lemma 3.7, both $\{x_n\}$ and $\{y_n\}$ are bounded. Then, there exists a positive number M such that

$$\|y_n\| = \|Jy_n\| \le M$$

for any $n \in \mathbb{N}$. Since $\{y_n\} \subset F$, we have $\langle x - x_n, Jy_m\rangle = 0$ for any $n, m \in \mathbb{N}$. So, we have

$$\begin{aligned} |\langle x - y_n, Jy_m\rangle| &= |\langle x - x_n, Jy_m\rangle - \langle y_n - x_n, Jy_m\rangle| \\ &= |\langle y_n - x_n, Jy_m\rangle| \\ &\le M\|x_n - y_n\|. \end{aligned}$$

Similarly, we have that $|\langle x - y_n, Jy_n\rangle| \le M\|x_n - y_n\|$ for any $n \in \mathbb{N}$. Then, we have that for any $n, m \in \mathbb{N}$,

$$\begin{aligned} |\langle x - y_n, Jy_n - Jy_m\rangle| &\le |\langle x - y_n, Jy_n\rangle| + |\langle x - y_n, Jy_m\rangle| \\ &\le 2M\|x_n - y_n\|. \end{aligned}$$

Since $\|x_n - y_n\|$ converges to 0 as $n \to \infty$, there exists a positive sequence t_n with $t_n \searrow 0$ such that

$$\langle y_n - x, Jy_n - Jy_m\rangle \le t_n$$

for all $n, m \in \mathbb{N}$. Similarly, we have

$$\langle y_m - x, Jy_m - Jy_n\rangle \le t_m$$

for all $n, m \in \mathbb{N}$. Then, we have

$$\frac{\phi(y_n, y_m) + \phi(y_m, y_n)}{2} = \langle y_n - y_m, Jy_n - Jy_m\rangle \le t_n + t_m.$$

Since E is uniformly convex and smooth, from Kamimura and Takahashi [33] there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$g(\|y_n - y_m\|) \le \phi(y_n, y_m)$$

and

$$g(\|y_n - y_m\|) = g(\|y_m - y_n\|) \le \phi(y_m, y_n)$$

for all $n, m \in \mathbb{N}$. Then, we have

$$g(\|y_n - y_m\|) \le t_n + t_m$$

for all $n, m \in \mathbb{N}$. Therefore, from the properties of g, $\{y_n\}$ is a Cauchy sequence in F. So, $\{x_n\}$ is also a Cauchy sequence in $R(x; F)$. Then both $\{y_n\}$ and $\{x_n\}$ converge to a same element $z \in E$. Since both F and $R(x; F)$ are closed, the limit z belongs to $F \cap R(x; F)$. □

Using Corollary 3.4, we have the following result.

Lemma 3.9. *Let E be a smooth Banach space, let M be a closed linear subspace of E and $x \in M$. For any homogeneous nonexpansive mapping $T : M \to M$, Tx is an element of $R(x; F(T)) \cap M$, where $F(T)$ is the set of all fixed points of T.*

The finite composition of homogeneous nonexpansive mappings is also a homogeneous nonexpansive mapping. Then, using Lemma 3.8 we have the following theorem.

Theorem 3.10. *Let E be a uniformly convex and smooth Banach space, let M be a closed linear subspace of E and let $\{T_n : n \in \mathbb{N}\}$ be a sequence of homogeneous nonexpansive mappings of M into itself such that $\cap_{n\in\mathbb{N}} F(T_n) \neq \emptyset$. Let $\{x_n\}$ be a sequence of M defined by $x \in M$ and*

$$x_n = T_n \circ T_{n-1} \circ \cdots T_1 x$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to an element of $\cap_{m\in\mathbb{N}} F(T_m)$ if and only if $\lim_{n\to\infty} \operatorname{dist}(x_n, \cap_{m\in\mathbb{N}} F(T_m)) = 0$.

Proof. Let $x \in M$ and put $S_n = T_n \circ T_{n-1} \circ \cdots T_1$ for all $n \in \mathbb{N}$. Since S_n is a homogenuous nonexpansive mapping of M into itself, we have that $\|x_n\| \leq \|x\|$ and $\langle x - x_n, Ju \rangle = 0$ for all $u \in F(S_n)$. So, we have $\|x_n\| \leq \|x\|$ and $\langle x - x_n, Ju \rangle = 0$ for all $u \in \cap_{m\in\mathbb{N}} F(S_m)$ and $n \in \mathbb{N}$. This implies $x_n \in R(x; \cap_{m\in\mathbb{N}} F(S_m))$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} \operatorname{dist}(x_n, \cap_{m\in\mathbb{N}} F(S_m)) = 0$, then from Lemma 3.8 we have $\{x_n\}$ converges strongly to a unique point z of $\cap_{m\in\mathbb{N}} F(S_m) \cap R(x; \cap_{m\in\mathbb{N}} F(S_m))$.

