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AN INTERIOR PROXIMAL CUTTING HYPERPLANE METHOD FOR MULTIVALUED VARIATIONAL INEQUALITIES

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Abstract. We present a new method for solving multivalued variational inequalities on polyhedra, where the underlying function is upper semicontinuous and generalized monotone. The method is based on the special interior proximal function which replaces the usual quadratic function. This leads to an interior proximal algorithm. The algorithm can be viewed as combining the cutting hyperplane method and the special interior proximal function. We also analyze the global convergence of the algorithm under minimal assumptions.

1. Introduction

Let C be a polyhedral set on $\mathbb{R}^n$ defined by

$$C := \{x \in \mathbb{R}^n \mid Ax \leq b\}, \tag{1.1}$$

where A is a $(p \times n)$ matrix, $b \in \mathbb{R}^p$. We suppose that $\text{int}C = \{x \mid Ax < b\}$ is nonempty. Let $F : C \to 2^{\mathbb{R}^n}$ be a multivalued mapping. We consider the following multivalued variational inequalities (shortly (MVI)):

$$\text{Find } x^* \in C, w^* \in F(x^*) \text{ such that } \langle w^*, x - x^*\rangle \geq 0, \quad \forall x \in C.$$

In this paper we suppose that F is upper semicontinuous, generalized monotone on C and $S \neq \emptyset$, where we denote by S the set of the solutions of (MVI).

In recent years, multivalued variational inequalities become an attractive field for many researchers both theory and applications (see [3, 5, 7, 15, 19]). Various methods have been developed for solving (MVI) when F is monotone and single valued (see [4, 6, 11, 16, 18]). In general, these methods can not be applied directly to the case when F is multivalued.

There exists several methods for solving (MVI) with monotone multivalued mapping F. A typical method is the projection one (see [13, 17]). At each iteration k of this method, a point $x^k \in C$ and a point $w^k \in F(x^k)$ are computed and then the vector $x^k - \rho_k w^k$, with stepsize $\rho_k > 0$, is projected on the closed convex feasible domain C. When F is strongly monotone and the stepsizes ρ_k tend to 0, the method is strongly convergent. Cohen [12] gave an example involving a monotone mapping F where the projection method does not converge. Recently, Khanh et al. [10] developed a projection-type algorithm for (MVI). This algorithm requires two

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projections on C only in a part of iterations (one third of the subcases). For the other iterations, only one projection is used. Weak convergence is proved when F is pseudomonone on C in a real Hilbert space.

The interior proximal regularization technique is a powerful tool for analyzing and solving optimization problems (see [13, 14]). Recently this technique has been used to develop proximal iterative algorithm for variational inequalities (see [11]). In our recent papers [1], we have used the interior proximal function for variational inequalities, and developed algorithms for solving them.

In this paper, we extend our results in [2] to generalized monotone nonlipschitzian multivalued variational inequalities. Namely, we first use the interior proximal function to develop a convergent algorithm for (MVI) with a generalized monotone multivalued function F. Next we construct an appropriate hyperplane which separates the current iterative point from the solution set. We combine this technique with Armijo-type linesearch technique to obtain a convergent algorithm for generalized monotone nonlipschitzian multivalued variational inequalities. Then the next iterate is obtained as the projection of the current iterate onto the intersection of the feasible set with the halfspace containing the solution set.

This paper is organized as follows. In the next section, we present an interior proximal algorithm by using the interior proximal function to implement the cutting hyperplane method. Section 3 is devoted to the proof of its global convergence to a solution of (MVI). An application to variational inequalities is discussed in the last section.

2. Generalized monotonicity and algorithm

Now we recall well known definition of *generalized monotonicity* of mappings which will be required in our following analysis (see [21]). We assume that the mapping F satisfies this condition.

