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EXISTENCE RESULTS FOR SYSTEMS OF GENERALIZED VECTOR VARIATIONAL INEQUALITIES WITH SET-VALUED SEMI-MONOTONE MAPPINGS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we introduce and study a new type of the system of generalized vector variational inequalities with set-valued semi-monotone mappings in reflexive Banach spaces. Under certain condition, some existence results for system of generalized vector variational inequalities with set-valued semi-monotone mappings are obtained by Kakutani-Fan-Glicksberg fixed point theorem.

1. Introduction

The vector variational inequality is a generalized of a variational inequality, having applications in different areas of optimization, optimal control, operations research, economics equilibrium and free boundary value problems. Giannessi [8] firstly introduced vector variational inequality (VVI) in a finite-dimensional Euclidean space. In 1987, Chen and Cheng [4] proposed the vector variational inequality in infinite-dimensional space and it was applied to some optimization problems. Recently, Huang and Fang [11] obtained some results for solutions of vector variational inequalities in reflexive Banach space. Since then, this problem has been a powerful tool in the study of vector optimization and traffic equilibrium problems; see [3, 4, 11, 16]. Due to its wide applications, and many existence results and algorithms for vector variational inequality problems have been established under various conditions (see, e.g. [1, 17] and the references therein).

It is well known that the monotonicity of a nonlinear mapping is one of the most frequently used hypotheses in the theory of the variational inequality. There are many kinds of generalized of monotone mappings in the literature of recent years, such as pseudo-monotone mappings, quasi-monotone mappings, etc. In 1999, Chen [5] introduced the concept of semi- monotonicity for a single-valued mapping, which occurred in the study of nonlinear partial differential equations of divergence type. Four years latter, a generalization of semi-monotonicity, the so called relaxed $\eta-\alpha$-semi-monotonicity was introduced by Fang and Huang [7] and a variational-like inequality problem related to it was studied. Recently, Fang [6] generalize the semi-monotonicity for a single-valued mapping to the case of a vector set-valued mapping,

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and investigate a generalized vector variational inequality problem related to this kind of vector set-valued mapping.

On the other hand, some interesting and important problems related to variational inequalities and complementarity problems were considered in recent papers. In 2003, Huang and Fang [10] introduced systems of order complementarity problems and established some existence results by fixed point theory. Kassay and Kolumbn [12] were introduced systems of variational inequalities and proved an existence theorem by used the Ky Fan lemma. Recently, Kassay et al. [13], introduced and studied Minty and Stampacchia variational inequality systems by the Kakutani-Fan-Glicksberg fixed point theorem. Very recently, in [20], Zhao and Xia introduced and studied systems of vector variational-like inequalities by the same fixed point theorem.

Motivated and inspired by these works, in this paper, we introduce and study a new type of systems of generalized vector variational inequalities with set-valued semi-monotone mappings in Banach spaces. By also using the Kakutani-Fan-Glicksberg fixed point theorem, we prove some existence results for system of generalized vector variational inequalities in Banach spaces.

2. Preliminaries

Let X and Y be two real Banach spaces, $L(X,Y)$ be the family of all linear bounded operators from X to Y, and K be a nonempty closed and convex subset of X. Recall that a subset C of Y is said to be a closed convex cone if C is closed and $C+C \subset C$, $\lambda C \subset C$ for $\lambda > 0$. In addition, if $C \neq Y$, then C is called a proper closed convex cone. A closed convex cone is pointed if $C \cap (-C) = \{0\}$. A mapping $C : K \longrightarrow 2^Y$ is said to be a cone mapping if $C(x)$ is a proper closed convex pointed cone and int $C(x) \neq \emptyset$ for each $x \in K$.

Next, we will introduce the concept of monotonicity and semi-monotonicity for set-valued mappings.