Conversely, if $\{x_n\}$ converges strongly to an element of $\cap_{m\in\mathbb{N}} F(S_m)$, then it is obvious that $\lim_{n\to\infty} \operatorname{dist}(x_n, \cap_{m\in\mathbb{N}} F(S_m)) = 0$. □

4. Strong convergence theorems

Let Y be a nonempty subset of a Banach space E and let Y^* be a nonempty subset of the dual space E^*. Then, we can define the annihilator $Y^*_\perp$ of Y^* and the annihilator $Y^\perp$ of Y as follows:

$$Y^*_\perp = \{x \in E : f(x) = 0 \text{ for all } f \in Y^*\}$$

and

$$Y^\perp = \{f \in E^* : f(x) = 0 \text{ for all } x \in Y\}.$$

We know the following result from Megginson [41].

Lemma 4.1 ([41]). *Let A be a nonempty subset of E. Then*

$$(A^\perp)_\perp = \overline{\operatorname{span}} A,$$

where $\overline{\operatorname{span}} A$ is the smallest closed linear subspace of E containing A.

Let $T : E \to E$ be a bounded linear operator. Then, the adjoint mapping $T^* : E^* \to E^*$ is defined as follows:

$$\langle x, T^* x^* \rangle = \langle Tx, x^* \rangle$$

for any $x \in E$ and $x^* \in E^*$. We know that T^* is also a bounded linear operator and $\|T\| = \|T^*\|$. If S and T are bounded linear operators form E into itself and $\alpha \in \mathbb{R}$, then $(S + T)^* = S^* + T^*$ and $(\alpha S)^* = \alpha\,(S)^*$. Let I be the identity operator on E. Then, I^* is the identity operator on E^*. Let $T^{**} : E^{**} \to E^{**}$ be the adjoint of T^*. Then we have $T^{**}(\pi(E)) \subset \pi(E)$ and $\pi^{-1}T^{**}\pi = T$, where π is the natural embedding from E into its second dual space E^{**}; see [41].

Lemma 4.2. *Let E be a strictly convex, smooth and reflexive Banach space, let T be a linear contractive operator of E into itself, i.e., $T : E \to E$ is a linear operator such that $\|T\| \leq 1$ and let $F(T)$ be the set of fixed points of T. Then $JF(T)$ is a closed linear subspace in E^* and $JF(T) = F(T^*) = \{z - Tz : z \in E\}^{\perp}$, where $J : E \to E^*$ is the normalized duality mapping and T^* is the adjoint operator of T.*

Proof. From Corollary 3.4, we have

$$\langle x - Tx, Ju\rangle = 0$$

for any $x \in E$ and $u \in F(T)$. We also have that

$$\begin{aligned}\langle x - Tx, Ju\rangle = 0 &\Leftrightarrow \langle x, Ju\rangle = \langle Tx, Ju\rangle \\ &\Leftrightarrow \langle x, Ju\rangle = \langle x, T^*Ju\rangle \\ &\Leftrightarrow \langle x, (I^* - T^*)Ju\rangle = 0,\end{aligned}$$

where I^* is the identity operator in E^*. Since this equation holds for all $x \in E$, we have $(I^* - T^*)Ju = 0$. Then, $T^*Ju = Ju$ and hence $JF(T) \subset F(T^*)$.

Since $\|T^*\| = \|T\| \leq 1$, we can get the same fact about T^*. So, we obtain that

$$J_*F(T^*) \subset F(T^{**}),$$

where $J_* : E^* \to E^{**}$ is the duality mapping in E^*. Under assumptions on E, we know that $J_* = J^{-1}$ and $T^{**} = T$. Then, we have

$$F(T^*) \subset JF(T^{**}) = JF(T).$$

So, we obtain that $F(T^*) = JF(T)$ and hence $JF(T)$ is a closed linear subspace of E^*.

Finally, we show that $F(T^*) = \{z - Tz : z \in E\}^{\perp}$. Let $S = I - T$, where $I : E \to E$ is the identity operator on E. If $x^* \in \{z \in E^* : S^*z = 0\}$, then we have

$$\langle Sy, x^*\rangle = \langle y, S^*x^*\rangle = 0$$

for any $y \in E$. This implies $x^* \in \{z - Tz : z \in E\}^{\perp}$. We know that $S^* = I^* - T^*$ and $\{z \in E^* : S^*z = 0\} = F(T^*)$. So, we have $F(T^*) \subset \{z - Tz : z \in E\}^{\perp}$. On the other hand, if $x^* \in \{z - Tz : z \in E\}^{\perp}$, then we have $\langle Sy, x^*\rangle = 0$ for all $y \in E$. Since

$$\langle y, S^*x^*\rangle = \langle Sy, x^*\rangle = 0$$

for all $y \in E$, we have $S^*x^* = 0$ and hence $x^* \in F(T^*)$. This implies $\{z - Tz : z \in E\}^{\perp} \subset F(T^*)$. Then, we have $F(T^*) = \{z - Tz : z \in E\}^{\perp}$. This completes the proof. □

Theorem 4.3. *Let E be a strictly convex, smooth and reflexive Banach space, let T be a linear contractive operator on E and let $\{S_n : n \in \mathbb{N}\}$ be a sequence of contractive linear operators on E such that $F(T) \subset F(S_n)$ for all $n \in \mathbb{N}$. Suppose $T \circ S_n = S_n \circ T$ for all $n \in \mathbb{N}$. Then, the following are equivalent:*

1. *S_nx converges to an element of $F(T)$ for each $x \in E$;*
2. *S_nx converges to 0 for each $x \in (JF(T))_{\perp}$;*
3. *$S_nx - T \circ S_nx$ converges to 0 for each $x \in E$.*

Furthermore, if (1) holds, then S_nx converges to $R_{F(T)}x \in F(T)$, where $R_{F(T)} = J^{-1}\Pi_{JF(T)}J$ and $\Pi_{JF(T)}$ is the generalized projection of E^* onto $JF(T)$.