Definition 2.1. Let x^* be an element of the solution set S. The function F is called *generalized monotone* on C, if

$$\langle w, x - x^* \rangle \geq 0, \quad \forall x \in C, w \in F(x).$$

It is clear that F is generalized monotone if F is *monotone*, i.e.,

$$\langle w - w', x - x' \rangle \geq 0, \quad \forall x, x' \in C, w \in F(x), w' \in F(x').$$

More generally, F is also generalized monotone if F is *pseudomonotone*, i.e., for all $x, x' \in C, w \in F(x), w' \in F(x')$

$$\langle w', x - x' \rangle \geq 0 \Rightarrow \langle w, x - x' \rangle \geq 0.$$

However, even if F is generalized monotone, F might not be monotone or pseudomonotone. It is not difficult to check such examples (see [21]).

A classical method to solve these problems is the proximal point algorithm (see [14]), which starting with any point $x^0 \in \mathbb{R}^n_+ := \{u \in \mathbb{R}^n \mid u_i \geq 0 \ \forall i = 1, \dots, n\}$ and $\lambda_k \geq \lambda > 0$, iteratively updates x^{k+1} conforming the following problem

$$0 \in \lambda_k T(x) + \nabla_x q(x, x^k), \tag{2.1}$$

where

$$q(x,x^k) = \frac{1}{2}\|x - x^k\|^2.$$

For $y \in \mathbb{R}^n_{++} := \{u \in \mathbb{R}^n \mid u_i > 0 \ \forall i = 1,\dots,n\}$, Auslender et al. in [9] have proposed a new type of proximal interior method through replacing function $q(x,x^k)$ by $d(x,x^k)$ which is defined as

$$(2.2) \qquad d(x,y) = \begin{cases} \frac{1}{2}\|x-y\|^2 + \mu \sum\limits_{i=1}^{n} y_i^2 (\frac{x_i}{y_i}\log\frac{x_i}{y_i} - \frac{x_i}{y_i} + 1) & \text{if } x > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

with $\mu \in (0,1)$. It is easy to verify that $d(\cdot,y)$ is a closed proper convex function, nonnegative and $d(x,y) = 0$ if and only if $x = y$. One of the motivation behind the specific form of the function $d(\cdot,\cdot)$ is as follows: The function $\frac{1}{2}\|x-y\|^2$ is an usual regularization term used in a proximal method, while the function $\sum\limits_{i=1}^{n} y_i^2(\frac{x_i}{y_i}\log\frac{x_i}{y_i} - \frac{x_i}{y_i} + 1)$ is added to enforce the method to become an interior one, i.e., to generate iterates staying in interior of $\mathbb{R}_{++}$.

Applying this idea to C defined by (1.1), for $y \in \text{int}C$, let a_i be the rows of the matrix A, we define the quantities,

$$l_i(x) = b_i - \langle a_i, x\rangle,$$

$$l(x) = \Big(l_1(x), l_2(x), \dots, l_p(x)\Big)^T,$$

$$D(x,y) = \begin{cases} \frac{1}{2}\|x-y\|^2 + \mu \sum\limits_{i=1}^{p} l_i^2(y)\Big(\frac{l_i(x)}{l_i(y)}\log\frac{l_i(x)}{l_i(y)} - \frac{l_i(x)}{l_i(y)} + 1\Big) & \text{if } x \in \text{int}C, \\ +\infty & \text{otherwise.} \end{cases}$$

We denote by $\nabla_1 D(x,y)$ the gradient of $D(\cdot,y)$ at x for every $y \in C$. It is easy to see that

$$\nabla_1 D(x,y) = x - y - \mu A^T X_y \log\frac{l(x)}{l(y)},$$

where $X_y = diag\Big(l_1(y),\dots,l_p(y)\Big)$ and $\log\frac{l(x)}{l(y)} = \Big(\log\frac{l_1(x)}{l_1(y)},\dots,\log\frac{l_p(x)}{l_p(y)}\Big)$.

