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SCHUADER'S CONJECTURE ON CONVEX METRIC SPACES

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Abstract. We first prove that the Schauder's conjecture holds for convex metric spaces, thus compact convex subsets of a CAT(0) space have the fixed point property for continuous mappings. We then obtain a continuous selection of a lower semi-continuous mapping with compact convex values defined on a compact convex subset of a convex metric space. Consequently, the Kakutani fixed point theorem is extended to a convex metric space.

1. Introduction

In this paper we mainly work on continuous selections, fixed points of single valued and multi-valued continuous mappings in metric spaces. One of the most famous results on selections known as Michael selection [11] states that:

Michael selection Let X be a Banach space and E a paracompact topological space. If $T : E \to 2^X \backslash \emptyset$ is a lower semi-continuous mapping with closed convex values. Then T has a continuous selection, i.e. there exists a continuous mapping $t : E \to X$ such that $tx \in Tx$, for each $x \in E$.

Various attempts had been made to modify or generalize this theorem. Examples of such results can be found in Park [14, 15] and referrences theirin. We want to point out one interesting observation from Yost [21] in which it had been shown that Lipschitz version of Michael selection was not possible, except for some finite dimensional cases.

One of the most resistant open problems in the theory of nonlocally convex linear metric spaces is:

Schauder's Conjecture Let E be a compact convex subset in a topological vector space. Then a continuous mapping $f : E \to E$ has a fixed point.

In [13] Nhu and Tri had shown that all Roberts spaces have fixed point property. We say that a metric space has the fixed point property for continuous mappings on compact convex sets if every continuous mapping from its compact convex subset into itself has a fixed point. In [12], Nhu continued his research toward the problem by introducing the notion of weak admissibility and proved that weakly admissible convex compact subsets have the fixed point property. Chen [3] solved this problem

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for convex continuous mappings. The problem was still open until 2001 when R. Cauty [2] finally gave to it an affirmative answer:

Theorem 1.1 ([2])**.** *Let E be an arbitrary convex subset of a topological vector space, every continuous mapping $f : E \to E$ such that $f(E)$ is contained in a compact subset of E (i.e. every relatively compact mapping $f : E \to E$) has the fixed point property.*

For multi-valued mappings, we have:

Kakutani Fixed Point Theorem Let E be a nonempty compact convex subset of R^n, and $T : E \to 2^E \backslash \emptyset$ be an upper semi-continuous mapping with compact convex values. Then T has a fixed point, i.e., a point x such that $x \in Tx$.

This theorem has been generalized to locally convex spaces by many authors to various types of mappings. See, e.g., Tychonoff [20], Brower [1], Fan [6], Glicksberg [7], Himelberg [8], Riech [16], [17].

Observe that the condition on the compactness of the domains of continuous mappings in the Schauder fixed point theorem can not be dropped. Indeed, this is the case even we consider only for Lipschitz mappings as shown in the following results (see also [4]):

Theorem 1.2 ([10])**.** *A convex set in a Banach space has the fixed point property for Lipschitz mappings if and only if it is compact.*

In this paper, we extend the Schauder fixed point theorem and the Kakutani fixed point theorem to the metric space setting.

2. Preliminaries and definitions

Let X be a topological space, E be a nonempty subset of X. A multi-valued mapping $T : E \to 2^X \backslash \emptyset$ is said to be *upper semi-continuous at* $x_0 \in E$ if for each neighborhood U of $T(x_0)$, there exists a neighborhood V of x_0 such that $T(x) \subset U$ for each $x \in V$, *lower semi-continuous at* $x_0 \in E$ if for each open set U such that $U \cap T(x_0) \neq \emptyset$, there exists a neighborhood V of x_0 such that $U \cap T(x) \neq \emptyset$ for each $x \in V$. T is *continuous* at $x_0 \in E$ if T is upper and lower semi-continuous at x_0. And T is *upper semi-continuous, lower semi-continuous, and continuous on* E if T is upper semi-continuous, lower semi-continuous, and continuous at each point of E, respectively. A mapping $t : E \to X$ is called a selection of the mapping T if $tx \in Tx$, for each $x \in E$. And a selection t is said to be a continuous selection if it is continuous.