Proof. Suppose (1). Then, for any $x \in E$, $S_nx \in R(x; F(S_n)) \subset R(x; F(T))$ for all $n \in \mathbb{N}$. We know from Lemma 3.7 that $R(x; F(T)) \cap F(T)$ consists of at most one point. Since $R(x; F(T))$ is closed and S_nx converges strongly to an element z of $F(T)$, $R(x; F(T)) \cap F(T) = \{z\}$. Let Rx be the unique element z of $R(x; F(T)) \cap F(T)$. Then, a mapping $R : E \to F(T)$ defined by $z = Rx$ is a retraction of E onto $F(T)$. Further, we know from Corollary 3.4 that $\langle x - S_nx, Ju\rangle = 0$ for all $u \in F(S_n)$ and $n \in \mathbb{N}$. So, we have

$$\langle x - Rx, Ju\rangle = 0 \tag{4.1}$$

From $Rx \in F(T)$, we also have $\langle x - Rx, JRx\rangle = 0$ and thus

$$\langle x - Rx, JRx - Ju\rangle = 0 \tag{4.2}$$

for any $u \in F(T)$. So, from Lemmas 2.3 and 2.4, R is the unique sunny generalized nonexpansive retraction of E onto $F(T)$. Therefore, from Theorem 2.6, we have

$$R = R_{F(T)} = J^{-1}\Pi_{JF(T)}J,$$

where $\Pi_{JF(T)}$ is the generalized projection of E^* onto $JF(T)$. If $x \in (JF(T))_\perp$, then we have $\langle x, Ju\rangle = 0$ for all $u \in F(T)$. From (4.1), we also have $\langle x-Rx, Ju\rangle = 0$ for all $u \in F(T)$. So, we get $\langle Rx, Ju\rangle = 0$ for all $u \in F(T)$. This implies $Rx \in (JF(T))_\perp$. From $Rx \in F(T) \cap (JF(T))_\perp$ and $F(T) \cap (JF(T))_\perp = \{0\}$, we have that $S_nx \to R_{F(T)}x = 0$ as $n \to \infty$. Then, we obtain (2).

Suppose (2). From Lemma 4.2, $JF(T)$ is a closed linear subspace of E^*. Then, we have from [2, 3, 20, 19] that for any $x \in E$,

$$x = R_{F(T)}x + P_{(JF(T))_\perp}x,$$

where $P_{(JF(T))_\perp}$ is the metric projection of E onto $(JF(T))_\perp$. So, we have from (2) that

$$\begin{aligned} S_nx &= S_n(R_{F(T)}x + P_{(JF(T))_\perp}x) \\ &= S_nR_{F(T)}x + S_nP_{(JF(T))_\perp}x \\ &= R_{F(T)}x + S_nP_{(JF(T))_\perp}x \\ &\to R_{F(T)}x \in F(T), \end{aligned}$$

as $n \to \infty$. Then, we obtain (1). Furthermore, we know from Corollary 3.4 that $x - Tx \in (JF(T))_\perp$ for all $x \in E$. So, we have from (2) that $S_n(x - Tx) \to 0$ as $n \to \infty$. So, we have from $T \circ S_n = S_n \circ T$ that for any $x \in E$,

$$\begin{aligned} S_nx - T \circ S_nx &= S_nx - S_n \circ Tx \\ &= S_n(x - Tx) \to 0, \end{aligned}$$

as $n \to \infty$. Then, we obtain (3).

Suppose (3). We have from (3) and $T \circ S_n = S_n \circ T$ that for any $x \in E$,

$$\begin{aligned} S_n(x - Tx) &= S_nx - S_n(Tx) \\ &= S_nx - S_n \circ T(x) \\ &= S_nx - T \circ S_n(x) \end{aligned}$$

$$\to 0.$$

So, we have $S_n y$ converges to 0 for any $y \in \{x - Tx : x \in E\}$. From Lemmas 4.2 and 4.1, we have

$$(JF(T))_\perp = (\{z - Tz : z \in E\}^\perp)_\perp = \overline{\text{span}}\{x - Tx : x \in E\}.$$

Take $x \in (JF(T))_\perp$. Then, for any $\epsilon > 0$, there exists an element $y \in \{x - Tx : x \in E\}$ such that $\|x - y\| < \epsilon$. So, we have

$$\begin{aligned} \|S_n x\| &= \|S_n y + (S_n x - S_n y)\| \\ &\le \|S_n y\| + \|S_n x - S_n y\| \\ &\le \|S_n y\| + \|x - y\| \\ &\le \|S_n y\| + \epsilon \end{aligned}$$

and hence

$$\limsup_{n\to\infty} \|S_n x\| \le \limsup_{n\to\infty} (\|S_n y\| + \epsilon) = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have that for any $x \in (JF(T))_\perp$, $S_n x$ converges to 0. Then, we obtain (2).