Otherwise, if F is a point-to-point mapping, then (MVI) can be formulated as the following variational inequalities, shortly (VIP), it can be written in:

Find $x^* \in C$ such that

$$\langle F(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C.$$

In this case, it is known that solutions coincide with zeros of the following projected residual function

$$T(x) = x - \text{Pr}_C(x - F(x)).$$

In other words, $x^0 \in C$ is a solution of (VIP) if and only if $T(x^0) = 0$ (see [20]). Let x^k be a current approximation to the solution. First, we compute $w^k = \text{argsup}\{\langle w, x^k\rangle \mid w \in F(x^k)\}$ and $\text{Pr}_C(x^k - cw^k)$ for some positive constant c. Next, we search the line segment between x^k and $\text{Pr}_C(x^k - cw^k)$ for a point $(\bar{w}^k, z^k)$ such that the hyperplane $\partial H_k = \{x \in \mathbb{R}^n \mid \langle \bar{w}^k, x - z^k\rangle = 0\}$ strictly separates x^k from the solution set S of the problem. To find such $(\bar{w}^k, z^k)$, we may use a computationally inexpensive Armijo-type procedure. Then we compute the next iterate x^{k+1} by projecting x^k onto the intersection of the feasible set C with the halfspace

$$H_k = \{x \in \mathbb{R}^n \mid \langle \bar{w}^k, x - z^k\rangle \le 0\}.$$

Using the interior proximal function $D(\cdot,\cdot)$ to implement the cutting hyperplane method for (MVI) is defined as follows.

Algorithm 2.2.

Step 0: Choose $x^0 \in C, w^0 \in F(x^0), k = 0, 0 < \sigma < \frac{\beta}{2}$, and $\gamma \in (0,1)$.

Step 1: Compute

$$\begin{aligned} &w^k := \text{argsup}\{\langle w, x^k\rangle \mid w \in F(x^k)\},\\ &y^k := \text{argmin}\{\langle w^k, y - x^k\rangle + \beta D(y, x^k) \mid y \in C\}, \quad (2.3)\\ &r(x^k) := x^k - y^k.\end{aligned}$$

Step 2: *(Cutting hyperplane)* Find the smallest nonnegative number m_k of m such that

$$v_k := \sup\{\langle w, r(x^k)\rangle \mid w \in F\big(x^k - \gamma^{m_k} r(x^k)\big)\} \ge \sigma\|r(x^k)\|^2. \quad (2.4)$$

Choose $\bar{w}^k \in F\big(x^k - \gamma^{m_k} r(x^k)\big)$ such that $\langle \bar{w}^k, r(x^k)\rangle = v_k$.
Set $z^k := x^k - \gamma^{m_k} r(x^k)$ and

$$H_k := \{x \in \mathbb{R}^n \mid \langle \bar{w}^k, x - z^k\rangle \le 0\}.$$

Step 3: Find $x^{k+1} := \text{Pr}_{C\cap H_k}(x^k)$. Set $k := k+1$, and go to Step 1.

3. Convergence of the algorithm

In the next lemma, we justify the stopping criterion.

Lemma 3.1. *If $r(x^k) = 0$, then (x^k, w^k) is a solution to (MVI).*

Proof. Since y^k is the solution to problem (2.3) and an optimization result in convex programming (see [13]), we have

$$\langle w^k + \beta\nabla_1 D(y^k, x^k), y - y^k\rangle \ge 0 \quad \forall y \in C.$$

Replacing $y^k = x^k$ in this inequality, we get

$$\langle w^k + \beta\nabla_1 D(x^k, x^k), y - x^k\rangle \ge 0 \quad \forall y \in C.$$

Since

$$\nabla_1 D(x, y) = x - y - \mu A^T X_y \log\frac{l(x)}{l(y)} \quad \forall x, y \in C, \quad (3.1)$$

we have

$$\nabla_1 D(x^k, x^k) = 0.$$

Thus

$$\langle w^k, y - x^k\rangle \ge 0 \quad \forall y \in C,$$

which implies that (x^k, w^k) is a solution to (MVI).

□

In Algorithm 2.2, we need to show the existence of the nonnegative integer m_k.

