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UNIFIED SCALARIZATION FOR SETS AND SET-VALUED KY FAN MINIMAX INEQUALITY

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Dedicated to the memory of Professor Ky Fan

Abstract. In the paper, we investigate some properties of nonlinear scalarizing functions for sets introduced by Kuwano, Tanaka, Yamada (2009) and prove four kinds of set-valued Ky Fan minimax inequality.

1. Introduction

Ky Fan minimax inequality in [1] is one of the important results in convex analysis as well as nonlinear analysis and it has many applications in those area. Recently, Georgiev and Tanaka [2, 3] generalize an equivalent form of scalar Ky Fan minimax inequality into set-valued four cases by using two types of nonlinear scalarizing functions for sets which are extensions of sublinear scalarizing functions for vectors used in [4]. The aim of this paper is to prove four kinds of set-valued Ky Fan minimax inequality as close to original as possible by using recent scalarizing method for sets.

In [14], Nishizawa, Tanaka and Georgiev introduce four types of nonlinear scalarizing functions for sets containing the above two scalarizing functions for sets, and observe sevaral properties of those scalarizing functions. Moreover, in [16], Shimizu, Nishizawa, and Tanaka obtain several optimality conditions for set-valued optimization problems via these four types of nonlinear scalarizing functions. On the other hand, Hamel and Löhne [7] define different types of scalarizing functions for sets, which evaluate a nonempty set as a real number or $\pm\infty$ by a certain comparison with a given reference set along a given direction based on set-relations introduced in [10]. By using these functions, they show generalized results on Ekeland's variational principle in an abstract space like topological vector space without such strong assumption as convexity. Moreover, a modified scalarizing function proposed in [17] gives a similar result to a minimal element theorem in [7] under different assumptions. Hernández and Marín [8] show two existence theorems of solutions for set-valued optimization problems by using these scalarizing functions.

Based on the approach of Hamel and Löhne, and on six kinds of set-relation introduced in [10], we propose twelve types of scalarizing functions for sets in [11]. These functions can be regarded as extensions of the four functions in [14] and Hamel and Löhne type functions in [7], and so we call them unified types of scalarizing

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functions for sets. In addition, in [11, 12], we propose the concavity of set-valued maps and characterize unified types of scalarizing functions. In this paper, we show several properties of those functions, which are used in order to apply the scalar Ky Fan minimax inequality in Section 4.

The organization of the paper is as follows. In Section 2, we introduce some basic concepts in set-relations. Also, we introduce definitions of convexity and continuity for set-valued maps. In Section 3, we introduce the definitions of unified types of scalarizing functions for sets, and investigate some properties of them. In Section 4, we show four kinds of set-valued Ky Fan minimax inequality.

2. Mathematical preliminaries

Throughout the paper, X and Y are a real topological vector space and a Euclidean space, that is, a finite dimensional real topological vector space, respectively. We assume that C is a proper closed convex cone in Y (that is, $C \neq Y$, $C + C = C$ and $\lambda C \subset C$ for all $\lambda \geq 0$) with nonempty topological interior. We define a partial ordering $\leq_C$ on Y as follows:

$$x \leq_C y \quad \text{if} \quad y - x \in C \quad \text{for} \quad x, y \in Y.$$

Let F be a set-valued map from $S \subset X$ into 2^Y where $S := \{x \in X | F(x) \neq \emptyset\}$ and assume that S is a convex set. For $A \in 2^Y \setminus \{\emptyset\}$, we denote the topological interior, topological closure, complement of A by int$A$, cl$A$, A^c, respectively. Also, we denote the algebraic sum, algebraic difference of A and B in Y by $A + B := \{a + b | a \in A, b \in B\}$, $A - B := \{a - b | a \in A, b \in B\}$, respectively. In addition, we denote the composite function of two functions f and g by $g \circ f$. When $x \leq_C y$ for $x, y \in Y$, we define the order interval between x and y by $[x, y] := \{z \in Y | x \leq_C z \text{ and } z \leq_C y\}$. When we choose $k \in \text{int}C$, we can construct an order interval $[-k, k]$ as an open neighborhood of the origin θ (the zero vector) of Y, but it is not always an absorbing set in an infinite dimensional space. This is one of the reasons why the dimension of Y is finite.

At first, we review some basic concepts of set-relation.

Definition 2.1 ([10])**.** For any $A, B \in 2^Y \setminus \{\emptyset\}$ and convex cone C in Y, we write

$A \leq_C^{(1)} B$ by $A \subset \bigcap_{b \in B}(b - C)$, equivalently $B \subset \bigcap_{a \in A}(a + C)$,
$A \leq_C^{(2)} B$ by $A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$,
$A \leq_C^{(3)} B$ by $B \subset (A + C)$,
$A \leq_C^{(4)} B$ by $\left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$,
$A \leq_C^{(5)} B$ by $A \subset (B - C)$,
$A \leq_C^{(6)} B$ by $A \cap (B - C) \neq \emptyset$, equivalently $(A + C) \cap B \neq \emptyset$.