An equivalence statement of being upper semi-continuous of a mapping T is that the mapping has a closed graph, i.e., for each sequence $\{x_n\}$ converging to x, for each $y_n \in Tx_n$ with $y_n \to y$, one has $y \in Tx$.

If E is a nonempty subset of a metric space (X, d), $x \in X$, we shall denote by $d(x, E) = \inf_{a \in E} d(x, a)$ the distance from the point x to the subset E, and $diam(E) = \sup_{a,b \in E} d(a, b)$ the diameter of E. If $\overline{E}$ is the closure of E, we can see that $d(x, E) = d(x, \overline{E})$, and $diam(E) = diam(\overline{E})$. Moreover, if E is compact, then there exists $a \in E$ such that $d(x,a) = d(x,E)$. Write $B(x,\epsilon)$ for the closed ball centered at x with radius ϵ.

Following Kirk [9], we suppose (X,d) is a metric space containing a family $\mathcal{L}$ of metric segments such that (a) each two points x,y in X are endpoints of exactly one member $[x,y]$ of $\mathcal{L}$ and (b) if $p,x,y \in X$ and if $m \in [x,y]$ satisfies $d(x,m) = \alpha d(x,y)$ for $\alpha \in [0,1]$, then

$$d(p,m) \leq (1-\alpha)d(p,x) + \alpha d(p,y). \tag{2.1}$$

Spaces of this type are said to be of hyperbolic type. Takahashi [19] called these spaces convex metric spaces, and for convenience, we shall follow his terminology throughout the rest of this paper. CAT(0) spaces as well as normed linear spaces and hyperconvex metric spaces are convex metric spaces. A subset of X is said to be *convex* if every segment in $\mathcal{L}$ joining two points in the set entirely lies in the set.

3. Main results

3.1. **Schauder's conjecture for convex metric spaces.** In this section we obtain the fixed point property for continuous mappings on compact convex subsets of a convex metric space. We introduce some more terminologies.

If $\alpha+\beta = 1$, $\alpha,\beta \in [0,1]$, and $x,y \in X$, we shall denote by $\alpha x \oplus \beta y$ the element m in $[x,y] \in \mathcal{L}$ such that $d(x,m) = \beta d(x,y)$ and $d(y,m) = \alpha d(x,y)$. Let $\{a_1,\dots,a_q\}$ be a subset of X. For $2 \leq n \leq q$, if $\sum_{i=1}^n \alpha_{in} = 1$, $\alpha_{in} \in [0,1]$, we write

$$\begin{aligned} \oplus_{i=1}^n \alpha_{in} a_i &= (1-\alpha_{nn})\Big(\frac{\alpha_{1n}}{1-\alpha_{nn}} a_1 \oplus \frac{\alpha_{2n}}{1-\alpha_{nn}} a_2 \oplus \cdots \oplus \frac{\alpha_{(n-1)n}}{1-\alpha_{nn}} a_{n-1}\Big) \oplus \alpha_{nn} a_n \\ &=: (1-\alpha_{nn}) k_n^n \oplus \alpha_{nn} a_n \qquad (3.1) \end{aligned}$$

as long as $\alpha_{nn} \neq 1$.

Thus, for examples,

$$\alpha_{12} a_1 \oplus \alpha_{22} a_2 \text{ is a point in the segment } [a_1,a_2] \text{ and}$$

$$\alpha_{13} a_1 \oplus \alpha_{23} a_2 \oplus \alpha_{33} a_3 = (\alpha_{13}+\alpha_{23})\Big(\frac{\alpha_{13}}{(\alpha_{13}+\alpha_{23})} a_1 \oplus \frac{\alpha_{23}}{(\alpha_{13}+\alpha_{23})} a_2\Big) \oplus \alpha_{33} a_3.$$

The defition of $\oplus$ in (3.1) is an ordered one in the sense that it depends on the order of points $a_1,\dots,a_q$. We can see that if A is a convex subset of X and $a_i \in A$, for each $i \in \{1,2,\dots,n\}$, then $\oplus_{i=1}^n \alpha_i a_i \in A$. Moreover, from relation (2.1), we have

$$d(\oplus_{i=1}^n \alpha_i a_i, x) \leq \sum_{i=1}^n \alpha_i d(a_i,x), \tag{3.2}$$

for each $x \in X$.