Lemma 3.2. *For $\gamma \in (0,1), 0 < \sigma < \frac{\beta}{2}$, if $r(x^k) > 0$ then there exists the smallest nonnegative integer m_k such that*

$$\sup\{\langle w, r(x^k)\rangle \mid w \in F\big(x^k - \gamma^{m_k} r(x^k)\big)\} \geq \sigma \|r(x^k)\|^2. \tag{3.2}$$

Proof. Assume on the contrary, (3.2) is not satisfied for any nonnegative integer i, i.e.,

$$\langle w, r(x^k)\rangle < \sigma \|r(x^k)\|^2 \quad \forall w \in F\big(x^k - \gamma^{m_k} r(x^k)\big).$$

As $k \to \infty$, from the upper semicontinuity of F, we have

$$\langle w, r(x^k)\rangle \leq \sigma \|r(x^k)\|^2 \quad \forall w \in F(x^k).$$

Substituting $w = \bar{w}^k \in F(x^k)$, we get

$$\langle \bar{w}^k, r(x^k)\rangle \leq \sigma \|r(x^k)\|^2. \tag{3.3}$$

For each $t > 0$, we have $1 - \frac{1}{t} \leq \log t$. We obtain after multiplication by $\frac{l_i(y^k)}{l_i(x^k)} > 0$ for each $i = 1, \dots, p$,

$$\frac{l_i(y^k)}{l_i(x^k)} - 1 \leq \frac{l_i(y^k)}{l_i(x^k)} \log \frac{l_i(y^k)}{l_i(x^k)}.$$

Then,

$$\begin{aligned} D(y^k, x^k) &= \frac{1}{2}\|x^k - y^k\|^2 + \mu \sum_{i=1}^{n} l_i^2(x^k)\Big(\frac{l_i(y^k)}{l_i(x^k)} \log \frac{l_i(y^k)}{l_i(x^k)} - \frac{l_i(y^k)}{l_i(x^k)} + 1\Big) \\ &\geq \frac{1}{2}\|r(x^k)\|^2. \end{aligned} \tag{3.4}$$

Since y^k is the solution to the strongly convex program (2.3), we have

$$\langle w^k, y - x^k\rangle + \beta D(y, x^k) \geq \langle w^k, y^k - x^k\rangle + \beta D(y^k, x^k) \quad \forall y \in C.$$

Substituting $y = x^k \in C$, we get

$$-\langle w^k, r(x^k)\rangle + \beta D(y^k, x^k) \leq 0. \tag{3.5}$$

Combinating (3.4) with (3.5), we obtain

$$\frac{\beta}{2}\|r(x^k)\|^2 \leq \langle w^k, r(x^k)\rangle. \tag{3.6}$$

Then, inequalities (3.3) and (3.6) imply that

$$\frac{\beta}{2}\|r(x^k)\|^2 \leq \langle w^k, r(x^k)\rangle \leq \sigma\|r(x^k)\|^2.$$

Hence it must be either $r(x^k) = 0$ or $\sigma \geq \frac{\beta}{2}$. The first case contracdicts to $r(x^k) \neq 0$, while the second one contracdicts to the fact $\sigma < \frac{\beta}{2}$. □

The following results perform some property of the cutting hyperplane H_k.

Lemma 3.3. *Let $\{x^k\}$ be a sequence generated by Algorithm 2.2. Then the followings hold:*

i) $x^k \notin H_k, S \subseteq C \cap H_k$.
ii) $x^{k+1} = Pr_{C \cap H_k}(\bar{y}^k)$, *where* $\bar{y}^k = Pr_{H_k}(x^k)$.

Proof. i) By noting $r(x^k) \neq 0$, we have

$$\begin{array}{rcl}\langle \bar{w}^k, x^k - z^k\rangle &=& \langle \bar{w}^k, x^k - (x^k - \gamma^{m_k} r(x^k))\rangle\\ &=& \langle \bar{w}^k, \gamma^{m_k} r(x^k)\rangle\\ &\geq& \sigma\gamma^{m_k}\|r(x^k)\|^2 > 0.\end{array}$$

This implies $x^k \notin H_k$. Since F is assumed to be generalized monotone,

$$\langle \bar{w}^k, z^k - x^*\rangle \geq 0 \Rightarrow \langle \bar{w}^k, x^* - z^k\rangle \leq 0 \Rightarrow x^* \in H_k.$$

ii) We know that

$$H = \{x \in \mathbb{R}^n \mid \langle w, x - x^0\rangle \leq 0\}, \mathrm{Pr}_H(y) = y - \frac{\langle w, y - x^0\rangle}{\|w\|^2} w.$$