Definition 2.1 ([15])**.** Let $T : K \longrightarrow 2^{L(X,Y)}$ is said to be monotone on K if for any $x, y \in K$, it holds that

$$\langle \xi - \eta, y - x \rangle \in C_-, \ \forall \ \xi \in T(x), \ \eta \in T(y),$$

where $C_- = \bigcap_{x \in K} C(x)$.

Definition 2.2 ([6])**.** A vector set-valued mapping $T : K \times K \longrightarrow 2^{L(X,Y)}$ is said to be a vector set-valued semi-monotone mapping on K if it satisfies the following conditions:

(1) for each $u \in K$, the mapping $T(u, \cdot) : K \longrightarrow 2^{L(X,Y)}$ is a vector set-valued monotone mapping in the sense of Definition 2.1;
(2) for each $v \in K$, the mapping $T(\cdot, v) : K \longrightarrow 2^{L(X,Y)}$ is lower semi-continuous on K, where K is equipped with the weak topology, and $L(X,Y)$ is equipped with the uniform convergence topology of operators.

For more details see, for instances, [6].

Let $S, T : K \times K \longrightarrow 2^{L(X,Y)}$ be two set-valued semi-monotone mappings on K. In this paper, we investigate the following system of generalized vector variational inequality problem (for short, the (SGVVIP). Find $(x_0, y_0) \in K \times K$ such that for each $z \in K$ there exist $\xi \in S(y_0, x_0)$ and $\zeta \in T(x_0, y_0)$ satisfying

$$\tag{2.1} \begin{cases} \langle \xi, z - x_0 \rangle \notin \text{-int } C(x_0) \\ \langle \zeta, z - y_0 \rangle \notin \text{-int } C(y_0). \end{cases}$$

If $T(x, y) \equiv S(y, y)$, then problem (2.1) reduces to the generalized vector variational inequality problem (for short, the (GVVIP)); Find $y_0 \in K$ such that for each $z \in K$ there exist $\zeta \in S(y_0, y_0)$ satisfying

$$\tag{2.2} \langle \zeta, z - y_0 \rangle \notin \text{-int } C(y_0).$$

Problem (2.2) was introduced by Fang [6].
If $S, T : K \times K \longrightarrow L(X, Y)$, then problem (2.1) reduces to finding $x_0 \in K$ such that for each $z \in K$,

$$\tag{2.3} \begin{cases} \langle S(y_0, x_0), z - x_0 \rangle \notin \text{-int } C(x_0) \\ \langle T(x_0, y_0), z - y_0 \rangle \notin \text{-int } C(y_0). \end{cases}$$

If $S, T : K \times K \longrightarrow L(X, Y)$, $S \equiv T$ and $x_0 = y_0$, then problem (2.1) reduces to finding $x_0 \in K$ such that for each $z \in K$,

$$\tag{2.4} \langle S(x_0, x_0), z - x_0 \rangle \notin \text{-int } C(x_0).$$

Problem (2.4) was introduced by Zheng [19].

Definition 2.3 ([14])**.** A mapping $T : K \longrightarrow 2^{(X,Y)}$ is said to be u-hemi-continuous on K if the set-valued mapping $F : [0, 1] \longrightarrow 2^Y$ defined by

$$F(t) = \langle T(z + t(y - z)), y - z \rangle = \{\langle \xi, y - z \rangle : \ \xi \in T(z + t(y - z))\}$$

is upper semi-comtinuous at 0^+ for each $y, \ z \in K$.

Lemma 2.4 ([14])**.** *Let X, Y be Banach spaces, K a nonempty weakly compact convex subset of X. Let $C : K \longrightarrow 2^Y$ be such that for each $x \in K$, $C(x)$ is a proper closed convex cone with int$C(x) \neq \emptyset$ and $W : K \longrightarrow 2^Y$ be defined by $W(x) = Y \backslash - int\ C(x)$such that the graph of W is weakly closed in $X \times Y$. If $T : K \longrightarrow 2^{L(X,Y)}$ is a vector set-valued monotone mapping and is u-hemi-continuous on K with nonempty values, then there exists $x_0 \in K$ such that for each $y \in K$ there exists $\xi \in T(x_0)$ satisfying*
$\langle \xi, y - x_0 \rangle \notin -int\ C(x_0)$.