We need the operation $\oplus$ to satisfy the following condition:

(3.3) *If $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{x_n\}$ is a sequence in X such that $\alpha_n \to \alpha$, $x_n \to x$, for some $\alpha \in [0,1]$ and $x \in X$, then $\alpha_n x_n \oplus (1-\alpha_n) y \to \alpha x \oplus (1-\alpha) y$, for each $y \in X$.*

From its definition in terms of comparison triangles, we can see that the operation $\oplus$ on CAT(0) spaces satisfies condition(3.3).

For any metric space (X, d), there defines a natural embedding $i : X \to l^\infty(X)$ by $i(x) = (d(x, y) - d(x_0, y))_{y\in X}$, where $x_0 \in X$ is fixed. Thus X can be isometrically embedded into the space $l^\infty(X)$ under supremum norm $\|\cdot\|_\infty$. For each finite subset $\{v_1, \dots, v_i\}$ of X, write

$$co\langle v_1, \dots, v_i\rangle = \Big\{ \oplus_{j=1}^i \alpha_j v_j : \{\alpha_1, \alpha_2, \dots, \alpha_i\} \subset [0, 1], \sum_{j=1}^{i} \alpha_j = 1\Big\}$$

as a subset of X, and

$$co(v_1, \dots, v_i) = \Big\{ \sum_{j=1}^{i} \alpha_j v_j : \{\alpha_1, \alpha_2, \dots, \alpha_i\} \subset [0, 1], \sum_{j=1}^{i} \alpha_j = 1\Big\}$$

as a subset of $l^\infty(X)$.

Define a mapping

$$\beta : co(v_1, \dots, v_i) \to co\langle v_1, \dots, v_i\rangle$$

by

$$\beta\Big(\sum_{j=1}^{i} \alpha_j v_j\Big) = \oplus_{j=1}^i \alpha_j v_j. \tag{3.4}$$

By condition(3.3), it can be shown that β is continuous.

Theorem 3.1. *Let E be a compact convex subset of a convex metric space X. Then E has the fixed point property for continuous mappings.*

Proof. Since E and $i(E)$ are isometric, it suffices to prove that $i(E)$ has the fixed point property for continuous mappings. Let $t : i(E) \to i(E)$ be continuous and let $\lambda > 0$. Since $i(E)$ is compact, there exists a finite subset $\{a_1, a_2, \dots, a_p\}$ of $i(E)$ such that $i(E) \subset \bigcup_{i=1}^p B(a_i, \lambda/8)$. Consider the convexhull $co(a_1, \dots, a_p)$. At each point a_i (latter will be called a vertex), draw a ball centered at a_i with radius $\lambda/4$. Color each of these balls in red. Draw segments (later will be called edges) of the form $[a_i, a_j]$ whenever $d(a_i, a_j) \le \lambda/2$, and color these edges in red. Moreover, if the convexhull $co(a_{i1}, \dots, a_{ij})$ (later called a red subface) for some $a_{ik} \in \{a_1, a_2, \dots, a_p\}, k = 1, \dots, j$, has all edges colored in red, we color the convexhull in red as well. Denote such the maximal convexhull by $C(A)$, where A is some set of vertices, and call it a component generated by A. Clearly, its diameter is at most $\lambda/4 + \lambda/2 + \lambda/4 = \lambda$. Let $\{C(A_1), \dots, C(A_q)\}$ be all distinct components where $A_i \subset \{a_1, \dots, a_p\}$ for each $i = 1, \dots, q$. Denote the set of points of intersection of edges and faces of $C(A_i)$ and $C(A_j)$ by $\{b_{ij}^1, \dots, b_{ij}^{k_{ij}}\}$ if it exists. Thus $\{b_{ij}^1, \dots, b_{ij}^{k_{ij}}\}$ is exactly the points of intersection of edges and faces of $co(A_i)$ and $co(A_j)$. On each $co(A_i) = co(v_1, \dots, v_m)$, each point of intersection b_{ij} lies on the boundary (with respect to the topological subspace $co(a_1, \dots, a_p)$) of $co(A_i)$. We can define a continuous function α from $co(a_1, ..., a_p)$ onto itself such that the restriction on $\bigcup_{i=1}^q C(A_i)$ is a retraction onto the union $\bigcup_{i=1}^q co(A_i)$. So