Hence,

$$\begin{array}{rcl}\bar{y}^k &=& \mathrm{Pr}_{H_k}(x^k)\\ &=& x^k - \dfrac{\langle \bar{w}^k, x^k - z^k\rangle}{\|\bar{w}^k\|^2}\bar{w}^k\\ &=& x^k - \dfrac{\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|^2}\bar{w}^k.\end{array}$$

Otherwise, for every $y \in C \cap H_k$ there exists $\lambda \in (0, 1)$ such that

$$\hat{x} = \lambda x^k + (1 - \lambda)y \in C \cap \partial H_k,$$

where $\partial H_k = \{x \in \mathbb{R}^n \mid \langle \bar{w}^k, x - z^k\rangle = 0\}$, because $x^k \in C$ but $x^k \notin H_k$.

Therefore,

$$\begin{array}{rcl}\|y - \bar{y}^k\|^2 &\geq& (1 - \lambda)^2\|y - \bar{y}^k\|^2\\ &=& \|\hat{x} - \lambda x^k - (1 - \lambda)\bar{y}^k\|^2\\ &=& \|(\hat{x} - \bar{y}^k) - \lambda(x^k - \bar{y}^k)\|^2\\ &=& \|\hat{x} - \bar{y}^k\|^2 + \lambda^2\|x^k - \bar{y}^k\|^2 - 2\lambda\langle \hat{x} - \bar{y}^k, x^k - \bar{y}^k\rangle\\ &=& \|\hat{x} - \bar{y}^k\|^2 + \lambda^2\|x^k - \bar{y}^k\|^2\\ &\geq& \|\hat{x} - \bar{y}^k\|^2,\end{array} \tag{3.7}$$

because $\bar{y}^k = \mathrm{Pr}_{H_k}(x^k)$. Also we have

$$\begin{array}{rcl}\|\hat{x} - x^k\|^2 &=& \|\hat{x} - \bar{y}^k + \bar{y}^k - x^k\|^2\\ &=& \|\hat{x} - \bar{y}^k\|^2 - 2\langle \hat{x} - \bar{y}^k, x^k - \bar{y}^k\rangle + \|\bar{y}^k - x^k\|^2\\ &=& \|\hat{x} - \bar{y}^k\|^2 + \|\bar{y}^k - x^k\|^2.\end{array}$$

Since $x^{k+1} = \mathrm{Pr}_{C\cap H_k}(x^k)$, using the Pythagorean theorem we can reduce that

$$\begin{array}{rcl}\|\hat{x} - \bar{y}^k\|^2 &=& \|\hat{x} - x^k\|^2 - \|\bar{y}^k - x^k\|^2\\ &\geq& \|x^{k+1} - x^k\|^2 - \|\bar{y}^k - x^k\|^2\\ &=& \|x^{k+1} - \bar{y}^k\|^2.\end{array} \tag{3.8}$$

From (2.3) and (3.8), we have

$$\|x^{k+1} - \bar{y}^k\| \leq \|y - \bar{y}^k\| \quad \forall y \in C \cap H_k,$$

which implies

$$x^{k+1} = \Pr_{C \cap H_k}(\bar{y}^k).$$

□

In order to prove the convergence of algorithm 2.2, we give the following key property of the sequence $\{x^k\}$ generated by the algorithm.

Lemma 3.4. *The sequence $\{x^k\}$ generated by Algorithm 2.2 satisfies the following inequality.*

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - y^k\|^2 - \left(\frac{\gamma^{m_k}\sigma}{\|\bar{w}^k\|}\right)^2 \|r(x^k)\|^4. \tag{3.9}$$

Proof. Since $x^{k+1} = \Pr_{C \cap H_k}(y^k)$, we have

$$\langle y^k - x^{k+1}, z - x^{k+1} \rangle \leq 0 \quad \forall z \in C \cap H_k.$$