Proposition 2.2 ([10])**.** *For any $A, B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:*

$A \leq_C^{(1)} B$ *implies* $A \leq_C^{(2)} B$, $\quad A \leq_C^{(1)} B$ *implies* $A \leq_C^{(4)} B$,
$A \leq_C^{(2)} B$ *implies* $A \leq_C^{(3)} B$, $\quad A \leq_C^{(4)} B$ *implies* $A \leq_C^{(5)} B$,
$A \leq_C^{(3)} B$ *implies* $A \leq_C^{(6)} B$, $\quad A \leq_C^{(5)} B$ *implies* $A \leq_C^{(6)} B$.

Proposition 2.3 ([11])**.** *For any $A, B \in 2^Y \setminus \{\emptyset\}$, the following statements hold:*

(1) *For each $j = 1, \dots, 6$,*
$A \leq_C^{(j)} B$ *implies* $(A + y) \leq_C^{(j)} (B + y)$ *for* $y \in Y$, *and*
$A \leq_C^{(j)} B$ *implies* $\alpha A \leq_C^{(j)} \alpha B$ *for* $\alpha \geq 0$.

(2) *For each $j = 1, \dots, 5$, $\leq_C^{(j)}$ is transitive.*

(3) *For each $j = 3, 5, 6$, $\leq_C^{(j)}$ is reflexive.*

Let us recall some definitions of C-notions ([13]). A subset A of Y is said to be *C-convex* (resp., *C-closed*) if $A + C$ is convex (resp., closed); *C-proper* if $A + C \neq Y$. Moreover, A is called *C-bounded* if for each neighborhood U of the zero vector in Y there exists $t \geq 0$ such that $A \subset tU + C$. Furthermore, we say that F is *C-notion on S* if $F(x)$ has the property C-notion for every $x \in S$.

Next, we introduce several definitions of cone-convexity and cone-continuity for set-valued maps. These notions are used in Sections 3 and 4.

Definition 2.4 ([11])**.** For each $j = 1, \dots, 5$,

(1) F is called a *type (j) naturally quasi C-convex function* if for each $x, y \in S$ and $\lambda \in (0, 1)$, there exists $\mu \in [0, 1]$ such that
$$F(\lambda x + (1 - \lambda)y) \leq_C^{(j)} \mu F(x) + (1 - \mu)F(y).$$

(2) F is called a *type (j) naturally quasi C-concave function* if for each $x, y \in S$ and $\lambda \in (0, 1)$, there exists $\mu \in [0, 1]$ such that
$$\mu F(x) + (1 - \mu)F(y) \leq_C^{(j)} F(\lambda x + (1 - \lambda)y).$$

Definition 2.5 ([11])**.** For each $j = 1, \dots, 5$,

(1) F is called a *type (j) properly quasi C-convex function* if for each $x, y \in S$ and $\lambda \in (0, 1)$,
$$F(\lambda x + (1 - \lambda)y) \leq_C^{(j)} F(x) \quad \text{or} \quad F(\lambda x + (1 - \lambda)y) \leq_C^{(j)} F(y).$$

(2) F is called a *type (j) properly quasi C-concave function* if for each $x, y \in S$ and $\lambda \in (0, 1)$,
$$F(x) \leq_C^{(j)} F(\lambda x + (1 - \lambda)y) \quad \text{or} \quad F(y) \leq_C^{(j)} F(\lambda x + (1 - \lambda)y).$$

Definition 2.6 ([13])**.** Let $x \in S$. Then,

(1) F is called *C-lower continuous at x* if for every open set V with $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x such that $F(y) \cap (V + C) \neq \emptyset$ for all $y \in U$. We shall say that F is *C-lower continuous on S* if it is C-lower continuous at every point $x \in S$,

(2) F is called *C-upper continuous at x* if for every open set V with $F(x) \subset V$, there exists an open neighborhood U of x such that $F(y) \subset V + C$ for all $y \in U$. We shall say that F is *C-upper continuous on S* if it is C-upper continuous at every point $x \in S$.

3. Unified scalarization for sets

In [11], we propose the following nonlinear scalarizing functions for sets. Let $V, V' \in 2^Y \setminus \{\emptyset\}$ and direction $k \in \mathrm{int}C$. For each $j = 1, \ldots, 6$, $I^{(j)}_{k,V'} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ and $S^{(j)}_{k,V'} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ are defined by

$$I^{(j)}_{k,V'}(V) := \inf \left\{ t \in \mathbb{R} \,\middle|\, V \leq^{(j)}_C (tk + V') \right\},$$

$$S^{(j)}_{k,V'}(V) := \sup \left\{ t \in \mathbb{R} \,\middle|\, (tk + V') \leq^{(j)}_C V \right\},$$

respectively. They are regarded as extensions of scalarizing functions for vectors and for sets. The original idea on the sublinear scalarization for vectors was dealt by Krasnosel'skij [9] in 1962 and by Rubinov [15] in 1977, and then it was applied to vector optimization with its concrete definition by Tammer (Gerstewitz) [5] in 1983, and to separation theorems for not necessary convex sets by Tammer (Gerstewitz) and Iwanow [6] in 1985. In recent years, several scalarization ideas for sets are proposed in [14, 7, 17], and all of them are special cases of unified types of scalarizing functions above.