α sends each ball $B(a_i, \lambda/4)$ into itself. For examples, an isolated ball $B(a_i, \lambda/4)$ (i.e., $B(a_i, \lambda/4) \cap B(a_j, \lambda/4) = \emptyset$ for all $j \neq i$) is sent to its center, and an isolated component $C(A_i)$ (i.e., $C(A_i) \cap C(A_j) = \emptyset$ for all $j \neq i$) is sent to $co(A_i)$. Thus, we observe that

(3.5) $d(x, \alpha x) \leq \lambda/4$ *for each point* x *in the red region.*

This follows from the fact that each point x in the ball $B(a, \lambda/4)$ satisfies $d(x, a) \leq \lambda/4$. We are going to map each point x in set $co(A)$ to βx in $i(E)$ so that $d(x, \beta x)$ is sufficiently small. Triangulate $\bigcup_{i=1}^q co(A_i)$ by induction on i using all vertices $\{a_1, \ldots, a_p\}$. Then extend this set of simplices to obtain a complete triangulation $\{\triangle^1, \ldots, \triangle^r\}$ of $co(a_1, \ldots, a_p)$. Order the vertices of $co(A_1), \ldots, co(A_q)$ in such a the way that the order of the vertices of faces of $\triangle^j$ common to the ones of $\triangle^i$ for $j < i$ are unchanged. This can be easily done by induction. If $b_{ij} = \sum \alpha_k v_{i_k}$ lies in a face $co(v_{i_1}, \ldots, v_{i_m})$ of $\triangle^i$, put $\overline{b}_{ij} = \oplus \alpha_k v_{i_k}$. Thus $\overline{a}_i = a_i$ for each i. Note by (3.2) that $d(v_{k_0}, \overline{b}_{ij}) \leq \lambda/2$ for each $k_0 \in \{i_1, \ldots, i_m\}$. Therefore, for b_{ij} and b_{ik} in a face of $\triangle^i$, we have

$$d(\overline{b}_{ij}, \overline{b}_{ik}) \leq d(\overline{b}_{ij}, v_{k_0}) + d(v_{k_0}, \overline{b}_{ik}) \leq \lambda/2 + \lambda/2 = \lambda, \tag{3.6}$$

where v_{k_0} is a vertex of $\triangle^i$.
Refine the triangulation $\{\triangle^1, \ldots, \triangle^r\}$ of $co(a_1, \ldots, a_q)$ by using intersection points b_{ij}'s appearing in each simplex in $\{\triangle^1, \ldots, \triangle^r\}$. Let $\{\triangle^1, \ldots, \triangle^s\}$ be a complete triangulation of $co(a_1, \ldots, a_q)$ using all vertices $\{a_1, \ldots, a_q\}$ and all b_{ij}'s.