Substituting $z = x^* \in C \cap H_k$, then we have

$$\langle y^k - x^{k+1}, x^* - x^{k+1} \rangle \leq 0 \Leftrightarrow \langle y^k - x^{k+1}, x^* - y^k + y^k - x^{k+1} \rangle \leq 0,$$

which implies

$$\|x^{k+1} - y^k\|^2 \leq \langle x^{k+1} - y^k, x^* - y^k \rangle.$$

Hence,

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|x^{k+1} - y^k + y^k - x^*\|^2 \\
&= \|x^{k+1} - y^k\|^2 + \|y^k - x^*\|^2 + 2\langle x^{k+1} - y^k, y^k - x^* \rangle \\
&\leq \langle x^* - y^k, x^{k+1} - y^k \rangle + \|y^k - x^*\|^2 + 2\langle x^{k+1} - y^k, y^k - x^* \rangle \\
&= \|y^k - x^*\|^2 + \langle x^{k+1} - y^k, y^k - x^* \rangle \\
&= \|y^k - x^*\|^2 - \|x^{k+1} - y^k\|^2. \qquad (3.10)
\end{aligned}$$

Since $z^k = x^k - \gamma^{m_k} r(x^k)$ and

$$y^k = \Pr_{H_k}(x^k) = x^k - \frac{\langle \bar{w}^k, x^k - z^k \rangle}{\|\bar{w}^k\|^2} \bar{w}^k,$$

we have

$$
\begin{aligned}
&\|y^k - x^*\|^2\\
= \;& \|x^k - x^*\|^2 + \frac{\langle \bar{w}^k, x^k - z^k\rangle^2}{\|\bar{w}^k\|^4}\|\bar{w}^k\|^2 - \frac{2\langle \bar{w}^k, x^k - z^k\rangle}{\|\bar{w}^k\|^2}\langle \bar{w}^k, x^k - x^*\rangle\\
= \;& \|x^k - x^*\|^2 + \left(\frac{\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|}\right)^2 - \frac{2\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|^2}\langle \bar{w}^k, x^k - x^*\rangle\\
= \;& \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|}\right)^2\\
& -2\left[\frac{\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|^2}\langle \bar{w}^k, x^k - x^*\rangle - \left(\frac{\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|}\right)^2\right]\\
= \;& \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|}\right)^2\\
& -\frac{2\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|^2}\left[\langle \bar{w}^k, x^k - x^*\rangle - \gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle\right]\\
= \;& \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|}\right)^2\\
& -\frac{2\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|^2}\langle \bar{w}^k, x^k - x^* - \gamma^{m_k} r(x^k)\rangle\\
= \;& \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|}\right)^2 - \frac{2\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|^2}\langle \bar{w}^k, z^k - x^*\rangle. \qquad (3.11)
\end{aligned}
$$

From the generalized monotonicity of F, we see that $\langle \bar{w}^k, z^k - x^*\rangle \geq 0$. This, together with $\bar{w}^k \in F(z^k)$, imply

$$\langle \bar{w}^k, r(x^k)\rangle \geq \sigma\|r(x^k)\|^2.$$

Thus, (3.11) reduces to

$$
\begin{aligned}
\|y^k - x^*\|^2 &\leq \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\langle \bar{w}^k, r(x^k)\rangle}{\|\bar{w}^k\|}\right)^2\\
&\leq \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k}\sigma}{\|\bar{w}^k\|}\right)^2\|r(x^k)\|^4. \qquad (3.12)
\end{aligned}
$$

Combining (3.10) and (3.12), we obtain (3.9) □

Theorem 3.5 (Convergence theorem)**.** *Let F be upper semicontinuous, compact valued and generalized monotone on C. Then the sequence $\{x^k\}$ generated by Algorithm 2.2 converges to a solution of (MVI).*

Proof. The inequality (3.9) implies that the sequence $\{\|x^k - x^*\|\}$ is nonincreasing and hence convergent. Consequently, the sequence $\{x^k\}$ is bounded.