Lemma 2.5 ([14])**.** *Suppose that $T : K \longrightarrow 2^{L(X,Y)}$ is a vector set-valued monotone mapping. If it is u-hemi-continuous on K, then the following two statements are equivalent:*
(i) *$x_0 \in K$ such that, for each $y \in K$, there exists $\xi \in T(x_0)$ such that*
$\langle \xi, y - x_0 \rangle \notin -int\ C(x_0)$;
(ii) *$x_0 \in K$ such that, for each $y \in K$ and for each $\xi \in T(y)$, it holds that*
$\langle \xi, y - x_0 \rangle \notin -int\ C(x_0)$.

Lemma 2.6 ([18, Kakutani-Fan-Glicksberg]). *Suppose that X is a Hausdorff locally convex space and K is a nonempty convex compact subset of X. If $T : K \longrightarrow 2^K$ is an upper semi-continuous mapping with nonempty convex closed values, then T has a fixed point in K, i.e., there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.*

The next lemma is the property for an upper semi-continuous mapping.

Lemma 2.7 ([6]). *Let X, Y be two Banach spaces, $K \subset X$. Suppose that the set-valued mapping $T : K \longrightarrow 2^Y$ is upper semi-continuous at x_0 with $T(x_0)$ compact. If $x_n \in K,\ n = 1, 2, ...$ with $x_n \to x_0$, and $y_n \in T(x_n)$, then there exists $y_0 \in T(x_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$.*

3. Existence results for systems of generalized vector variational inequalities

In this section, we will prove two existence theorems for system of generalized vector variational inequalities in Banach spaces.

Theorem 3.1. *Let X be a real reflexive Banach space, Y a Banach space, K a nonempty bounded closed convex subset of X. Suppose that the mapping $C : K \to 2^Y$ is a cone mapping and the mapping $W : K \to 2^Y$ defined by $W(x) = F \setminus -int\ C(x)$ is weakly closed and satisfies $\lambda W(x) + (1-\lambda)W(y) \subset W(\lambda x + (1-\lambda)y)$. Let $S,\ T : K \times K \longrightarrow 2^{L(X,Y)}$ with nonempty convex compact values satisfies the following conditions:*

1. *S, T are set valued semi-monotone mapping on K;*
2. *for each $u \in K$, the mappings $S(u,\cdot) : K \longrightarrow 2^{L(X,Y)}$ and $T(u,\cdot) : K \longrightarrow 2^{L(X,Y)}$ are continuous on each finite dimensional subspace of X.*

Then the (SGVVI) has a solution in K.

Proof. Let F be a finite dimentional subspace of X such that $K_F = K \cap F \neq \emptyset$. For any $(x, y) \in K \times K$, consider the auxiliary problem: $(AP)_F$ Find $(x_0, y_0) \in K_F \times K_F$ such that for each $z \in K_F$ there exist $\xi_0 \in S(y, x_0)$ and $\zeta_0 \in T(x, y_0)$ satisfying

$$\begin{cases} \langle \xi_0, z - x_0 \rangle \notin \text{-int } C(x_0) \\ \langle \zeta_0, z - y_0 \rangle \notin \text{-int } C(y_0). \end{cases} \tag{3.1}$$

By condition (1) and (2), we know that $S(y,\cdot)$ and $T(x,\cdot)$ satisfy the condition of Lemma 2.4. It follows from of Lemma 2.4 that the problem $(AP)_F$ is solvable.
Define a multi-valued mapping $G : K_F \times K_F \longrightarrow 2^{K_F \times K_F}$ by

$$G(x, y) = \{(x_0, y_0) \in K_F \times K_F\ :\ (x_0, y_0) \text{ solve problem } (AP)_F\},\ \forall (x, y) \in K_F \times K_F.$$