Define a mapping β on $co(a_1, \ldots, a_p)$ so that the restriction of β over $\triangle^i = co(v_{i1}, \ldots, v_{in_i})$ is the mapping defined by

$$\beta\Big(\sum_{j=1}^{n_i} \alpha_j v_{ij}\Big) = \oplus_{j=1}^{n_i} \alpha_j \overline{v}_{ij}. \tag{3.7}$$

As above, we can show that β is continuous. Moreover,

$$d(x, \beta x) \leq 3\lambda/2 \text{ for each } x \text{ that lies in } \bigcup_{i=1}^{q} co(A_i). \tag{3.8}$$

To verify (3.8), we let $x \in \triangle^j \subset co(A_i)$ for some i and j. Let $x = \sum \alpha_k u_{i_k}$, where for each k, u_{i_k} is a vertex in $A_i = \{v_{i1}, \ldots, v_{in_i}\}$ or a point of intersection b_{ij} lying in a face $\triangle^j$. Thus (3.6) implies that

$$\begin{aligned} d(x, \beta x) &\leq d(x, u_{i_{k_0}}) + \sum_k \alpha_k d(u_{i_{k_0}}, u_{i_k}) \\ &\leq \lambda/2 + \lambda = 3\lambda/2. \end{aligned}$$

Let $\pi : i(E) \to co(a_1, \ldots, a_p)$ be the mapping defined by $\pi(x) = co(\{y \in co(a_1, \ldots, a_p) \cap i(E) : d(x, y) = d(x, co(a_1, \ldots, a_p) \cap i(E))\})$. Note by the compactness of $co(a_1, \ldots, a_p) \cap i(E)$ that $\pi(x) \neq \emptyset$. Note also that $d(x, y) \leq \lambda/8$ for all $y \in \pi(x)$. Clearly, $\pi(x)$ is compact and convex and $\pi(x) = \{x\}$ for $x \in co(a_1, \ldots, a_p) \cap i(E)$. Using the fact that each neighborhood of $\pi(x)$ contains a neighborhood of the form $N_\epsilon(\pi(x)) := \bigcup_{y \in \pi(x)} B(y, \epsilon)$, it is straightforward to show that π is upper semi-continuous. For each $x \in i(E)$, $\pi(x)$ lies in the red region. For suppose $x \in B(a_i, \lambda/8)$ for some i and for each $y \in \pi(x)$, we have $d(x, y) \leq d(x, a_i) \leq \lambda/8$ and hence $d(a_i, y) \leq d(a_i, x) + d(x, y) \leq \lambda/8 + \lambda/8 = \lambda/4$. Therefore $\pi(x) \subset B(a_i, \lambda/4)$ which is a part of the red region.

Now the mapping $\pi t\beta\alpha$ is upper semi-continuous on $co(a_1, \ldots, a_p)$ and has compact convex values. Kakutani fixed point theorem then guarantees that $x \in \pi t\beta\alpha x$ for some x. Since $t\beta\alpha x \in i(E)$, so $\pi t\beta\alpha x$ lies in the red region and hence x lies in the red region.

Now, by (3.5) and (3.8), $d(t\beta\alpha x, \beta\alpha x) \leq d(t\beta\alpha x, x) + d(x, \alpha x) + d(\alpha x, \beta\alpha x) \leq \lambda/8 + \lambda/4 + 3\lambda/2 = 15\lambda/8$. Finally, as λ is arbitary and E is compact, t has a fixed point. □

For R-trees, and for nonexpansive mappings, the domain of a continuous mapping can only assume to be closed and convex. For example, Espinola and Kirk [5] had shown that:

Theorem 3.2 ([5])**.** *Let (X, d) be a complete R-tree, and suppose E is a closed convex subset of X which does not contain a geodesic ray. Then every commuting family $\Im$ of nonexpansive mappings of $E \to E$ has a nonempty common fixed point set.*

This result was extended by Shahzad [18] by proving that:

Theorem 3.3 ([18])**.** *Let X be a nonempty geodesically bounded closed convex subset of a complete R-tree. $\Im$ a commuting family of nonexpansive self-mappings of X, and $T : X \to 2^X$ almost lower semi-continuous, where for any $x \in X$, Tx is nonempty closed bounded and convex. If $\Im$ and T commute weakly, then there exists an element $z \in X$ such that $z = t(z) \in T(z)$ for all $t \in \Im$.*