Furthermore, if (1) holds, then we have from the proof of (1) that for any $x \in E$, $S_n x$ converges strongly to $R_{F(T)} x \in F(T)$. □

Using Theorem 4.3, we have the following useful result.

Theorem 4.4. *Let E be a strictly convex, smooth and reflexive Banach space, let T be a linear contactive operator on E, let $\{T_i : i \in \mathbb{N}\}$ be a sequence of linear contractive operators on E such that $F(T) \subset F(T_i)$ for all $i \in \mathbb{N}$. Let $S_n = T_n \circ T_{n-1} \circ \cdots \circ T_1$ for all $n \in \mathbb{N}$ and suppose that $T \circ S_n = S_n \circ T$ for all $n \in \mathbb{N}$. Then, the following are equivalent:*

1. *$S_n x$ converges to an element of $F(T)$ for each $x \in E$;*
2. *$S_n x$ converges to 0 for each $x \in (JF(T))_\perp$;*
3. *$S_n x - T \circ S_n x \to 0$ for each $x \in E$;*

Furthermore, if (1) holds, then $S_n x$ converges to $R_{F(T)} x \in F(T)$, where $R_{F(T)} = J^{-1}\Pi_{JF(T)} J$ and $\Pi_{JF(T)}$ is the generalized projection of E^* onto $JF(T)$.

Proof. For any $n \in \mathbb{N}$, $S_n = T_n \circ T_{n-1} \circ \cdots \circ T_1$ is a linear contractive operator on E and $F(T) \subset F(S_n)$. for all $i \in \mathbb{N}$. Further, from the assumption, $T \circ S_n = S_n \circ T$ for all $n \in \mathbb{N}$. So, we have the desired result from Theorem 4.3 □

5. Applications

In this section, using Theorems 4.3 and 4.4, we obtain some strong convergence theorems for linear contractive mappings in a Banach space.

In 2003, Bauschk, Deutsch, Hundal and Park showed the following theorem [7].

Theorem 5.1. *Let T be a contractive linear operator on a Hilbert space H; i.e. $\|T\| \le 1$, and let M be a closed linear subspace of H. Consider the following statements;*

1. *$\lim_{n\to\infty} \|T^n x - P_M x\| = 0$ for each $x \in H$;*
2. *$M = F(T)$ and $T^n x$ converges to 0 for each $x \in M^\perp$;*

(3) *$M = F(T)$ and $T^n x - T^{n+1}x \to 0$ for each $x \in E$.*

Then, all statements are equivalent.

Using Theorem 4.3, we can obtain an extension of the above theorem to a Banach space.

Theorem 5.2. *Let E be a strictly convex, smooth and reflexive Banach space, let M be a closed linear subspace of E such that there exists a sunny generalized nonexpansive retraction R of E onto M and let T be a contractive linear operator on E. Then the following are equivalent:*

(1) *$T^n x$ converges to the element Rx of M for each $x \in E$;*
(2) *$M = F(T)$ and $T^n x$ converges to 0 for each $x \in (JM)_\perp$;*
(3) *$M = F(T)$ and $T^n x - T^{n+1}x \to 0$ for each $x \in E$.*

Furthermore, if (1) holds, then $R = R_{F(T)} = J^{-1}\Pi_{JF(T)}J$, where $\Pi_{JF(T)}$ is the generalized projection of E^* onto $JF(T)$.

Proof. If (1) holds, then it is obvious that $F(T) \subset M$. Conversely, take $z \in M$. Then we have $Rz = z$. Since $T^n z$ converges to the element $Rz = z$ and T is continuous, we have $T^{n+1}z$ converges to the element Tz. On the other hand, $T^{n+1}z$ converges to the element z. So, we have $Tz = z$. This implies $M \subset F(T)$. Then we get $M = F(T)$. Define $S_n = T^n$ for all $n \in \mathbb{N}$. Then, we have $F(T) \subset F(T_i)$ and $T \circ S_n = S_n \circ T$ for all $n \in \mathbb{N}$. So, we have the desired result from Theorem 4.3. □

Remark 5.3. If M is a closed linear subspace of a Hilbert space H, then there exists the metric projection P of H onto M. In a Hilbert space, the metric projection P of H onto M is coincident with the sunny generalized nonexpansive retraction R_M of H onto M.

Applying Theorem 4.4, we obtain a strong convergence theorem of Mann type for contractive linear mappings in a Banach space.

Theorem 5.4. *Let E be a smooth and uniformly convex Banach space and let T be a contractive linear operator on E. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then a sequence $\{x_n\}$ generated by $x_1 = x \in E$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n = 1, 2, 3, \dots,$$

converges strongly to the element Rx of $F(T)$, where $R = R_{F(T)} = J^{-1}\Pi_{JF(T)}J$ and $\Pi_{JF(T)}$ is the generalized projection of E^* onto $JF(T)$.