Next, we will show that this mapping has at least one fixed point in K_F.
Step 1. It is clear that $G(x, y)$ is nonempty and bounded for each $(x, y) \in K_F \times K_F$.
Step 2. Show that $G(x, y)$ is convex for each $(x, y) \in K_F \times K_F$.
Let $(x_1, y_1),\ (x_2, y_2) \in G(x, y)$. Then by the definition of $G(x, y)$, we note for each $z \in K_F$ there exist $\xi_i \in S(y, x_i),\ i = 1, 2$ and $\zeta_j \in T(x, y_j),\ j = 1, 2$ satisfying

$$\begin{cases} \langle \xi_i, z - x_i \rangle \notin \text{-int } C(x_i),\ i = 1, 2 \\ \langle \zeta_j, z - y_j \rangle \notin \text{-int } C(y_j),\ j = 1, 2, \end{cases}$$

for all $z \in K_F$. By lemma 2.5, we have

$$\begin{cases} \langle \xi, z - x_i \rangle \notin \text{-int } C(x_i), \; i = 1,2 \\ \langle \zeta, z - y_j \rangle \notin \text{-int } C(y_j), \; j = 1,2, \end{cases}$$

for each $\xi \in S(y,z)$ and $\zeta \in T(x,z)$. Thus, for each $\lambda \in [0,1]$, we obtain

$$\begin{aligned} \langle \xi, z - (\lambda x_1 + (1-\lambda)x_2) \rangle &= \lambda \langle \xi, z - x_1 \rangle + (1-\lambda)\langle \xi, z - x_2 \rangle \\ &\in \lambda W(x_1) + (1-\lambda)W(x_2) \\ &\subset W(\lambda x_1 + (1-\lambda)x_2) \\ &= Y \setminus - int\ C(\lambda x_1 + (1-\lambda)x_2) \end{aligned}$$

and

$$\begin{aligned} \langle \zeta, z - (\lambda y_1 + (1-\lambda)y_2) \rangle &= \lambda \langle \zeta, z - y_1 \rangle + (1-\lambda)\langle \zeta, z - y_2 \rangle \\ &\in \lambda W(y_1) + (1-\lambda)W(y_2) \\ &\subset W(\lambda y_1 + (1-\lambda)y_2) \\ &= Y \setminus - int\ C(\lambda y_1 + (1-\lambda)y_2) \end{aligned}$$

By using Lemma 2.5 again, we note for each $z \in K_F$ that there exist $\bar{\xi} \in S(y, \lambda x_1 + (1-\lambda)x_2)$ and $\bar{\zeta} \in T(x, \lambda y_1 + (1-\lambda)y_2)$ such that

$$\begin{cases} \langle \bar{\xi}, z - (\lambda x_1 + (1-\lambda)x_2) \rangle \notin \text{-int } C(\lambda x_1 + (1-\lambda)x_2) \\ \langle \bar{\zeta}, z - (\lambda y_1 + (1-\lambda)y_2) \rangle \notin \text{-int } C(\lambda y_1 + (1-\lambda)y_2). \end{cases}$$

This mean that $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in G(x,y)$. Thus, we note that $G(x,y)$ is convex.

Step 3. Show that $G(x,y)$ is closed for each $(x,y) \in K_F \times K_F$.

