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A CLASS OF NONLINEAR EIGENVALUE PROBLEMS WITH FOUR SOLUTIONS

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Dedicated to the memory of Ky Fan, a true giant of Mathematics

ABSTRACT. In this paper, we present a class of C^1 functionals on reflexive Banach spaces which possess at least four critical points for a value of a real parameter from which they depend. Applications to the Neumann problem are also given.

The best way of introducing the main result of this paper is to consider the Neumann problem

$$(P_\lambda) \qquad \begin{cases} -u'' + u = \lambda\alpha(t)f(u) & \text{in } [0,1] \\ u'(0) = u'(1) = 0 \end{cases}$$

where $\lambda \in \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$, $\alpha : [0,1] \to [0,+\infty[$ are two continuous non-constant functions.

Also, consider the following two conditions:

(a) there exists $\sigma > 0$ and $\xi_1 \in \left]-\frac{\sigma}{\sqrt{2}}, \frac{\sigma}{\sqrt{2}}\right[$ such that

$$0 < \int_0^{\xi_1} f(t)dt = \sup_{|\xi|\leq\sigma} \int_0^{\xi} f(t)dt < \sup_{\xi\in\mathbf{R}} \int_0^{\xi} f(t) \ ;$$

(b) one has

$$\max\left\{\limsup_{|\xi|\to+\infty} \frac{\int_0^\xi f(t)dt}{\xi^2}, \limsup_{\xi\to 0} \frac{\int_0^\xi f(t)dt}{\xi^2}\right\} \leq 0 < \sup_{\xi\in\mathbf{R}} \int_0^\xi f(t)dt \ .$$

Under condition (a), by Theorem 1.1 of [3], problem (P_λ) has at least one solution for each $\lambda > 0$. On the other hand, if condition (b) is satisfied, then one necessarily has $f(0) = 0$ and, by Theorem 1 of [4] (which holds for $n = 1$ too), problem (P_λ) has at least two non-zero solutions for each $\lambda > 0$ large enough. So, a natural question arises. Namely, what happens when (a) and (b) are simultaneously satisfied ? *A priori*, there is no evident reason to get a better conclusion than that holding under (b) only. However, thanks to the main result of this paper, we can prove that if, besides (a) and (b), we also have $\frac{\xi_1^2}{2\int_0^{\xi_1} f(t)dt} \leq \int_0^1 \alpha(t)dt < \frac{\sigma^2}{4\sup_{\xi\in\mathbf{R}}\int_0^\xi f(t)dt}$, there is some $\hat{\lambda} > 1$ such that problem $(P_{\hat{\lambda}})$ has at least three non-zero solutions.

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Here is our main result:

Theorem 1. *Let X be a reflexive real Banach space; $\Phi : X \to \mathbf{R}$ a coercive and sequentially weakly lower semicontinuous C^1 functional whose derivative admits a continuous inverse on X^*; $\Psi, J : X \to \mathbf{R}$ two C^1 functionals with compact derivative. Assume that there exist two points $u_0, u_1 \in X$ with the following properties:*
(i) u_0 is a strict local minimum of Φ and $\Phi(u_0) = \Psi(u_0) = J(u_0) = 0$;
(ii) $\Phi(u_1) \le \Psi(u_1)$ and $J(u_1) > 0$;
Moreover, suppose that, for some $\rho \in \mathbf{R}$, one has either

$$(1) \quad \sup_{\lambda>0}\inf_{x\in X}(\lambda(\Phi(x)-\Psi(x)-\rho)-J(x)) < \inf_{x\in X}\sup_{\lambda>0}(\lambda(\Phi(x)-\Psi(x)-\rho)-J(x))$$

or

$$(2) \quad \sup_{\lambda>0}\inf_{x\in X}(\Phi(x)-\Psi(x)-\lambda(\rho+J(x))) < \inf_{x\in X}\sup_{\lambda>0}(\Phi(x)-\Psi(x)-\lambda(\rho+J(x))) \ .$$

Finally, assume that

$$(3) \qquad \max\left\{\limsup_{\|u\|\to+\infty}\frac{\Psi(u)}{\Phi(u)}, \limsup_{u\to u_0}\frac{\Psi(u)}{\Phi(u)}\right\} < 1$$

and

$$(4) \qquad \max\left\{\limsup_{\|u\|\to+\infty}\frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0}\frac{J(u)}{\Phi(u)}\right\} \le 0 \ .$$

Under such hypotheses, there exists $\lambda^* > 0$ such that the equation

$$\Phi'(u) = \Psi'(u) + \lambda^* J'(u)$$

has at least four solutions in X, u_0 being one of them.