Proposition 3.1 ([11])**.** *Let $A, B, V, V' \in 2^Y \setminus \{\emptyset\}$ and $k \in \mathrm{int}C$. Then, the following statements hold:*

(1) *For each $j = 1, \ldots, 6$,*

$$V \leq^{(j)}_C (tk + V') \quad \textit{implies} \quad V \leq^{(j)}_C (sk + V') \quad \textit{for any } s \geq t,$$

$$(tk + V') \leq^{(j)}_C V \quad \textit{implies} \quad (sk + V') \leq^{(j)}_C V \quad \textit{for any } s \leq t.$$

(2) *For each $j = 1, \ldots, 6$ and $\alpha \in \mathbb{R}$,*

$$I^{(j)}_{k,V'}(V + \alpha k) = I^{(j)}_{k,V'}(V) + \alpha,$$

$$S^{(j)}_{k,V'}(V + \alpha k) = S^{(j)}_{k,V'}(V) + \alpha.$$

(3) *For each $j = 1, \ldots, 5$,*

$$A \leq^{(j)}_C B \quad \textit{implies} \quad I^{(j)}_{k,V'}(A) \leq I^{(j)}_{k,V'}(B) \quad \textit{and} \quad S^{(j)}_{k,V'}(A) \leq S^{(j)}_{k,V'}(B).$$

Proposition 3.2. *Let $A \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(1) *For any $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$,*

$$I^{(6)}_{k,V'}(A) \leq I^{(3)}_{k,V'}(A) \leq I^{(2)}_{k,V'}(A) \leq I^{(1)}_{k,V'}(A),$$

(2) *For any $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$,*

$$I^{(6)}_{k,V'}(A) \leq I^{(5)}_{k,V'}(A) \leq I^{(4)}_{k,V'}(A) \leq I^{(1)}_{k,V'}(A).$$

Proof. Let $t_1 := I^{(1)}_{k,V'}(A)$ and assume that $t_1 < I^{(2)}_{k,V'}(A)$. Then, there exists $\epsilon > 0$ such that

$$t_1 < t_1 + \epsilon < I^{(2)}_{k,V'}(A) \quad \text{and} \quad A \leq^{(1)}_C (t_1 + \epsilon)k + V'.$$

By Proposition 2.2, we obtain $A \leq_C^{(2)} (t_1+\epsilon)k+V'$ and then $I^{(2)}_{k,V'}(A) \leq t_1+\epsilon$. This is a contradiction. Hence, we obtain $I^{(2)}_{k,V'}(A) \leq I^{(1)}_{k,V'}(A)$. Similarly, we can prove $I^{(3)}_{k,V'}(A) \leq I^{(2)}_{k,V'}(A)$. Thus, we have

$$I^{(3)}_{k,V'}(A) \leq I^{(2)}_{k,V'}(A) \leq I^{(1)}_{k,V'}(A).$$

Statement (2) can be proved in the same way. □

Proposition 3.3. *Let $A \in 2^Y \setminus \{\emptyset\}$. Then, the following statements hold:*

(1) *For any $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$,*

$$S^{(1)}_{k,V'}(A) \leq S^{(2)}_{k,V'}(A) \leq S^{(3)}_{k,V'}(A) \leq S^{(6)}_{k,V'}(A),$$

(2) *For any $k \in \mathrm{int}C$ and $V' \in 2^Y \setminus \{\emptyset\}$,*

$$S^{(1)}_{k,V'}(A) \leq S^{(4)}_{k,V'}(A) \leq S^{(5)}_{k,V'}(A) \leq S^{(6)}_{k,V'}(A).$$

Proof. We can prove the statements in a similar way to the proof of Proposition 3.2. □

Proposition 3.4. *Let $A, V' \in 2^Y \setminus \{\emptyset\}$ and $k \in \mathrm{int}C$. Then, the following statements hold:*

(1) *If A is $(-C)$-bounded and V' is C-bounded then $I^{(1)}_{k,V'}(A) \in \mathbb{R}$.*

(2) *For each $j = 2, 3$, if A is C-proper and V' is C-bounded then $I^{(j)}_{k,V'}(A) \in \mathbb{R}$.*

(3) *For each $j = 4, 5$, if A is $(-C)$-bounded and V' is $(-C)$-proper then $I^{(j)}_{k,V'}(A) \in \mathbb{R}$.*

(4) *If A is C-proper and V' is $(-C)$-bounded then $I^{(6)}_{k,V'}(A) \in \mathbb{R}$.*

Proof. At first, we prove (1). Assume that A is $(-C)$-bounded and V' is C-bounded. Then, by the definition of C-boundedness, it is easy to check that there exist $\bar{a}, \bar{v} \in Y$ such that

$$A \subset \bar{a} - C \quad \text{and} \quad V' \subset \bar{v} + C.$$

Now, we consider the set $[-k,\, k]$. Since $k \in \mathrm{int}C$, we obtain $\mathrm{int}([-k,\, k]) \neq \emptyset$ and $\mathrm{int}([-k,\, k])$ contains the zero vector. As a result, $\mathrm{int}([-k,\, k])$ is absorbing. Let $U := \mathrm{int}([-k,\, k])$. Then, there exists $\bar{t} \geq 0$ such that

$$\bar{a} - \bar{v} \in \bar{t}U \subset \bar{t}k - C.$$

Moreover, by the definition of $\leq_C^{(1)}$ and (1) of Proposition 2.3,

$$A \leq_C^{(1)} \{\bar{a}\} \leq_C^{(1)} \{\bar{t}k + \bar{v}\} \leq_C^{(1)} \bar{t}k + V'.$$