Let $\{(x_n, y_n)\}$ be a sequence in $G(x,y)$ such that $(x_n, y_n) \longrightarrow (x_0, y_0)$. Then it follows from the definition of $G(x,y)$ that for each $z \in K_F$ there exist $\xi_n \in S(y, x_n)$ and $\zeta_n \in T(x, y_n)$ such that

$$\begin{cases} \langle \xi_n, z - x_n \rangle \notin \text{-int } C(x_n) \\ \langle \xi_n, z - y_n \rangle \notin \text{-int } C(y_n), \end{cases}$$

for all $n \in \mathbb{N}$. According to Lemma 2.7, there exist $\xi_0 \in S(y, x_0)$, $\zeta_0 \in T(x, y_0)$ and subsequences $\{\xi_{n_k}\}$ of $\{\xi_n\}$, $\{\zeta_{n_j}\}$ of $\{\zeta_n\}$ such that $\xi_{n_k} \longrightarrow \xi_0$ and $\zeta_{n_j} \longrightarrow \zeta_0$. Thus, letting $k \longrightarrow \infty$ and $j \longrightarrow \infty$, we get $\langle \xi_0, z - x_0 \rangle \notin$ -int $C(x_0)$ and $\langle \zeta_0, z - y_0 \rangle \notin$ -int $C(y_0)$ since W is weakly closed. Hence $(x_0, y_0) \in G(x,y)$ and therefore $G(x,y)$ is closed.

Step 4. Show that the mapping $G : K_F \times K_F :\longrightarrow 2^{K_F \times K_F}$ is upper semi-continuous. Since $K_F \times K_F$ is compact, we only need to show that the mapping $G : K_F \times K_F :\longrightarrow 2^{K_F \times K_F}$ is closed. Suppose that $(x_n, y_n) \in K_F \times K_F$ for all $n = 1, 2, 3, \ldots$ with $(x_n, y_n) \longrightarrow (x_0, y_0)$ and $(u_n, v_n) \in G(x_n, y_n)$ with $(u_n, v_n) \longrightarrow (u_0, v_0)$. We will show that $(u_0, v_0) \in G(x_0, y_0)$. It is clear from the definition of $G(x,y)$ that for each $z \in K_F$ there exist $\xi_n \in S(y_n, u_n)$ and $\zeta_n \in T(x_n, v_n)$ such that

$$\begin{cases} \langle \xi_n, z - u_n \rangle \notin \text{-int } C(u_n) \\ \langle \xi_n, z - v_n \rangle \notin \text{-int } C(v_n), \end{cases}$$

for all $n = 1,\ 2,\ 3, \ldots$. Thus for all $\varphi_n \in S(y_n, z)$ and $\phi_n \in T(x_n, z)$, we have

$$\begin{cases} \langle \varphi_n, z - u_n \rangle \notin \text{-int } C(u_n) \\ \langle \phi_n, z - v_n \rangle \notin \text{-int } C(v_n), \end{cases}$$

for all $n = 1,\ 2,\ 3, \ldots$. Since $S(\cdot, z)$ and $T(\cdot, z)$ are lower semi-continuous, for each $\varphi \in S(y_0, z)$ and $\phi \in T(x_0, z)$, there exist $\varphi_n \in S(y_n, z)$ and $\phi_n \in T(x_n, z)$ such that $\varphi_n \longrightarrow \varphi$ and $\phi_n \longrightarrow \phi$. Hence, letting $n \longrightarrow \infty$, it follows by the closedness of W that

$$\begin{cases} \langle \varphi, z - u_0 \rangle \notin \text{-int } C(u_0) \\ \langle \phi, z - v_0 \rangle \notin \text{-int } C(v_0). \end{cases}$$

By Lemma 2.5, there exist $\xi_0 \in S(y_0, u_0)$ and $\zeta_0 \in T(x_0, v_0)$ such that

$$\begin{cases} \langle \xi_0, z - u_0 \rangle \notin \text{-int } C(u_0) \\ \langle \zeta_0, z - v_0 \rangle \notin \text{-int } C(v_0). \end{cases}$$