Let us recall that, when X is a topological space (resp. a convex subset of a vector space), a function $f : X \to \mathbf{R}$ is said to be inf-compact (resp. quasi-concave) if the set $f^{-1}(]-\infty, c])$ (resp. $f^{-1}([c, +\infty[)$) is compact (resp. convex) for all $c \in \mathbf{R}$.

The following result, established in [5], is a key tool in the proof of Theorem 1.

Theorem A. *Let X be a topological space, $I \subseteq \mathbf{R}$ an open interval and $P : X \times I \to \mathbf{R}$ a function satisfying the following conditions:*
(a_1) for each $x \in X$, the function $P(x, \cdot)$ is quasi-concave and continuous ;
(a_2) for each $\lambda \in I$, the function $P(\cdot, \lambda)$ is lower semicontinuous and inf-compact ;
(a_3) one has

$$\sup_{\lambda\in I}\inf_{x\in X} P(x,\lambda) < \inf_{x\in X}\sup_{\lambda\in I} P(x,\lambda) \ .$$

Under such hypotheses, there exists $\lambda^* \in I$ such that the function $P(\cdot, \lambda^*)$ has at least two global minima.

Now, we can give the

Proof of Theorem 1. Fix $\mu > 0$. By (i), we also can fix a neighbourhood U of u_0 such that $\Phi(u) > 0$ for all $u \in U \setminus \{u_0\}$. From (3) and (4), it follows that

$$\limsup_{u\to u_0}\frac{\Psi(u)+\frac{1}{\mu}J(u)}{\Phi(u)} < 1 \ .$$

Consequently, there exists a neighbourhood V of u_0, with $V \subseteq U$, such that

$$\frac{\Psi(u) + \frac{1}{\mu} J(u)}{\Phi(u)} < 1$$

for all $u \in V \setminus \{u_0\}$. So, we have

$$\mu(\Phi(u) - \Psi(u)) - J(u) > 0$$

for all $u \in V \setminus \{u_0\}$. In other words, u_0 is a strict local minimum of the functional $\mu(\Phi - \Psi) - J$. From (ii), we have

$$\mu(\Phi(u_1) - \Psi(u_1)) - J(u_1) < 0 \ ,$$

and so u_0 is not a global minimum for that functional. From (3) and (4) again, we have

$$\limsup_{\|u\| \to +\infty} \frac{\Psi(u) + \frac{1}{\mu} J(u)}{\Phi(u)} < 1 \ .$$

From this, recalling that Φ is coercive and observing that

$$\mu(\Phi(u) - \Psi(u)) - J(u) = \mu\Phi(u) \left(1 - \frac{\Psi(u) + \frac{1}{\mu} J(u)}{\Phi(u)} \right) \ ,$$

we clearly infer that

$$\lim_{\|u\| \to +\infty} (\mu(\Phi(u) - \Psi(u)) - J(u)) = +\infty \ . \tag{5}$$

Since X is reflexive, the functionals Ψ, J are sequentially weakly continuous, being with compact derivative ([6], Corollary 41.9). From this and (5), in view of the reflexivity of X again and of the Eberlein-Smulyan theorem, we then infer that the sub-level sets of the functional $\mu(\Phi - \Psi) - J$ are weakly compact. Consequently, we can apply Theorem A, with $I =]0, +\infty[$, considering X with the weak topology and taking either

$$P(x, \lambda) = \lambda(\Phi(x) - \Psi(x) + \rho) - J(x)$$

or

$$P(x, \lambda) = \Phi(x) - \Psi(x) + \lambda(\rho - J(x))$$

according to whether either (1) or (2) holds, respectively. Therefore, in any case, there exists $\mu^* > 0$ such that the functional $\mu^*(\Phi - \Psi) - J$ has at least two gobal minima. We already know that u_0 is a local, not global minimum for $\mu^*(\Phi - \Psi) - J$. Now, we remark that this functional, due to (5) and to our assumptions on Φ, Ψ and J, turns out to satisfy the Palais-Smale condition ([6], Example 38.25). At this point, we can apply Theorem (1.ter) of [2] which ensures the existence of at least four critical points of the functional $\mu^*(\Phi - \Psi) - J$. Our conclusion then follows taking $\lambda^* = \frac{1}{\mu^*}$. □