Thus, from (2) of Proposition 2.3 and the definition of $I^{(1)}_{k,V'}$, we have

$$I^{(1)}_{k,V'}(A) \leq \bar{t} < \infty.$$

On the other hand, for any $a \in A - C$, there exists $t_a \geq 0$ such that

$$a - \bar{v} \in t_a U \subset -t_a k + C.$$

Hence, by (3) of Proposition 3.1 we obtain

$$I^{(1)}_{k,V'}(\{-t_a k + \bar{v}\}) \leq I^{(1)}_{k,V'}(\{a\}). \tag{3.1}$$

Furthermore, since U is absorbing, for any $v \in V'$ there exists $t_{\bar{v}} \geq 0$ such that

$$v - \bar{v} \in t_{\bar{v}} U \subset t_{\bar{v}} k - C.$$

Therefore, we have $v - t_{\bar{v}} k \in \bar{v} - C$ and so

$$-t_{\bar{v}} = I^{(3)}_{k,V'}(-t_{\bar{v}} k + V') \leq I^{(3)}_{k,V'}(\{-t_{\bar{v}} k + v\}) \leq I^{(3)}_{k,V'}(\{\bar{v}\}).$$

From (1) of Proposition 3.2, we obtain

$$-t_{\bar{v}} \leq I^{(3)}_{k,V'}(\{\bar{v}\}) \leq I^{(1)}_{k,V'}(\{\bar{v}\}). \tag{3.2}$$

By (3.1), (3.2) and (2) of Proposition 3.1,

$$-\infty < -t_{\bar{v}} - t_a \leq -t_a + I^{(1)}_{k,V'}(\{\bar{v}\}) \leq I^{(1)}_{k,V'}(\{a\}). \tag{3.3}$$

Moreover, since $a \in A - C$ and C is a convex cone, $I^{(1)}_{k,V'}(\{a\}) \leq I^{(1)}_{k,V'}(A)$. For this result and (3.3), we obtain $-\infty < I^{(1)}_{k,V'}(A)$. Consequently, $I^{(1)}_{k,V'}(A) \in \mathbb{R}$.

Next, we prove (2). We consider the case of $j = 3$. Assume that A is C-proper and V' is C-bounded. Since A is C-proper, there exists $\bar{a} \in Y$ such that $\bar{a} \notin A + C$. It follows from the C-boundedness of V' that there exists $\bar{v} \in Y$ such that $V' \subset \bar{v} + C$. Let $U := \mathrm{int}([-k,\, k])$. Then, U is an absorbing neighborhood of the zero vector of Y, and so for any $a \in A$, there exists $t_a \geq 0$ such that

$$a - \bar{v} \in t_a U - C \subset t_a k - C.$$

Thus, we obtain

$$t_a k + V' \subset t_a k + \bar{v} + C \subset a + C \subset A + C,$$

and hence $I^{(3)}_{k,V'}(A) \leq t_a < \infty$. Next, we prove $-\infty < I^{(3)}_{k,V'}(A)$. Since U is absorbing, for any $v \in V'$, there exists $\hat{t} \geq 0$ such that

$$v - \bar{a} \in \hat{t} U \subset \hat{t} k - C.$$

Thus, we obtain $-\hat{t} k + v \in \bar{a} - C$ and so

$$\left(-\hat{t} k + V'\right) \cap (\bar{a} - C) \neq \emptyset.$$

For this result and $\bar{a} \notin A + C$, it is easy to check that $-\hat{t} k + V' \not\subset A + C$. Therefore, we have

$$-\infty < -\hat{t} \leq I^{(3)}_{k,V'}(A).$$

Consequently, $I^{(3)}_{k,V'}(A) \in \mathbb{R}$. The remainder cases of $j = 2$, (3), and (4) can be proved similarly. $\square$

Proposition 3.5. *Let $A, V' \in 2^Y \setminus \{\emptyset\}$ and $k \in \mathrm{int} C$. Then, the following statements hold:*

(1) *If A is C-bounded and V' is $(-C)$-bounded then $S^{(1)}_{k,V'}(A) \in \mathbb{R}$.*

(2) *For each $j = 2, 3$, if A is C-bounded and V' is C-proper then $S^{(j)}_{k,V'}(A) \in \mathbb{R}$.*

(3) *For each $j = 4, 5$, if A is $(-C)$-proper and V' is $(-C)$-bounded then $S^{(j)}_{k,V'}(A) \in \mathbb{R}$.*

(4) *If A is $(-C)$-proper and V' is C-bounded then $S^{(6)}_{k,V'}(A) \in \mathbb{R}$.*

Proof. We can prove the statements in a similar way to the proof of Proposition 3.4. □

Proposition 3.6. *Let $A, B, V' \in 2^Y \setminus \{\emptyset\}$, θ the zero vector of Y, and $k \in \mathrm{int}C$. Then, the following statements hold:*