Proof. Let $T_i = \alpha_i I + (1-\alpha_i)T$ for all $i \in \mathbb{N}$, where I is the identity operator on E and let $S_n = T_n \circ T_{n-1} \circ \cdots \circ T_1$ for all $n \in \mathbb{N}$. Then, we have that $x_{n+1} = S_n x$. Since T is a linear cotractive operator, $F(T)$ is a closed linear subspace of E. For any $i \in \mathbb{N}$, we have $\|T_i\| \le 1$ and $F(T) = F(T_i)$. Indeed, we have

$$\|T_i\| = \|\alpha_i I + (1-\alpha_i)T\| \le \alpha_i\|I\| + (1-\alpha_i)\|T\| \le 1.$$

We show $F(T) = F(T_i)$. If $x \in F(T)$, then $T_i x = \alpha_{i-1}Ix + (1-\alpha_{i-1})Tx = x$ and hence $F(T) \subset F(T_i)$. Conversely, if $x \in F(T_i)$, then since $1-\alpha_{i-1} > 0$, we have

$$x = T_i x \Rightarrow x = \alpha_i Ix + (1-\alpha_i)Tx$$

$$\begin{aligned}&\Rightarrow (1-\alpha_i)x = (1-\alpha_i)Tx\\&\Rightarrow x = Tx.\end{aligned}$$

Then, we have $F(T_i) \subset F(T)$. Using these results, we obtain that $\|S_n\| \leq 1$ and $F(T) \subset F(S_n)$ for any $n \in \mathbb{N}$.

Next, we shall show that $T \circ S_n = S_n \circ T$. When $n = 1$, we have

$$\begin{aligned}T \circ S_1 &= T \circ T_1\\&= T\left(\alpha_1 I + (1-\alpha_1)T\right)\\&= \alpha_1 T + (1-\alpha_1)T^2\\&= T_1 \circ T = S_1 \circ T.\end{aligned}$$

Suppose that for some $k \in \mathbb{N}$, $T \circ S_k = S_k \circ T$. Then, we have

$$\begin{aligned}T \circ S_{k+1} &= T \circ T_{k+1} \circ S_k\\&= T\left(\alpha_k S_k + (1-\alpha_k)T \circ S_k\right)\\&= \alpha_k T \circ S_k + (1-\alpha_k)T^2 \circ S_k\\&= \alpha_k S_k \circ T + (1-\alpha_k)T \circ S_k \circ T\\&= T_{k+1} \circ S_k \circ T = S_{k+1} \circ T.\end{aligned}$$

Then, by induction, we have that $T \circ S_n = S_n \circ T$ for any $n \in \mathbb{N}$.

By Theorem 4.4, it is sufficient to show that

$$\|x_n - Tx_n\| \to 0 \text{ as } n \to \infty$$

for any $x_0 = x \in E$. Let $u \in M$. Then, for fixed $x_0 = x \in E$, we have

$$\begin{aligned}\|x_{n+1} - u\| &= \|\alpha_n x_n + (1-\alpha_n)Tx_n - u\|\\&\leq \alpha\|x_n - u\| + (1-\alpha_n)\|Tx_n - u\|\\&\leq \alpha\|x_n - u\| + (1-\alpha_n)\|x_n - u\|\\&= \|x_n - u\|.\end{aligned}$$

So, $\lim_{\to\infty} \|x_n - u\|$ exists. Putting $\lim_{\to\infty} \|x_n - u\| = c$, without loss of generality, we can assume that $c \neq 0$.

Using Lemma 2.9, we have that

$$\begin{aligned}\|x_{n+1} - u\| &= \|\alpha x_n + (1-\alpha_n)Tx_n - u\|\\&= \|\alpha(x_n - u) + (1-\alpha_n)(Tx_n - u)\|\\&\leq \|x_n - u\|\left\{1 - 2\alpha_n(1-\alpha_n)\delta\left(\frac{\|Tx_n - x_n\|}{\|x_n - u\|}\right)\right\}.\end{aligned}$$

Then, we obtain

$$2\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)\delta\left(\frac{\|Tx_n - x_n\|}{\|x_n - u\|}\right) \leq \|x_1 - u\| - c < +\infty.$$

From the assumpusions of $\{\alpha_n\}$, we have $\liminf_{n\to\infty} \delta\left(\frac{\|Tx_n - x_n\|}{\|x_n - u\|}\right) = 0$. Then we have

$$\liminf_{n\to\infty} \|Tx_n - x_n\| = 0.$$

On the other hand, we have

$$\begin{aligned}\|Tx_{n+1} - x_{n+1}\| &\le \alpha_n \|Tx_{n+1} - x_n\| + (1-\alpha_n)\|Tx_{n+1} - Tx_n\| \\ &\le \alpha_n \|Tx_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\le \alpha_n \|Tx_{n+1} - x_{n+1}\| + (1-\alpha_n)\|Tx_n - x_n\|.\end{aligned}$$

Then, we have $\|Tx_{n+1} - x_{n+1}\| \le \|Tx_n - x_n\|$. So, we obtain that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \liminf_{n\to\infty} \|Tx_n - x_n\| = 0.$$

By Theorem 4.4, $\{x_n\}$ converges strongly to the element Rx of $F(T)$, where $R = R_{F(T)} = J^{-1}\Pi_{JF(T)}J$ and $\Pi_{JF(T)}$ is the generalized projection of E^* onto $JF(T)$. This completes the proof. □

From Theorem 4.3, we can show a mean strong convergence theorem for contractive linear operators in a Banach space; see Yosida [59].