As a corollary of Theorem 3.1 we autometically have the following result:

Corollary 3.4. *Compact convex subsets of a CAT(0) space have the fixed point property for continuous mappings.*

3.2. **Continuous selections.** The following known result plays a major role in the proof of our main result in this section:

Lemma 3.5. *Let X be a metric space and $\psi : X \to \mathbb{R}$ be an upper semi-continuous mapping. Then there exits a decreasing sequence of continuous mappings $\{\phi_n\}$ conveinges pointwise to ψ.*

Remark 3.6. It is easy to see that if $|\psi| < M$ on a compact subset of X, then $|\phi_n| < M$ on that set for all large n.

We now present a continuous selection theorem for mappings which are lower semi-continuous:

Theorem 3.7. *Let E be a nonempty compact convex subset of a convex metric space (X, d), and $T : E \to 2^E \backslash \emptyset$ be a lower semi-continuous mapping with compact convex values. Then there exists a continuous function $t : E \to E$ such that $tx \in Tx$, for each $x \in E$.*

Proof. Given $\varepsilon > 0$. Since E is compact, so there exists a finite subset $\{z_1, z_2, \dots, z_N\}$ of E such that

$$E \subset \cup_{i=1}^{N} B(z_i, \varepsilon/4). \tag{3.9}$$

Define $\psi_{1i}(x) = d(z_i, Tx)$ for $x \in E$ and for $i = 1, \dots, N$. Since T is lower semi-continuous, it is easy to see that ψ_{1i} is an upper semi-continuous mapping.

Observe that the set $E_{1i} := \{x \in E : \psi_{1i}(x) < \varepsilon\}$ is open, and it is seen by (3.9) that $E = \cup_{i=1}^{N} E_{1i}$. Writing E_{1i} as a union of open balls in E we see by the compactness of E that $E \subset \cup_{j=1}^{M} B(w_{1j}, r_{1j}) = \cup_{n=1}^{\infty} \cup_{j=1}^{M} B(w_{1j}, r_{1j} - 1/n)$. Again by compactness, $E \subset \cup_{j=1}^{M} \overline{B}(w_{1j}, r_{1j} - 1/n_0) = \cup_{i=1}^{N} F_{1i}$, for some n_0, where F_{1i} is closed in X and $F_{1i} \cap E \subset E_{1i}$. Remark 3.6 provides us the existence of a continuous mapping φ_1 satisfying $\varphi_1 \geq \psi_{11}$ and $\varphi_1 < \varepsilon$ on $F_{11} \cap E$.

Define $S_1 x = \overline{B}(z_1, \varphi_1(x) + \varepsilon) \cap Tx$. Notice that $S_1 x$ is compact and convex for each x. We now show that S_1 is lower semi-continuous.

Let $\{x_n\}$ be a sequence in E converging to some $x \in E$. Let $y \in S_1 x$ and choose $\omega_n \in Tx_n$ such that $d(y, Tx_n) = d(y, \omega_n)$ for each n. Since Tx is convex, we may assume that $d(z_1, y) < \varphi_1(x) + \varepsilon$. Since T is lower semi-continuous, and since $\omega_n \longrightarrow y$, we have $d(z_1, \omega_n) \leq \varphi_1(x) + \varepsilon$ for all large n. Consequencely $\omega_n \in S_1 x_n$ for all large n. Thus for those n, $d(y, S_1 x_n) \leq d(y, \omega_n)$, and $d(y, S_1 x_n) \longrightarrow 0$ as desired.