(1) *If A and B are C-proper and there is an open neighborhood G of θ such that $A \leq_C^{(3)} B + G$ then*

$$I^{(3)}_{k,\{\theta\}}(A) < I^{(3)}_{k,\{\theta\}}(B) \quad \textit{and}$$

(2) *If A and B are C-bounded and there is an open neighborhood G of θ such that $A \leq_C^{(3)} B + G$ then*

$$S^{(3)}_{k,\{\theta\}}(A) < S^{(3)}_{k,\{\theta\}}(B).$$

(3) *If A and B are $(-C)$-bounded and there is an open neighborhood G of θ such that $A + G \leq_C^{(5)} B$ then*

$$I^{(5)}_{k,\{\theta\}}(A) < I^{(5)}_{k,\{\theta\}}(B) \quad \textit{and}$$

(4) *If A and B are $(-C)$-proper and there is an open neighborhood G of θ such that $A + G \leq_C^{(5)} B$ then*

$$S^{(5)}_{k,\{\theta\}}(A) < S^{(5)}_{k,\{\theta\}}(B).$$

Proof. We prove statements (1) and (2); the others can be proved in each similar way. Let $t_A := I^{(3)}_{k,\{\theta\}}(A)$ and $t_B := I^{(3)}_{k,\{\theta\}}(B)$. By (2) of Proposition 3.4 and the definition of t_B, for any $\epsilon > 0$ there exists $t(\epsilon) \in \mathbb{R}$ such that

$$t_B < t(\epsilon) < t_B + \epsilon \quad \text{and} \quad B \leq_C^{(3)} t(\epsilon)k,$$

that is, $t(\epsilon)k \in B + C$. From G is absorbing, there exists $t_0 > 0$ such that $-t_0 k \in G$ and so we obtain

$$(t(\epsilon) - t_0)k \in B + C + G \subset A + C,$$

that is, $A \leq_C^{(3)} ((t(\epsilon) - t_0)\, k + \{\theta\})$. By the definition of t_A, we get $t_A \leq t(\epsilon) - t_0$, and hence $t_A < t_B + \epsilon - t_0$. Since ϵ is an arbitrary positive real number, we obtain $t_A \leq t_B - t_0 < t_B$. Consequently, $I^{(j)}_{k,\{\theta\}}(A) < I^{(j)}_{k,\{\theta\}}(B)$.

Next, let $s_A := S^{(3)}_{k,\{\theta\}}(A)$ and $s_B := S^{(3)}_{k,\{\theta\}}(B)$. By (2) of Proposition 3.5 and the definition of s_A, for any $\epsilon > 0$ there exists $s(\epsilon) \in \mathbb{R}$ such that

$$s_A - \epsilon < s(\epsilon) \leq s_A \quad \text{and} \quad s(\epsilon)k \leq_C^{(3)} A,$$

that is, $A \subset s(\epsilon)k + C$. From G is absorbing, there exists $s_0 > 0$ such that $-s_0 k \in G$ and so we obtain

$$B - s_0 k \subset B + G \subset A + C,$$

Thus we have

$$B \subset A + s_0 k + C \subset (s(\epsilon) + s_0)\, k + C,$$

that is, $((s(\epsilon)+s_0)\,k+\{\theta\}) \leq_C^{(3)} B$. By the definition of s_B, we get $s(\epsilon)+s_0 \leq s_B$, and hence $s_A-\epsilon+s_0 < s_B$. Since ϵ is an arbitrary positive real number, we obtain $s_A < s_A+s_0 \leq s_B$. Consequently, $S^{(3)}_{k,\{\theta\}}(A) < S^{(3)}_{k,\{\theta\}}(B)$. □

Next, we introduce inherited properties on cone-convexity and cone-continuity of set-valued maps. At first, we remark that the unified types of scalarizing functions have an important merit on the inheritance properties in contrast with the approach of [14].

Let $V' \in 2^Y \setminus \{\emptyset\}$ and direction $k \in \mathrm{int}C$. For any set-valued map $F : S \to 2^Y$ and for each $j = 1, \dots, 6$, we consider the following composite functions:

$$(I^{(j)}_{k,V'} \circ F)(x) := I^{(j)}_{k,V'}(F(x)), \quad x \in S,$$

$$(S^{(j)}_{k,V'} \circ F)(x) := S^{(j)}_{k,V'}(F(x)), \quad x \in S.$$

Then, we can directly discuss inherited properties on cone-convexity and cone-continuity of parent set-valued map F to $I^{(j)}_{k,V'} \circ F$ and $S^{(j)}_{k,V'} \circ F$ in an analogous fashion to linear scalarizing function like linear functional.