Theorem 5.5. *Let E be a smooth, strictly convex and reflexive Banach space and let T be a contractive linear operator on E. Then, for each $x \in E$, the Cesàro means*

$$S_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x$$

converge strongly to the element Rx of $F(T)$, where $R = R_{F(T)} = J^{-1}\Pi_{JF(T)}J$ and $\Pi_{JF(T)}$ is the generalized projection of E^* onto $JF(T)$.

Proof. For any $n \in \mathbb{N}$, the operator $S_n : E \to E$ is a contractive linear operator. Further, we have $F(T) \subset F(S_n)$ and $T \circ S_n = S_n \circ T$ for any $n \in \mathbb{N}$. In fact, for any $x \in E$ and $n \in \mathbb{N}$, we have

$$TS_n x = \frac{1}{n}\sum_{k=1}^{n} T^{k+1}x = \frac{1}{n}\sum_{k=1}^{n} T^k Tx = S_n Tx.$$

To complete the proof, it is sufficient to show that $S_n x - T \circ S_n x \to 0$ for each $x \in E$. We have

$$\begin{aligned}S_n x - T \circ S_n x &= \frac{1}{n}\sum_{k=1}^{n} T^k x - T\left(\frac{1}{n}\sum_{k=1}^{n} T^k x\right) \\ &= \frac{1}{n}\sum_{k=1}^{n} T^k x - \frac{1}{n}\sum_{k=2}^{n+1} T^k x \\ &= \frac{1}{n}\left(Tx - T^{n+1}x\right).\end{aligned}$$

Then, for any $n \in \mathbb{N}$, we have

$$\|S_n x - T \circ S_n x\| = \frac{1}{n}\|Tx - T^{n+1}x\| \le \frac{1}{n}(\|Tx\| + \|T^{n+1}x\|) = \frac{2}{n}\|Tx\|.$$

So, we obtain that $S_n x - T \circ S_n x \to 0$ for each $x \in E$. Using Theorem 4.3, $\{S_n x\}$ converges strongly to the element Rx of $F(T)$, where $R = R_{JF(T)} = J^{-1}\Pi_{JF(T)}J$ and $\Pi_{JF(T)}$ is the generalized projection of E^* onto $JF(T)$. This completes the proof. □

Remark 5.6. In Theorem 5.5, note that the point $z = \lim_{n\to\infty} S_n x$ is characterlized by the sunny generalized nonexpansive retraction $R = R_{F(T)} = J^{-1}\Pi_{JF(T)}J$ of E onto $F(T)$. Such a result is still new even if the operator T is linear.

In [10], Bruck introduced and discussed a firmly nonexpansive mapping in a Banach space. All norm one linear projections, all sunny nonexpansive retractions, and all resolvents of an accretive operator are firmly nonexpansive; see also [4]. Let E be a Banach space and let C be a nonempty closed convex subset of E. Then a mapping $T : C \to E$ is said to be firmly nonexpansive [10] if

$$\|t(x-y)+(1-t)(Tx-Ty)\| \geq \|Tx-Ty\|$$

for all $x, y \in C$ and $t \geq 0$. It E is smooth, it is not hard to check that a mapping $T : C \to E$ is firmly nonexpansive if and only if

$$\langle x - Tx - (y - Ty), J(Tx - Ty)\rangle \geq 0$$

for all $x, y \in C$, where J is the duality mapping on E. In a smooth Banach space, a linear operator $T : E \to E$ is firmly nonexpansive if it satisfies

$$\|Tx\|^2 \leq \langle x, JTx\rangle$$

for any $x \in E$. For any nonexpansive mappings S on a Hilbert space, the mapping $T = \frac{1}{2}(I + S)$ is firmly nonexpansive, where I is the identity mapping; see [17]. From Theorem 3.5, we have the following result.

Theorem 5.7. *Let E be a smooth Banach space and let T be a linear operator on E. If T is a firmly generalized nonexpansive type, then T is firmly nonexpansive.*

Finally, we prove strong convergence theorems for firmly nonexpansive linear operators in a Banach space.

Theorem 5.8. *Let E be a strictly convex, smooth and reflexive Banach space, let T be a linar firmly nonexpansive operator on E, let $F(T)$ be the set of all fixed points of T. Then, for any $x \in E$, $\lim_{n\to\infty} T^n x = R_{F(T)}x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction of E onto $R_{F(T)}$.*

Proof. From Ibaraki and Takahashi [27] and [28], we can define the sunny generalized nonexpansive retraction $R_{F(T)}$ of E onto $F(T)$. We also know from Reich and Shafrir [47] that if T is firmly nonexpansive, then for all $x \in E$ and $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = \lim_{n\to\infty} \frac{\|T^{n+k}x - T^n x\|}{k} = \lim_{n\to\infty} \left\| \frac{T^n x}{n} \right\|.$$

Since $\{T^n x\}$ is bounded for any $x \in E$, we have

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = \lim_{n\to\infty} \left\| \frac{T^n x}{n} \right\| = 0.$$

Then, by Theorem 5.2, we obtain $\lim_{n\to\infty} T^n x = R_{F(T)}x$ for any $x \in E$. □

From Theorem 3.10, we also obtain the following theorem.