Consider a closed set $E_2 := \{x \in E : \varphi_1(x) \geq \varepsilon\}$. Note that

$$E_2 \subset \cup_{i=2}^{N} \{x \in E : \psi_{1i}(x) < \varepsilon\}. \tag{3.10}$$

The validity of (3.10) follows from the observation that $E_{1i} = \cup_{j=1}^{N} (F_{1j} \cap E_{1i})$. Define upper semi-continuous mappings $\psi_{2i}(x) = d(z_i, S_1 x)$ for $x \in E$ and for $i = 2, \dots, N$. The sets $E_{2i} := \{x \in E : \psi_{2i}(x) < \varepsilon\}$ is open for each $i = 2, \dots, N$. Moreover, $E_2 = \cup_{i=2}^{N} E_{2i}$. To see this, let $x \in E_2$. By (3.10) we see that $\psi_{1i}(x) < \varepsilon$, for some $i = 2, \dots, N$. If for this i, $\psi_{2i}(x) \geq \varepsilon$, then $B(z_i, \varepsilon) \cap Tx \cap B(z_1, \varphi_1(x) + \varepsilon)^c \neq \emptyset$. Therefore by (3.9) and by convexity of Tx there must be some $j \neq i$ such that $\psi_{2j}(x) < \varepsilon, i.e., x \in E_{2j}$. Since E_2 is compact, the above argument garuantees that $E_2 \subset \cup_{i=2}^{N} F_{2i}$, where F_{2i} is closed in X and $F_{2i} \cap E \subset E_{2i}$. Choose a continuous mapping φ_2 such that $\varphi_2 \geq \psi_{22}$ and $\varphi_2 < \epsilon$ on $F_{22} \cap E$.

Then define $S_2 x = \overline{B}(z_2, \varphi_2(x) + \epsilon) \cap S_1 x$. Observe that $S_2 x$ is compact and convex for each x and S_2 is lower semi-continuous.

By induction, we can define mappings $S_i, i = 1, \dots, N$, such that

$$Tx =: S_0 x \supset S_1 x \supset S_2 x \supset \cdots \supset S_N x,$$

for each $x \in E$.

Write $T_\varepsilon = S_N$. Since for each $x \in E$, if $\varphi_i(x) < \varepsilon$, then $T_\varepsilon x \subset S_i x \subset \overline{B}(z_i, \varphi_i(x) + \varepsilon) \subset \overline{B}(z_i, 2\varepsilon)$. Therefore $diam(T_\varepsilon x) \leq 4\varepsilon$, for each $x \in E$.

Taking $\varepsilon = 1/n$ and write T_n for $T_{1/n}$ so that $T_n x \subset T_{n-1} x$ for all n and x. Since $diam(T_n x) \leq 4/n$, $\cap_n T_n x = \{t_x\}$ is a singleton for each x.

Define $t : E \to 2^E \backslash \emptyset$ by $x \mapsto t_x$. Clearly $tx \in Tx$ for each $x \in E$. It remains to show that t is continuous. Let $\varepsilon > 0$ and let $\{x_n\}$ be a sequence in E with $x_n \to x_0$ for some $x_0 \in E$. Choose $n_0 \in \mathbb{N}$ so that $8/n < \varepsilon$. Since $tx_0 \in T_{n_0} x_0$ and T_{n_0} is lower semi-continuous, there exists $n_1 \in \mathbb{N}$ such that $d(tx_0, T_{n_0} x_n) < \varepsilon/2$, for each

$n \geq n_1$. Let $\{a_n\}$ be in X such that $a_n \in T_{n_0}x_n$, and

$$d(tx_0, a_n) = d(tx_0, T_{n_0}x_n),$$

for each $n \in \mathbb{N}$. Hence

$$\begin{aligned} d(tx_0, tx_n) &\leq d(tx_0, a_n) + d(a_n, tx_n) \\ &\leq d(tx_0, T_{n_0}x_n) + diam(T_{n_0}x_n) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

for each $n \geq n_1$. Thus $tx_n \to tx_0$. So we conclude that t is continuous. □

It is a common argument showing that an upper semi-continuous mapping can be approximated by lower semi-continuous mappings. Theorem 3.7 and Theorem 3.1 combine give us the Kakutani fixed point theorem on CAT(0) spaces:

Corollary 3.8. *Let E be a nonempty compact convex subset of a CAT(0) space X. Let $T : E \to 2^X \backslash \emptyset$ be an upper semi-continuous mapping with compact convex values. Then T has a fixed point.*

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