Thus $(u_0, v_0) \in G(x_0, y_0)$. Therefore G is upper semi-continuous. By the Kakutani-Fan-Glicksberg fixed point theorem, there exist $(x_0, y_0) \in K_F \times K_F$ such that $(x_0, y_0) \in G(x_0, y_0)$. That is for each $z \in K_F$ there exist $\xi \in S(y_0, x_0)$ and $\zeta \in T(x_0, y_0)$ such that

$$\begin{cases} \langle \xi, z - x_0 \rangle \notin \text{-int } C(x_0) \\ \langle \zeta, z - y_0 \rangle \notin \text{-int } C(y_0). \end{cases}$$

Now, we generalize this result to the whole space.
Let $\Phi = \{M : M$ is a finite dimennsional subspace of X with $K_M = K \cap M \neq \emptyset\}$ and S_M be the solution set of the following problem: Find $(\bar{x}, \bar{y}) \in K \times K$ such that for each $z \in K_M$ there exist $\xi \in S(\bar{y}, \bar{x})$ and $\zeta \in T(\bar{x}, \bar{y})$ such that

$$\begin{cases} \langle \xi, z - \bar{x} \rangle \notin \text{-int } C(\bar{x}) \\ \langle \zeta, z - \bar{y} \rangle \notin \text{-int } C(\bar{y}). \end{cases}$$

From the previous discussion, we know that S_M is nonempty and bounded for all $M \in \Phi$. Let $\overline{S}^w_M$ denote the weak closure of S_M. Obviously, we have

$$S_{\bigcup_{i=1}^n M_i} \subset \bigcap_{i=1}^n S_{M_i} \subset \bigcap_{i=1}^n \overline{S}^w_{M_i}.$$

Since X is reflexive, we have $\overline{S}^w_M$ is weakly compact for all $M \in \Phi$. Thus $\{\overline{S}^w_M : M \in \Phi\}$ has the finite intersection property. It implies that $\bigcap_{M \in \Phi} \overline{S}^w_M \neq \phi$. Let $(x_0, y_0) \in \bigcap_{M \in \Phi} \overline{S}^w_M$. Then for each $z \in K_M$, there exist $\xi \in S(y_0, x_0)$ and $\zeta \in T(x_0, y_0)$ such that

$$\begin{cases} \langle \xi, z - x_0 \rangle \notin \text{-int } C(x_0) \\ \langle \zeta, z - y_0 \rangle \notin \text{-int } C(y_0). \end{cases}$$

Next, for any given $z \in K$, choose $M \in \Phi$ such that $z,\ x_0,\ y_0 \in K_M$. Since $(x_0, y_0) \in \overline{S}^w_M$, there exists $(x_n, y_n) \in S_M$ such that (x_n, y_n) converse weakly to

(x_0, y_0). Therefore for each $z \in K_M$ and for all $\xi_n \in S(y_n, z)$, $\zeta \in T(x_n, z)$, we have

$$\begin{cases} \langle \xi_n, z - x_n \rangle \notin \text{-int } C(x_n) \\ \langle \zeta_n, z - y_n \rangle \notin \text{-int } C(y_n). \end{cases}$$

Since $S(\cdot, z)$ and $T(\cdot, z)$ are lower semi-continuous, for each $\xi \in S(x_0, z)$ and $\zeta \in T(y_0, z)$ there exist $\xi_n \in S(y_n, z)$ and $\zeta_n \in T(x_n, z)$ such that $\xi_n \longrightarrow \xi$ and $\zeta_n \longrightarrow \zeta$. Letting $n \longrightarrow \infty$ and as w is weakly closed, we have

$$\begin{cases} \langle \xi, z - x_0 \rangle \notin \text{-int } C(x_0) \\ \langle \zeta, z - y_0 \rangle \notin \text{-int } C(y_0). \end{cases}$$

By Lemma 2.5, there exist $\xi_0 \in S(y_0, x_0)$ and $\zeta_0 \in T(x_0, y_0)$ such that

$$\begin{cases} \langle \xi_0, z - x_0 \rangle \notin \text{-int } C(x_0) \\ \langle \zeta_0, z - y_0 \rangle \notin \text{-int } C(y_0). \end{cases}$$

This complete the proof. □

If S and T are single value mappings, then we get the following corollary.