Theorem 5.9. *Let E be a strictly convex, smooth and reflexive Banach space, let M be a closed hyperplane of E such that for some $z^* \in E^* \setminus \{0\}$*

$$M = \{z \in E : \langle z, z^* \rangle = 0\}$$

and let $\{T_n : n \in \mathbb{N}\}$ be a sequence of homogeneous nonexpansive mappings of E into itself with $F(T_n) = M$, $n \in \mathbb{N}$. For $x \in E$, define a sequence $\{x_n\}$ in E by

$$x_n = T_n \circ T_{n-1} \circ \cdots T_1 x.$$

Then $\{x_n\}$ converges strongly to an element $y \in M$ if and only if it converges weakly to $y \in M$.

Proof. It is sufficient to show that if $\{x_n\}$ converges weakly to $y \in M$, then it converges to $y \in M$ strongly. Let P_M be the metric projection of E onto M. Suppose $x_n \rightharpoonup y \in M$. If $x_m \in M$ for some $m \in \mathbb{N}$, then $x_n = x_m$ for any $n \geq m$. So, we have $x_n \to y \in M$. If $\{x_n\} \subset E \setminus M$, then we have from [49] that

$$P_M x_n = x_n - \frac{1}{\|z^*\|^2} \langle x_n, z^* \rangle J^{-1} z^*.$$

So, we have

$$\|x_n - P_M x_n\| = \frac{1}{\|z^*\|} |\langle x_n, z^* \rangle|.$$

Since $x_n \rightharpoonup y \in M = \{z \in E : \langle z, z^* \rangle = 0\}$, we have $\|x_n - P_M x_n\| \to 0$. From Theorem 3.10, $\{x_n\}$ converges to an element of M strongly. Then, we have $x_n \to y$. □

References

[1] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, 1996, 15–50.

[2] Ya. I. Alber, *Generalized Projections, Decompositions, and the Pythagorean-Type Theorem in Banach Spaces*, Appl. Math. Lett. **11** (1998), 115–121.

[3] Ya. I. Alber, *James orthogonality and orthogonal decompositions of Banach spaces*, J. Math. Anal. Appl. **312** (2005), 330–342.

[4] K. Aoyama, F. Kohsaka and W. Takahashi, *Three generalizations of firmly nonexpansive mappings: Their relations and continuous properties*, J. Nonlinear Convex Anal. **10** (2009), 131–147.

[5] K. Aoyama, F. Kohsaka and W. Takahashi, *Strongly relatively nonexpansive sequences in Banach spaces and applications*, in Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka Eds.), submitting.

[6] J. B. Baillon, R. E. Bruck and S. Reich, *On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces*, Houston J. Math. **4** (1978), 1–9.

[7] H. H. Bauschke, F. Deutsch, H. Hundal and S. H. Park, *Accelerating the convergence of the method of alternating projections*, Trans. Amer. Math. Soc. **355** (2003), 3433–3461.

[8] R. E. Bruck, *Nonexpansive retracts of Banach spaces*, Bull. Amer. Math. Soc. **76** (1970), 384–386.

[9] R. E. Bruck, *Properties of fixed-point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc. **179** (1973), 251–262.

[10] R. E. Bruck, *Nonexpansive projections on subsets of Banach spaces*, Paciffic J. Math. **47** (1973), 341–355.

[11] D. Butnariu, S. Reich, and A. J. Zaslavski, *Weak convergence of orbits of nonlinear operators in reflexive Banach spaces*, Numer. Funct. Anal. Optim. **24** (2003), 489–508.

[12] F. Deutsch, *Linear selections for the metric projection*, J. Funct. Anal. **49** (1982), 269–292.

[13] F. Deutsch, *A survey of metric selections*, Contemp. Math. **18** (1983), 49-71.

[14] F. Deutsch, *Best approximation in inner product spaces*, Springer Verlag, 2001.

[15] S. Dhompongsa, W. Fupinwong, W. Takahashi and J. C. Yao, *Fixed point theorems for nonlinear mappings and strict convexity of Banach spaces*, J. Nonlinear Convex Anal. **11** (2010), to appear.

[16] M.J. Fabian, P. Habala, P. Hajek and J. Pelant, *Functional Analysis and Infinite-Dimensional Geometry*, Springer Verlag, 2001.

[17] K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, 1990.

[18] H. Honda, T. Ibaraki and W. Takahashi, *Duality theorems and convergence theorems for nonlinear mappings in Banach spaces and applications*, Int. J. Math. Stat. **6** (2010), 46–64.

[19] T. Honda and W. Takahashi, *Norm One Projections and Generalized Conditional Expectations*, Sci. Math. Jpn. **69** (2009), 303–313.

[20] T. Honda and W. Takahashi, *Nonlinear projections and generalized conditional expectations in Banach spaces*, Taiwanese J. Math., to appear.

[21] H. Honda, W. Takahashi and J. C. Yao, *Nonexpansive retractions onto closed convex cones in Banach spaces*, Taiwanese J. Math. **14** (2010), 1023–1046.

[22] H. Hudzik, Y. Wang and R. Sha, *Orthogonally complemented subspaces in Banach spaces* , Numer. Funct. Anal. Optim. **29** (2008), 779–790.