Remark 2. It is important to remark that, in view of Theorem 1 of [1], condition (1) is equivalent to the existence of $u_2, u_3 \in X$ satisfying

$$\Phi(u_2) - \Psi(u_2) < \rho < \Phi(u_3) - \Psi(u_3)$$

and
$$\frac{\sup_{(\Phi-\Psi)^{-1}(]-\infty,\rho])} J - J(u_2)}{\rho - \Phi(u_2) + \Psi(u_2)} < \frac{\sup_{(\Phi-\Psi)^{-1}(]-\infty,\rho])} J - J(u_3)}{\rho - \Phi(u_3) + \Psi(u_3)} \ .$$
Likewise, condition (2) is equivalent to the existence of $u_2, u_3 \in X$ satisfying
$$J(u_3) < \rho < J(u_2)$$
and
$$\frac{\Phi(u_2) - \Psi(u_2) - \inf_{J^{-1}([\rho,+\infty[)}(\Phi - \Psi)}{J(u_2) - \rho} < \frac{\Phi(u_3) - \Psi(u_3) - \inf_{J^{-1}([\rho,+\infty[)}(\Phi - \Psi)}{J(u_3) - \rho} \ .$$

Remark 3. We also remark that, on the basis of Theorem A, condition (1) can be replaced by the formally more general one:

there exist an interval $A \subseteq \mathbf{R}$ and two functions $\gamma : A \to]0, +\infty[$, $\eta : A \to \mathbf{R}$ such that
$$\sup_{\lambda\in A} \inf_{x\in X} (\gamma(\lambda)(\Phi(x)-\Psi(x))-J(x)+\eta(\lambda)) < \inf_{x\in X} \sup_{\lambda\in A} (\gamma(\lambda)(\Phi(x)-\Psi(x))-J(x)+\eta(\lambda))$$
and, for each $x \in X$, the function $\gamma(\cdot)(\Phi(x) - \Psi(x)) + \eta(\cdot)$ is quasi-concave and continuous in A.

However, we do not know, at present, some significant situations where (1) fails while such a condition applies. A similar remark holds for condition (2).

We now give an application of Theorem 1 from which, in turn, the result implicitly stated at the beginning of the paper follows.

In the sequel, the Sobolev space $H^1(0, 1)$ is considered with the usual norm
$$\|u\| = \left(\int_0^1 (|u'(t)|^2 + |u(t)|^2)dt\right)^{\frac{1}{2}} \ .$$
Note that
$$\sup_{[0,1]} |u| \leq \sqrt{2}\|u\|$$
for all $u \in H^1(0, 1)$.

We will use the following lemma.

Lemma 4. *Let $h : \mathbf{R} \to \mathbf{R}$ and $\gamma : [0, 1] \to [0, +\infty[$ be two continuous non-zero functions. For each $u \in H^1(0, 1)$, put*
$$T(u) = \int_0^1 \gamma(t)h(u(t))dt \ .$$
Then, one has
$$\limsup_{u\to 0} \frac{T(u)}{\|u\|^2} \leq \sup_{[0,1]} \gamma \max\left\{0, \limsup_{\xi\to 0} \frac{h(\xi)}{\xi^2}\right\} \tag{6}$$
and
$$\limsup_{\|u\|\to+\infty} \frac{T(u)}{\|u\|^2} \leq \sup_{[0,1]} \gamma \max\left\{0, \limsup_{|\xi|\to+\infty} \frac{h(\xi)}{\xi^2}\right\} \ . \tag{7}$$

Proof. For brevity, put

$$\rho_1 = \max\left\{0, \limsup_{\xi\to 0} \frac{h(\xi)}{\xi^2}\right\}$$

and

$$\rho_2 = \max\left\{0, \limsup_{|\xi|\to +\infty} \frac{h(\xi)}{\xi^2}\right\} .$$

Fix $\eta > \rho_1$. So, there is some $\delta > 0$ such that

$$h(\xi) \le \eta\xi^2$$

for all $\xi \in [-\delta, \delta]$. For $u \in H^1(0,1)$ with $\|u\| \le \frac{\delta}{\sqrt{2}}$, we have $\sup_{[0,1]} |u| \le \delta$ and so

$$h(u(t)) \le \eta |u(t)|^2$$

for all $t \in [0,1]$. Multiplying by γ (recall that $\gamma \ge 0$) and integrating, we then get

$$T(u) \le \eta \sup_{[0,1]} \gamma \|u\|^2 .$$

From this it follows that

$$\limsup_{u\to 0} \frac{T(u)}{\|u\|^2} \le \eta \sup_{[0,1]} \gamma$$

and so (6), by the arbitrariness of η.