Since $w^k \in F(x^k)$, $r(x^k) = x^k - \mathrm{Pr}_C(x^k - cw^k)$, $z^k = x^k - \gamma^{m_k} r(x^k)$ and F is upper semicontinuous and compact valued on C, the sequence $\{z^k\}$ is also bounded (see [8]). Hence, the sequence $\{w^k\}$ is bounded, i.e., there exists $M > 0$ such that

$$\|w^k\| \leq M \quad \forall k = 1, \ldots.$$

This, together with (3.9), implies

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - y^k\|^2 - \left(\frac{\gamma^{m_k}\sigma}{M}\right)^2 \|r(x^k)\|^4. \tag{3.13}$$

Since $\{\|x^k - x^*\|\}$ converges to zero, it is easy to see that

$$\lim_{k\to\infty} \gamma^{m_k}\|r(x^k)\| = 0.$$

The cases remaining to consider are the following.

Case 1. $\limsup_{k\to\infty} \gamma^{m_k} > 0$. This case must follow that $\liminf_{k\to\infty} \|r(x^k)\| = 0$. Since $\{x^k\}$ is bounded, there exists an accumulation point $\bar{x}$ of $\{x^k\}$. In other words, a subsequence $\{x^{k_i}\}$ converges to some $\bar{x}$ such that $r(\bar{x}) = 0$, as $i \to \infty$. Then we see from Lemma 3.3 that $\bar{x} \in S$, and besides we can take $x^* = \bar{x}$, in particular in (3.13). Thus $\{\|x^k - \bar{x}\|\}$ is a convergent sequence. Since $\bar{x}$ is an accumulation point of $\{x^k\}$, the sequence $\{\|x^k - x^*\|\}$ converges to zero, i.e., $\{x^k\}$ converges to $\bar{x} \in S$.

Case 2. $\lim_{k\to\infty} \gamma^{m_k} = 0$. Since m_k is the smallest nonnegative integer, $m_k - 1$ does not satisfy (2.4). Hence, we have

$$\langle w, r(x^k)\rangle < \sigma\|r(x^k)\|^2 \quad \forall w \in F\big(x^k - \gamma^{m_k-1}r(x^k)\big),$$

and besides

$$\langle w, r(x^{k_i})\rangle < \sigma\|r(x^{k_i})\|^2 \quad \forall w \in F\big(x^{k_i} - \gamma^{m_{k_i}-1}r(x^{k_i})\big). \tag{3.14}$$

Passing onto the limit in (3.14) as $i \to \infty$ and using the upper semicontinuity of F, we have

$$\langle w, r(\bar{x})\rangle \leq \sigma\|r(\bar{x})\|^2 \quad \forall w \in F(\bar{x}). \tag{3.15}$$

From (3.6) we have

$$\frac{\beta}{2}\|r(x^{k_i})\|^2 \leq \langle w^{k_i}, r(x^{k_i})\rangle.$$

Since F is upper semicontinuous, passing onto the limit as $i \to \infty$ we obtain

$$\frac{\beta}{2}\|r(\bar{x})\|^2 \leq \langle \bar{w}, r(\bar{x})\rangle.$$

Combining this with (3.15), we have

$$\frac{\beta}{2}\|r(\bar{x})\|^2 \leq \langle \bar{w}, r(\bar{x})\rangle \leq \sigma\|r(\bar{x})\|^2,$$

which implies $r(\bar{x}) = 0$ or $\sigma \geq \frac{\beta}{2}$. The second case contracdicts to the fact $0 < \sigma < \frac{\beta}{2}$ and hence $r(\bar{x}) = 0$, $\bar{x} \in S$. Letting $x^* = \bar{x}$ and repeating the previous arguments, we conclude that the whole sequence $\{x^k\}$ converges to $\bar{x} \in S$.

□

4. Applications to variational inequalities

The aim of this section is to consider the proposed algorithm on a class of the multivalued variational inequalities, where

$$C := \{x \in \mathbb{R}^n \mid Ax \le b\},$$

and the function $F : C \to \mathbb{R}^n$ is of the form:

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^*\rangle \ge 0 \quad \forall x \in C.$$

We now apply Algorithm 2.2 to the variational inequalities (VIP). Note that in this case, at iteration k, we have

$$\begin{aligned} w^k &= F(x^k),\\ \bar{w}^k &= F\big(x^k - \gamma^{m_k} r(x^k)\big),\\ v^k &= \langle F\big(x^k - \gamma^{m_k} r(x^k)\big), r(x^k)\rangle. \end{aligned}$$

Then, the algorithm for (VIP) can be written in the following.