[23] T. Ibaraki and W. Takahashi, *Convergence theorems for new projections in Banach spaces* (in Japanese), RIMS Kokyuroku **1484** (2006), 150–160.

[24] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory **149** (2007), 1–14.

[25] T. Ibaraki and W. Takahashi, *Weak convergence theorem for new nonexpansive mappings in Banach spaces and its applications*, Taiwanese J. Math. **11** (2007), 929–944.

[26] T. Ibaraki and W. Takahashi, *Strong convergence theorem by hybrid method for generalized resolvents of maximal monotone operators in Banach spaces*, J. Nonlinear Convex Anal. **9** (2008), 71–81.

[27] T. Ibaraki and W. Takahashi, *Fixed point theorems for nonlinear mappings of nonexpansive type in Banach spaces*, J. Nonlinear Convex Anal. **10** (2009), 21–32.

[28] T. Ibaraki and W. Takahashi, *Generalized nonexpansive mappings and a proximal-type algorithm in Banach spaces*, in Nonlinear Analysis and Optimization I: Nonlinear Analysis, Contemporary Mathematics, Vol. 513, Amer. Math. Soc., Providence, RI, 2010, pp. 169–180.

[29] T. Ibaraki and W. Takahashi, *Strong convergence theorems for a finite family of nonlinear operators of firmly nonexpansive type in Banach spaces*, in Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka Eds.), to appear.

[30] T. Ibaraki and W. Takahashi, *Strong convergence theorem by hybrid method for generalized resolvents of maximal monotone operators in Banach spaces*, J. Nonlinear Convex Anal. **9** (2008), 71–81.

[31] S. Itoh and W. Takahashi, *The common fixed point theory of singlevalued mappings and multivalued mappings*, Pacific J. Math. **79** (1978), 493–508.

[32] R. C. James, *Inner products in normed linear spaces*, Bull. Amer. Math. Soc. **53** (1947), 559–566.

[33] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.

[34] K. Kido and W. Takahashi, *Mean ergodic theorems for semigroups of linear operators*, J. Math. Anal. Appl. **103** (1984), 387–394.

[35] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstr. Appl. Anal. **2004** (2004), 239–249.

[36] F. Kohsaka and W. Takahashi, *Block iterative methods for a finite family of relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. **2007** (2007), Article ID 21972, 18 pp.

[37] F. Kohsaka and W. Takahashi, *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 197–209.
[38] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
[39] S. Matsushita and W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach space*, Fixed Point Theory Appl. **2004** (2004), 37–47.
[40] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in Banach space*, J. Approx. Theory **134** (2005), 257–266.
[41] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer–Verlag, Berlin–Heidelberg–New York, 1998.
[42] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372–379.
[43] B. Randrianantoanina, *Norm one projections in Banach spaces*, Taiwanese J. Math. **5** (2001), 35-95.
[44] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach space*, J. Math. Anal. Appl. **67** (1979), 274–276.
[45] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.
[46] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., 178, Dekker, New York, 1996, 313–318.
[47] S. Reich, I. Shafrir, *The asymptotic behavior of firmly nonexpansive mappings*, Proc. Amer. Math. Soc. **101** (1987), 246–250.
[48] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer–Verlag, Berlin–Heidelberg–New York, 1970.
[49] W. Song and Z. Cao, *The generalized decomposition theorem in Banach spaces and its applications*, J. Approx. Theory **129** (2004), 167–181.
[50] W. Takahashi, *Nonlinear Functional Analysis – Fixed Point Theory and Its Applications*, Yokohama Publishers, 2000.
[51] W. Takahashi, *Convex Analysis and Approximation of Fixed Points* (in Japanese), Yokohama Publishers, 2000.
[52] W. Takahashi, *Viscosity approximation methods for resolvents of accretive operators in Banach spaces*, J. Fixed Point Theory Appl. **1** (2007), 135–147.
[53] W. Takahashi, *Proximal point algorithms and four resolvents of nonlinear operators of monotone type in Banach spaces*, Taiwanese J. Math. **12** (2008), 1883–1910.
[54] W. Takahashi, *Viscosity approximation methods for countable families of nonexpansive mappings in Banach spaces*, Nonlinear Anal. **70** (2009), 719–734.
[55] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.
[56] W. Takahashi and J.-C. Yao, *Weak and strong convergence theorems for positively homogeneous nonexpansive mappings in Banach spaces*, Taiwanese J. Math. **15** (2011), to appear.
[57] W. Takahashi and J.-C. Yao, *Nonlinear operators of monotone type and convergence theorems with equilibrium problems in Banach spaces*, Taiwanese J. Math. **15** (2011), to appear.
[58] R. Wittmann, *Hopfs ergodic theorem for nonlinear operators*, Math. Ann. **289** (1991), 239–253.
[59] K. Yosida, *Mean ergodic theorem in Banach spaces*, Proc. Imp. Acad. Tokyo **14** (1938), 292–294.

Manuscript received May 31, 2010
revised December 14, 2010

Wataru Takahashi
Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan;
and
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan
E-mail address: wataru@is.titech.ac.jp

Jen-Chih Yao
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan
E-mail address: yaojc@math.nsysu.edu.tw

Takashi Honda
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan
E-mail address: honda@mail.math.nsysu.edu.tw