Theorem 3.7 ([11])**.** *Let $F : S \to 2^Y$, $V' \in 2^Y \setminus \{\emptyset\}$, and $k \in \mathrm{int}C$. Then, the following statements hold:*

(1) *For each $j = 1, 2, 3$, if F is type (j) naturally quasi C-convex, then $I^{(j)}_{k,V'} \circ F$ is quasi convex.*
(2) *For each $j = 4, 5$, if F is type (j) naturally quasi C-convex and V' is $(-C)$-convex, then $I^{(j)}_{k,V'} \circ F$ is quasi convex.*

Theorem 3.8 ([11])**.** *Let $F : S \to 2^Y$, $V' \in 2^Y \setminus \{\emptyset\}$, and $k \in \mathrm{int}C$. Then, the following statements hold:*

(1) *For each $j = 1, 4, 5$, if F is type (j) naturally quasi C-concave, then $S^{(j)}_{k,V'} \circ F$ is quasi concave.*
(2) *For each $j = 2, 3$, if F is type (j) naturally quasi C-concave and V' is C-convex, then $S^{(j)}_{k,V'} \circ F$ is quasi concave.*

Theorem 3.9. *Let $F : S \to 2^Y$, $V' \in 2^Y \setminus \{\emptyset\}$, and $k \in \mathrm{int}C$. Then, the following statements hold:*

(1) *For each $j = 1, \dots, 5$, if F is type (j) properly quasi C-convex, then $I^{(j)}_{k,V'} \circ F$ and $S^{(j)}_{k,V'} \circ F$ are quasi convex.*
(2) *For each $j = 1, \dots, 5$, if F is type (j) properly quasi C-concave, then $I^{(j)}_{k,V'} \circ F$ and $S^{(j)}_{k,V'} \circ F$ are quasi concave.*

Proof. We prove statement (2) only. Assume that F is type (j) properly quasi C-concave, that is, for every $x, y \in X$ and $\lambda \in (0, 1)$,

$$F(x) \leq_C^{(j)} F(\lambda x + (1-\lambda)y) \quad \text{or} \quad F(y) \leq_C^{(j)} F(\lambda x + (1-\lambda)y).$$

Hence, by (3) of Proposition 3.1, we obtain

$$\min\{(I^{(j)}_{k,V'} \circ F)(x), (I^{(j)}_{k,V'} \circ F)(y)\} \leq \left(I^{(j)}_{k,V'} \circ F\right)(\lambda x + (1-\lambda)y),$$

and then $I^{(j)}_{k,V'} \circ F$ is quasi concave. Similarly, $S^{(j)}_{k,V'} \circ F$ is quasi concave and the proof is completed. $\square$

Theorem 3.10 ([18]). *Let $F : S \to 2^Y$, $V' \in 2^Y \setminus \{\emptyset\}$, and $k \in \mathrm{int}C$. Then, the following statements hold:*

(1) *For each $j = 1, 4, 5$, if F is C-lower continuous on S then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on S. Moreover, if F is $(-C)$-upper continuous on S then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on S.*

(2) *For each $j = 2, 3, 6$, if F is $(-C)$-lower continuous on S then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on S. Moreover, if F is C-upper continuous on S then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on S.*

Theorem 3.11 ([18]). *Let $F : S \to 2^Y$, $V' \in 2^Y \setminus \{\emptyset\}$, and $k \in \mathrm{int}C$. Then, the following statements hold:*

(1) *For each $j = 4, 5, 6$, if F is C-lower continuous on S then $S^{(j)}_{k,V'} \circ F$ is lower semicontinuous on S. Moreover, if F is $(-C)$-upper continuous on S then $S^{(j)}_{k,V'} \circ F$ is upper semicontinuous on S.*

(2) *For each $j = 1, 2, 3$, if F is $(-C)$-lower continuous on S then $S^{(j)}_{k,V'} \circ F$ is upper semicontinuous on S. Moreover, if F is C-upper continuous on S then $S^{(j)}_{k,V'} \circ F$ is lower semicontinuous on S.*

4. Set-valued Ky Fan minimax inequality

The following theorem is equivalent to Theorem 1 in [1] of Ky Fan minimax inequality; this equivalence was proved by Takahashi [19] firstly in 1976.

Theorem 4.1. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $f : X \times X \to \mathbb{R}$. If f satisfies the following conditions:*

(1) *for each fixed $y \in X$, $f(\cdot, y)$ is lower semicontinuous,*
(2) *for each fixed $x \in X$, $f(x, \cdot)$ is quasi concave,*
(3) *for all $x \in X$, $f(x, x) \le 0$,*

then there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \le 0$ for all $y \in Y$.

Based on the above theorem, we shall show four kinds of Ky Fan minimax inequality for set-valued maps by using several results in Section 3.

Lemma 4.2. *Let Y be a real topological vector space, C a proper closed convex cone in Y with $\mathrm{int}C \neq \emptyset$, $k \in \mathrm{int}C$ and $A, V' \in 2^Y \setminus \{\emptyset\}$. Assume that $r \in \mathbb{R}$. Then the following statements hold:*

(1) *If A is C-closed, then*

$$I^{(3)}_{k,V'}(A) \le r \quad \text{implies} \quad A \le^{(3)}_C rk + V',$$

(2) *If V' is $(-C)$-closed, then*

$$I^{(5)}_{k,V'}(A) \le r \quad \text{implies} \quad A \le^{(5)}_C rk + V'.$$

Proof. Firstly, we show statement (1). Assume that A is C-closed, $I^{(3)}_{k,V'}(A) \leq r$ and $A \not\leq^{(3)}_C rk + V'$. Then, there exists $v' \in rk + V'$ such that $v' \in (A + C)^c$. Since A is C-closed, $(A + C)^c$ is open. Hence, there exists $\delta > 0$ such that

$$v' + \delta k \notin A + C.$$

Thus, $(r + \delta)k + V' \not\subset A + C$ and then

$$r < r + \delta < I^{(3)}_{k,V'}(A).$$

This contradicts $I^{(3)}_{k,V'}(A) \leq r$. Consequently, $A \leq^{(3)}_C rk + V'$. Similarly, we can prove statement (2) and the proof is completed. □