Corollary 3.2. *Let X be a real reflexive Banach space, Y a Banach space, K a nonempty bounded closed convex subset of X. Suppose that the mapping $C : K \to 2^Y$ is a cone mapping and the mapping $W : K \to 2^Y$ defined by $W(x) = F \setminus - int\ C(x)$ is weakly closed and satisfies $\lambda W(x) + (1 - \lambda)W(y) \subset W(\lambda x + (1 - \lambda)y)$. Let $S,\ T : K \times K \longrightarrow L(X, Y)$ satisfies the following conditions:*

(1) *$S,\ T$ are semi-monotone mapping on K;*
(2) *for each $u \in K$, the mappings $S(u, \cdot) : K \longrightarrow 2^{L(X,Y)}$ and $T(u, \cdot) : K \longrightarrow 2^{L(X,Y)}$ are continuous on each finite dimensional subspace of X.*

Then the (SGVVI) has a solution in K.

Next, we consider the system of generalized vector variational inequality problem in which K is an unbounded. We have the following result.

Theorem 3.3. *Let X be a real reflexive Banach space, Y a Banach space, K a nonempty unbounded closed convex subset of E. Suppose that the mapping $C : K \longrightarrow 2^Y$ is cone mapping and the mapping $W : K \longrightarrow 2^Y$ defined by $W(x) = F \setminus -intC(x)$ is weakly closed and $\lambda W(x) + (1 - \lambda)W(y) \subset W(\lambda x + (1 - \lambda)y)$. Let $S,\ T : K \times K \longrightarrow 2^{L(X,Y)}$ with nonempty convex compact values satisfies the following conditions:*

(1) *$S,\ T$ are set valued semi-monotone mapping on K;*
(2) *for each $u \in K$, the mappings $S(u, \cdot) : K \longrightarrow 2^{L(X,Y)}$ and $T(u, \cdot) : K \longrightarrow 2^{L(X,Y)}$ are continuous on each finite dimensional subspace of X;*
(3) *there exists $u_0 \in K$ such that if $(x_n, y_n) \in K \times K$ with $(x_n, y_n) \longrightarrow \infty$ as $n \longrightarrow \infty$, then for each n large enough it holds that $\exists \xi_n \in S(y_n, u_0)$ and $\zeta_n \in T(x_n, u_0)$ satisfying*

$$\begin{cases} \langle \xi_n, u_0 - x_n \rangle \in \textit{-int } C(x_n) \\ \langle \zeta_n, u_0 - y_n \rangle \in \textit{-int } C(y_n) \end{cases}$$

Then the (SGVVI) has a solution in K.

Proof. For each $n \in \mathbb{N}$, let $K_n = K \cap B(\theta, n)$, where $B(\theta, n)$ is the closed ball with center at θ and radius n. Hence, from Theorem 3.1, we note that there exists $(x_n, y_n) \in K_n \times K_n$ such that for each $z \in K_n$ there exists $\xi_n \in S(y_n, x_n)$ and $\zeta_n \in T(x_n, y_n)$ satisfying

$$\begin{cases} \langle \xi_n, z - x_n \rangle \notin \text{-int } C(x_n) \\ \langle \zeta_n, z - y_n \rangle \notin \text{-int } C(y_n). \end{cases}$$

By Lemma 2.5, for all $\varphi_n \in S(y_n, z)$ and $\phi_n \in T(x_n, z)$, we have

$$\begin{cases} \langle \varphi_n, z - x_n \rangle \notin \text{-int } C(x_n) \\ \langle \phi_n, z - y_n \rangle \notin \text{-int } C(y_n). \end{cases}$$