Now, let $\nu > \rho_2$. So, for some $\omega > 0$, we have

$$h(\xi) \le \nu\xi^2$$

for all $\xi \in \mathbf{R} \setminus [-\omega, \omega]$. For each $u \in H^1(0,1) \setminus \{0\}$, we have

$$\begin{aligned} \frac{T(u)}{\|u\|^2} &= \frac{\int_{u^{-1}([-\omega,\omega])} \gamma(t)h(u(t))dt}{\|u\|^2} + \frac{\int_{u^{-1}(\mathbf{R}\setminus[-\omega,\omega])} \gamma(t)h(u(t))dt}{\|u\|^2} \\ &\le \frac{\sup_{[0,1]} \gamma \sup_{[-\omega,\omega]} h}{\|u\|^2} + \nu \sup_{[0,1]} \gamma . \end{aligned}$$

Hence

$$\limsup_{\|u\|\to +\infty} \frac{T(u)}{\|u\|^2} \le \nu \sup_{[0,1]} \gamma$$

and (7) follows by the arbitrariness of ν. $\square$

Here is the application of Theorem 1.

Theorem 5. *Let $f, g : \mathbf{R} \to \mathbf{R}$, $\alpha, \beta : [0,1] \to [0, +\infty[$ be four non-zero continuous functions. Assume that*

$$\max\left\{\limsup_{|\xi|\to+\infty} \frac{\int_0^\xi f(t)dt}{\xi^2}, \limsup_{\xi\to 0} \frac{\int_0^\xi f(t)dt}{\xi^2}\right\} \le 0 \tag{8}$$

and

$$\sup_{\xi\in\mathbf{R}} \int_0^\xi g(t)dt < +\infty,\ \limsup_{\xi\to 0} \frac{\int_0^\xi g(t)dt}{\xi^2} < \frac{1}{2\sup_{[0,1]} \beta} . \tag{9}$$

Finally, suppose that there exist $\sigma > 2\sqrt{\int_0^1 \beta(t)dt \sup_{\xi\in\mathbf{R}} \int_0^\xi g(t)dt}$ *and* $\xi_1 \in \mathbf{R}$ *such that*

$$0 < \int_0^{\xi_1} f(t)dt = \sup_{|\xi|\leq\sigma} \int_0^\xi f(t)dt < \sup_{\xi\in\mathbf{R}} \int_0^\xi f(t) \tag{10}$$

and

$$\xi_1^2 \leq 2\int_0^1 \beta(t)dt \int_0^{\xi_1} g(t)dt \ . \tag{11}$$

Under such hypotheses, there exists $\lambda^* > 0$ *such that the problem*

$$\begin{cases} -u'' + u = \lambda^*\alpha(t)f(u) + \beta(t)g(u) & \text{in} \ \ [0,1] \\ u'(0) = u'(1) = 0 \end{cases}$$

has at least three non-zero solutions.

Proof. For each $u \in H^1(0,1)$, put

$$\Phi(u) = \frac{1}{2}\|u\|^2 \ ,$$

$$\Psi(u) = \int_0^1 \beta(t)G(u(t))dt$$

and

$$J(u) = \int_0^1 \alpha(t)F(u(t))dt \ ,$$

where $F(\xi) = \int_0^\xi f(t)dt$, $G(\xi) = \int_0^\xi g(t)dt$. By classical results, the functionals Φ, Ψ, J are C^1, Φ', J' are compact and, for $\lambda \in \mathbf{R}$, the solutions of the problem

$$\begin{cases} -u'' + u = \lambda\alpha(t)f(u) + \beta(t)g(u) & \text{in} \ \ [0,1] \\ u'(0) = u'(1) = 0 \end{cases}$$

are exactly the critical points in $H^1(0,1)$ of the functional $\Phi - \Psi - \lambda J$. To apply Theorem 1, we take $u_0 = 0$ and $u_1 = \xi_1$. Of course, (i) is evident and (ii) follows directly from (10) and (11). Moreover, thanks to Lemma 4, conditions (3) and (4) are direct consequences of (9) and (8) respectively. Finally, let us check that (1) holds. To this end, choose

$$\rho = \frac{\sigma^2}{4} - \int_0^1 \beta(t)dt \sup_{\mathbf{R}} G \ .$$

Now, observe that

$$\{u \in H^1(0,1) : \Phi(u) - \Psi(u) \leq \rho\} \subseteq \left\{u \in H^1(0,1) : \|u\|^2 \leq 2\left(\rho + \int_0^1 \beta(t)dt \sup_{\mathbf{R}} G\right)\right\}$$

$$= \left\{u \in H^1(0,1) : \|u\|^2 \leq \frac{\sigma^2}{2}\right\} \subseteq \left\{u \in H^1(0,1) : \sup_{[0,1]} |u| \leq \sigma\right\} \ .$$