Theorem 4.3. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, C a proper closed convex cone in Y with* int$C \neq \emptyset$ *and $F : X \times X \to 2^Y \setminus \{\emptyset\}$. If F satisfies the following conditions:*

(1) *F is $(-C)$-bounded on $X \times X$,*
(2) *for each fixed $y \in X$, $F(\cdot, y)$ is C-lower continuous,*
(3) *for each fixed $x \in X$, $F(x, \cdot)$ is type* (5) *properly quasi C-concave,*
(4) *for all $x \in X$, $F(x, x) \subset -C$,*

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \subset -C$ for all $y \in Y$.

Proof. For this end, we consider the function $\left(I^{(5)}_{k,\{\theta\}} \circ F\right)$ where $k \in \text{int}C$. From (3) of Proposition 3.4, $\left(I^{(5)}_{k,\{\theta\}} \circ F\right)(x, y) \in \mathbb{R}$ for any $x, y \in X$. Moreover, by (1) of Theorem 3.10 and (2) of Theorem 3.9, we obtain

- for each fixed $y \in X$, $\left(I^{(5)}_{k,\{\theta\}} \circ F\right)(\cdot, y)$ is lower semicontinuous,
- for each fixed $x \in X$, $\left(I^{(5)}_{k,\{\theta\}} \circ F\right)(x, \cdot)$ is quasi concave.

Also, by the definition of $I^{(5)}_{k,\{\theta\}}$, $\left(I^{(5)}_{k,\{\theta\}} \circ F\right)(x, x) \leq 0$ for all $x \in X$. Hence, we can apply the result of Theorem 4.1 to $\left(I^{(5)}_{k,\{\theta\}} \circ F\right)$, that is, there exists $\bar{x} \in X$ such that $\left(I^{(5)}_{k,\{\theta\}} \circ F\right)(\bar{x}, y) \leq 0$ for all $y \in Y$. Clearly, $\{\theta\}$ is $(-C)$-closed and then from (2) of Lemma 4.2, we have $F(\bar{x}, y) \subset -C$. □

Theorem 4.4. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, C a proper closed convex cone in Y with* int$C \neq \emptyset$ *and $F : X \times X \to 2^Y \setminus \{\emptyset\}$. If F satisfies the following conditions:*

(1) *F is C-proper and C-closed on $X \times X$,*
(2) *for each fixed $y \in X$, $F(\cdot, y)$ is C-upper continuous,*
(3) *for each fixed $x \in X$, $F(x, \cdot)$ is type* (3) *properly quasi C-concave,*
(4) *for all $x \in X$, $F(x, x) \cap (-C) \neq \emptyset$,*

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \cap (-C) \neq \emptyset$ for all $y \in Y$

Proof. In a similar way to the proof of Theorem 4.3, by (2) of Theorem 3.10, (2) of Theorem 3.9 and Theorem 4.1, there exists $\bar{x} \in X$ such that

$$\left(I^{(3)}_{k,\{\theta\}} \circ F\right)(\bar{x},\, y) \leq 0$$

for all $y \in Y$. By (1) of Lemma 4.2, it is equivalent to $\{\theta\} \subset F(\bar{x},\, y) + C$ and then $F(\bar{x},\, y) \cap (-C) \neq \emptyset$ for all $y \in Y$. □

Theorem 4.5. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, C a proper closed convex cone in Y with $\mathrm{int}C \neq \emptyset$ and $F : X \times X \to 2^Y \setminus \{\emptyset\}$. If F satisfies the following conditions:*

(1) *F is $(-C)$-proper on $X \times X$,*
(2) *for each fixed $y \in X$, $F(\cdot,\, y)$ is C-lower continuous,*
(3) *for each fixed $x \in X$, $F(x,\, \cdot)$ is type (5) naturally quasi C-concave,*
(4) *for all $x \in X$, $F(x,\, x) \cap \mathrm{int}C = \emptyset$,*

then there exists $\bar{x} \in X$ such that $F(\bar{x},\, y) \cap \mathrm{int}C = \emptyset$ for all $y \in Y$.

Proof. In a similar way to the proof of Theorem 4.3, by (1) of Theorem 3.11, (1) of Theorem 3.8 and Theorem 4.1, there exists $\bar{x} \in X$ such that

$$\left(S^{(5)}_{k,\{\theta\}} \circ F\right)(\bar{x},\, y) \leq 0$$

for all $y \in Y$. It is equivalent to $\left(S^{(5)}_{k,\{\theta\}} \circ F\right)(\bar{x},\, y) \not> 0$. Hence, by (4) of Proposition 3.6 we have $G \not\leq^{(5)}_C F(\bar{x},\, y)$ for any open neighborhood G of the zero vector of Y. Therefore we get $F(\bar{x},\, y) \cap \mathrm{int}C = \emptyset$ for all $y \in Y$. □

Theorem 4.6. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, C a proper closed convex cone in Y with $\mathrm{int}C \neq \emptyset$ and $F : X \times X \to 2^Y \setminus \{\emptyset\}$. If F satisfies the following conditions:*

(1) *F is compact-valued on $X \times X$,*
(2) *for each fixed $y \in X$, $F(\cdot,\, y)$ is C-upper continuous,*
(3) *for each fixed $x \in X$, $F(x,\, \cdot)$ is type (3) naturally quasi C-concave,*
(4) *for all $x \in X$, $F(x,\, x) \not\subset \mathrm{int}C$,*

then there exists $\bar{x} \in X$ such that $F(\bar{x},\, y) \not\subset \mathrm{int}C$ for all $y \in Y$.