Algorithm 4.1.

Step 0: Choose $x^0 \in C, k = 0, 0 < \sigma < \frac{\beta}{2}$, and $\gamma \in (0, 1)$.

Step 1: Compute

$$\begin{aligned} y^k &:= \operatorname{argmin}\{\langle F(x^k), y - x^k\rangle + \beta D(y, x^k) \mid y \in C\},\\ r(x^k) &:= x^k - y^k. \end{aligned} \tag{4.1}$$

Step 2: (Cutting hyperplane) Find the smallest nonnegative number m_k of m such that

$$v_k \ge \sigma \|r(x^k)\|^2. \tag{4.2}$$

Set $z^k := x^k - \gamma^{m_k} r(x^k)$, and

$$H_k := \{x \in \mathbb{R}^n \mid \langle \bar{w}^k, x - z^k\rangle \le 0\}.$$

Step 3: Find $x^{k+1} := \mathrm{Pr}_{C\cap H_k}(x^k)$. Set $k := k + 1$, and go to Step 1.

Validity and convergence of this algorithm is immediate from Algorithm 2.2. Subproblems (4.1) and (4.2) can then be solved efficiently, for example, by the Matlab Optimization Toolbox.

To illustrate our algorithm, we consider an academic numerical test of the function F with $n = 7$ and $F(x) := Mx + G(x) + q$, where G is defined by the components of the $G(x)$ are $G_j(x) = d_j \arctan(x_j)\ \forall j \ge 1$, d_j is chosen by $d = (3, 2, 1, 4, 9, 1, 2)^T$.

$$M := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7\\ 1 & 0 & 0 & 5 & 6 & 2 & 3\\ 0 & 0 & 9 & 2 & 3 & 4 & 5\\ 3 & 3 & 3 & 4 & 1 & 1 & 1\\ 3 & 4 & 5 & 1 & 2 & 2 & 3\\ 0 & 1 & 0 & 0 & 1 & 1 & 9\\ 2 & 3 & 4 & 1 & 2 & 3 & 4 \end{pmatrix}, q := \begin{pmatrix} 1\\ 3\\ 0\\ 1\\ 5\\ 6\\ 9 \end{pmatrix}, d := \begin{pmatrix} 3\\ 2\\ 0\\ 4\\ 9\\ 1\\ 2 \end{pmatrix}$$

and

$$C := \begin{cases} x \in \mathbb{R}^7_+, \\ 4 \leq x_1 + 2x_2 + x_3 + 3x_5 + x_7 \leq 10, \\ 9 \leq \sum_{i=1}^{7} x_i \leq 15, \\ 6 \leq x_2 + x_3 + 2x_4 + x_7 \leq 13, \\ 1 \leq x_2 + x_3 \leq 5. \end{cases}$$

Then, the function F is generalized monotone (but not monotone), continuous and differentiable on C.

Lemma 3.1 shows that if $r(x^k) = 0$ then x^k is a solution to (VIP). So that we can say that x^k is an ϵ-solution to (VIP) if we have $\|r(x^k)\| \leq \epsilon$ with $\epsilon > 0$. Take $\mu = 0.5, \epsilon = 10^{-6}, \gamma = 0.7, \beta = 2, \sigma = 1$ and $x^0 = (1, 1, 0.5, 1, 3, 1, 3)^T \in C$. We perform the Algorithm 2.2 in Matlab R2008a running on a PC Desktop Intel(R) Core(TM)2 Duo CPU T5750@ 2.00GHz 1.32 GB, 2Gb RAM. The tolerance is taken by $\epsilon = 10^{-6}$. The approximate solution obtained after 9 iterations is

$$x^* = (0.5836, 1.2555, 0.4890, 0.7737, 2.6514, 0.7825, 2.7080)^T.$$

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