From this, taking (10) into account, we get

$$\sup_{(\Phi-\Psi)^{-1}(]-\infty,\rho])} J \leq J(u_1) \ .$$

On the other hand, by (11), one has

$$\Phi(u_1) - \Psi(u_1) \leq 0 < \rho \ .$$

At this point, we see that to satisfy the equivalent formulation of (1) recalled in Remark 2, we can choose $u_2 = u_1$ and take as u_3 any constant c such that $F(c) > \sup_{[-\sigma,\sigma]} F$. Such a c does exist by (10). So, each assumption of Theorem 1 is satisfied, and the conclusion follows. □

Remark 6. It is very important to remark that if, instead of (10), we assume that there exist $\sigma > 2\sqrt{\int_0^1 \beta(t)dt \sup_{\xi\in\mathbf{R}} \int_0^\xi g(t)dt}$ and $\xi_1, \xi_2 \in \mathbf{R}$, with $\xi_1\xi_2 > 0$, such that

$$0 < \int_0^{\xi_1} f(t)dt = \sup_{|\xi|\leq\sigma} \int_0^\xi f(t)dt < \int_0^{\xi_2} f(t) \ ,$$

then, in the conclusion of Theorem 5, we can ensure that the three non-zero solutions are non-negative (resp. non-positive) provided $\xi_1 > 0$ (resp. $\xi_2 < 0$). To see this, it suffices to apply Theorem 5 to the functions $f_0, g_0 : \mathbf{R} \to \mathbf{R}$ defined by

$$f_0(\xi) = \begin{cases} f(\xi) & \text{if } \ \xi \geq 0 \\ 0 & \text{if } \ \xi < 0 \ , \end{cases}$$

$$g_0(\xi) = \begin{cases} g(\xi) & \text{if } \ \xi \geq 0 \\ 0 & \text{if } \ \xi < 0 \ , \end{cases}$$

when $\xi_1 > 0$ or by

$$f_0(\xi) = \begin{cases} f(\xi) & \text{if } \ \xi \leq 0 \\ 0 & \text{if } \ \xi > 0 \ , \end{cases}$$

$$g_0(\xi) = \begin{cases} g(\xi) & \text{if } \ \xi \leq 0 \\ 0 & \text{if } \ \xi > 0 \ , \end{cases}$$

when $\xi_1 < 0$.

From Theorem 5, applied with $f = g$ and $\alpha = \beta$, via Remark 6, we get:

Corollary 7. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that*

$$\sup_{\xi\in\mathbf{R}} \int_0^\xi f(t)dt < +\infty, \ \limsup_{\xi\to 0} \frac{\int_0^\xi f(t)dt}{\xi^2} \leq 0 \ .$$

Moreover, suppose that there exist $\sigma > 0$ and $\xi_1, \xi_2 \in \mathbf{R}$, with $\xi_1\xi_2 > 0$, such that

$$0 < \int_0^{\xi_1} f(t)dt = \sup_{|\xi|\leq\sigma} \int_0^\xi f(t)dt < \int_0^{\xi_2} f(t)$$

and

$$\frac{\xi_1^2}{\int_0^{\xi_1} f(t)dt} < \frac{\sigma^2}{2\sup_{\xi\in\mathbf{R}} \int_0^\xi f(t)dt} \ .$$

Under such hypotheses, for every continuous function $\alpha : [0,1] \to [0,+\infty[$ *satisfying*

$$\frac{\xi_1^2}{2\int_0^{\xi_1} f(t)dt} \leq \int_0^1 \alpha(t)dt < \frac{\sigma^2}{4\sup_{\xi\in\mathbf{R}}\int_0^{\xi} f(t)dt} ,$$

there exists $\hat{\lambda} > 1$ *such that the problem*

$$\begin{cases} -u'' + u = \hat{\lambda}\alpha(t)f(u) & \text{in} \quad [0,1] \\ u'(0) = u'(1) = 0 \end{cases}$$

has at least three non-zero solutions which are non-negative or non-positive according to whether $\xi_1 > 0$ *or* $\xi_1 < 0$*.*