By conditin (iii), we know that $\{(x_n, y_n)\}$ is bounded. If not, without loss of generality, we assume that $(x_n, y_n) \longrightarrow \infty$. Thus for $z = u_0$, $\varphi_n \in S(y_n, u_0)$ and $\phi_n \in T(x_n, u_0)$, we have

$$\begin{cases} \langle \varphi_n, u_0 - x_n \rangle \notin \text{-int } C(x_n) \\ \langle \phi_n, u_0 - y_n \rangle \notin \text{-int } C(y_n). \end{cases}$$

This is a contradiction according to condition (iii). Thus $\{(x_n, y_n)\}$ is bounded. Without loss of generality, we assume that $(x_n, y_n) \longrightarrow^w (x_0, y_0)$. We shall show that (x_0, y_0) is the solution of the (SGVVI). Consider, for each $z \in K$ and each $\xi \in S(y_0, z)$ and $\zeta_n \in T(x_0, z)$, it follow from the lower semi-continuity of $S(\cdot, z)$ and $T(\cdot, z)$ that there exist $\xi_n \in S(y_n, z)$ and $\zeta \in T(x_n, z)$ such that $\xi_n \longrightarrow \xi$ and $\zeta_n \longrightarrow \zeta$ satisfying

$$\begin{cases} \langle \xi_n, z - x_n \rangle \notin \text{-int } C(x_n) \\ \langle \zeta_n, z - y_n \rangle \notin \text{-int } C(y_n). \end{cases}$$

Now, letting $n \longrightarrow \infty$, we have

$$\langle \xi_n, z - x_n \rangle \longrightarrow^w \langle \xi, z - x_0 \rangle$$

and

$$\langle \zeta_n, z - y_n \rangle \longrightarrow^w \langle \zeta, z - y_0 \rangle.$$

By the weak closedness of W, we obtain

$$\langle \xi, z - x_0 \rangle \notin \text{-int } C(x_0)$$

and

$$\langle \zeta, z - y_0 \rangle \notin \text{-int } C(y_0).$$

Using Lemma 2.5 again, we note that, for each $z \in K$, there exist $\xi_0 \in S(y_0, x_0)$ and $\zeta_0 \in T(x_0, y_0)$ satisfying

$$\begin{cases} \langle \xi_0, z - x_0 \rangle \notin \text{-int } C(x_0) \\ \langle \zeta_0, z - y_0 \rangle \notin \text{-int } C(y_0). \end{cases}$$

This complete the proof. $\square$

Corollary 3.4. *Let X be a real reflexive Banach space, Y a Banach space, K a nonempty unbounded closed convex subset of E. Suppose that the mapping $C : K \longrightarrow 2^Y$ is cone mapping and the mapping $W : K \longrightarrow 2^Y$ defined by $W(x) = F \setminus -intC(x)$ is weakly closed and $\lambda W(x) + (1-\lambda)W(y) \subset W(\lambda x + (1-\lambda)y)$. Let $S,\ T : K \times K \longrightarrow L(X, Y)$ satisfies the following conditions:*

(1) *$S,\ T$ are semi-monotone mapping on K;*
(2) *for each $u \in K$, the mappings $S(u, \cdot) : K \longrightarrow 2^{L(X,Y)}$ and $T(u, \cdot) : K \longrightarrow 2^{L(X,Y)}$ are continuous on each finite dimensional subspace of X;*
(3) *there exists $u_0 \in K$ such that if $(x_n, y_n) \in K \times K$ with $(x_n, y_n) \longrightarrow \infty$ as $n \longrightarrow \infty$, then for each n large enough it holds that $\exists \xi_n \in S(y_n, u_0)$ and $\zeta_n \in T(x_n, u_0)$ satisfying*

$$\begin{cases} \langle \xi_n, u_0 - x_n \rangle \in \text{-}int\ C(x_n) \\ \langle \zeta_n, u_0 - y_n \rangle \in \text{-}int\ C(y_n) \end{cases}$$

Then the (SGVVI) has a solution in K.

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