Proof. In a similar way to the proof of Theorem 4.3, by (2) of Theorem 3.11, (2) of Theorem 3.8 and Theorem 4.1, there exists $\bar{x} \in X$ such that

$$\left(S^{(3)}_{k,\{\theta\}} \circ F\right)(\bar{x},\, y) \leq 0$$

for all $y \in Y$. It is equivalent to $\left(S^{(3)}_{k,\{\theta\}} \circ F\right)(\bar{x},\, y) \not> 0$. Hence, by (2) of Proposition 3.6, we have $\{\theta\} \not\leq^{(3)}_C F(\bar{x},\, y) + G$ for any open neighborhood G of the zero vector of Y. Therefore we get $F(\bar{x},\, y) \not\subset \mathrm{int}C$ for all $y \in Y$. □

Remark 4.7. It is easy to check that if F is a single-valued function into the real numbers then Theorems 4.3, 4.4, 4.5, and 4.6 are reduced to Theorem 4.1.

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References

[1] K. Fan, *A minimax inequality and its applications*, Inequalities III, O. Shisha (ed.), Academic Press, New York, 1972, pp.103–113.

[2] P. G. Georgiev and T. Tanaka, *Vector-valued set-valued variants of Ky Fan's inequality*, J. Nonlinear and Convex Anal. **1** (2000), 245–254.

[3] P. G. Georgiev and T. Tanaka, *Fan's inequality for set-valued maps*, Nonlinear Anal. **47** (2001), 607–618.

[4] C. Gerth (Tammer) and P. Weidner, *Nonconvex Separation Theorems and Some Applications in Vector Optimization*, J. Optim. Theory Appl. **67** (1990), 297–320.

[5] C. Gerstewitz (Tammer), *Nichtkonvexe dualität in der vektoroptimierung*, Wiss. Zeitschr. TH Leuna-Merseburg **25** (1983), 357–364 (in German).

[6] C. Gerstewitz (Tammer) and E. Iwanow, *Dualität für nichtkonvexe vektoroptimierungs probleme*, Wiss. Z. Tech. Hochsch Ilmenau **2** (1985), 61–81 (in German).

[7] A. Hamel and A. Löhne, *Minimal element theorems and Ekeland's principle with set relations*, J. Nonlinear and Convex Anal. **7** (2006), 19–37.

[8] E. Hernández, L. Rodríguez-Marín, *Nonconvex scalarization in set-optimization with set-valued maps*, J. Math. Anal. Appl. **325** (2007), 1–18.

[9] M. A. Krasnosel'skij, *Positive solutions of operator equations*, Fizmatgiz, Moskow, 1962 (in Russian).

[10] D. Kuroiwa, T. Tanaka, and T.X.D. Ha, *On cone convexity of set-valued maps*, Nonlinear Anal. **30** (1997), 1487–1496.

[11] I. Kuwano, T. Tanaka, and S. Yamada, *Characterization of nonlinear scalarizing functions for set-valued maps*, Nonlinear Analysis and Optimization, S. Akashi, W. Takahashi and T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2009, pp.193–204.

[12] I. Kuwano, T. Tanaka, and S. Yamada, *Inherited properties of nonlinear scalarizing functions for set-valued maps*, to appear in the proceedings of Nonlinear Analysis and Convex Analysis, S. Akashi, Y. Kimura, and T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2010.

[13] D. T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, 319, Springer, Berlin, 1989.

[14] S. Nishizawa, T. Tanaka, and P. G. Georgiev, *On inherited properties of set-valued maps*, Nonlinear Analysis and Convex Analysis, W. Takahashi and T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2003, pp.341–350.

[15] A. M. Rubinov, *Sublinear operators and their applications*, Russian Math. Surveys **32** (1977), 115–175 (in Russian).

[16] A. Shimizu, S Nishizawa and T. Tanaka, *Optimality conditions in set-valued optimization using nonlinear scalarization methods*, Nonlinear Analysis and Convex Analysis, W. Takahashi and T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2007, pp.565–574.

[17] A. Shimizu and T. Tanaka, *Minimal element theorem with set-relations*, J. Nonlinear and Convex Anal. **9** (2008), 249–253.

[18] Y. Sonda, I. Kuwano and T. Tanaka, *Cone-semicontinuity of set-valued maps by analogy with real-valued semicontinuity*, to appear in Nihonkai Math. J., 2010.

[19] W. Takahashi, *Nonlinear variational inequalities and fixed point theorems*, J. Math. Soc. Japan **28** (1976), 168–181.

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