Another consequence of Theorem 5 is as follows:

Proposition 8. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a continuous function and* a, b, c, σ *four positive constants, with* $a < b < \sigma < c$ *and* $\sigma > \sqrt{2}b$*, such that*

$$f(\xi) \geq 0$$

for all $\xi \in]-\infty, -c] \cup [-\sigma, 0] \cup [a, b]$*, while*

$$f(\xi) \leq 0$$

for all $\xi \in [-c, -\sigma] \cup [0, a] \cup [b, +\infty[$*, and*

$$0 < \int_0^b f(t)dt < \int_0^{-c} f(t)dt \ .$$

Moreover, let $g : \mathbf{R} \to \mathbf{R}$ *and* $\beta : [0,1] \to [0,+\infty[$ *be two continuous functions such that*

$$\int_0^b g(t)dt > 0, \ \sup_{\xi\in\mathbf{R}}\int_0^{\xi} g(t)dt < +\infty, \ \limsup_{\xi\to 0}\frac{\int_0^{\xi} g(t)dt}{\xi^2} < \frac{1}{2\sup_{[0,1]}\beta}$$

and

$$\frac{b^2}{2\int_0^b g(t)dt} \leq \int_0^1 \beta(t)dt < \frac{\sigma^2}{4\sup_{\xi\in\mathbf{R}}\int_0^{\xi} g(t)dt} \ .$$

Under such hypotheses, for each continuous non-zero function $\alpha : [0,1] \to [0,+\infty[$*, there exists* $\lambda^* > 0$ *such that the problem*

$$\begin{cases} -u'' + u = \lambda^*\alpha(t)f(u) + \beta(t)g(u) & \text{in} \quad [0,1] \\ u'(0) = u'(1) = 0 \end{cases}$$

has at least three non-zero solutions.

Proof. The assumptions on the sign of f readily imply that

$$\int_0^b f(t)dt = \sup_{|\xi|\leq\sigma}\int_0^{\xi} f(t)dt < \sup_{\xi\in\mathbf{R}}\int_0^{\xi} f(t)dt = \int_0^{-c} f(t)dt \ .$$

The same assumptions also imply that $\int_0^{\xi} f(t)dt \leq 0$ for all $\xi \in [-\sigma, a]$, and so (8) holds. Consequently, if we take $\xi_1 = b$, the assumptions of Theorem 5 are satisfied and the conclusion follows. $\square$

Finally, notice the following two particular cases of Proposition 8.

Example 9. Let a, b, c, σ be four constants as in Proposition 8, and let $\varphi : [a, b] \to [0, +\infty[$, $\psi : [-c, -\sigma] \to]-\infty, 0]$ be two continuous non-constant functions, with $\psi(-c) = \psi(-\sigma) = \varphi(a) = \varphi(b) = 0$, such that

$$\int_a^b \varphi(t)dt < \int_{-\sigma}^{-c} \psi(t)dt \ .$$

Define the continuous function $f : \mathbf{R} \to \mathbf{R}$ by

$$f(\xi) = \begin{cases} \psi(\xi) & \text{if} \quad \xi \in [-c, -\sigma] \\ \varphi(\xi) & \text{if} \quad \xi \in [a, b] \\ 0 & \text{otherwise} \ . \end{cases}$$

Then, for each triple of continuous functions $g : \mathbf{R} \to \mathbf{R}$, $\alpha, \beta : [0, 1] \to [0, +\infty[$ as in Proposition 8, the conclusion of Proposition 8 does hold.

Example 10. Let a, b, c, σ be four constants as in Proposition 8. Furthermore, assume that

$$0 < b^4 - 2ab^3 < c^4 - 2\sigma c^3 \ .$$

Define the function $f : \mathbf{R} \to \mathbf{R}$ by

$$f(\xi) = \begin{cases} -\xi^3 + (a + b)\xi^2 - ab\xi & \text{if} \quad \xi \geq 0 \\ -\xi^3 - (\sigma + c)\xi^2 - \sigma c\xi & \text{if} \quad \xi < 0 \ . \end{cases}$$

Then, for each triple of continuous functions $g : \mathbf{R} \to \mathbf{R}$, $\alpha, \beta : [0, 1] \to [0, +\infty[$ as in Proposition 8, the conclusion of Proposition 8 does hold.

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