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KRASNOSEL′SKII AND KY FAN TYPE FIXED POINT THEOREMS IN ORDERED BANACH SPACES

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Abstract. A common fixed point theorem for a condensing map S and a 1-set contractive map T, defined on a closed convex subset of an ordered Banach space, is proved. As applications, a number of Krasnosel′skii type fixed point theorems, iterative approximation of common fixed points and Ky Fan type approximation theorems for various classes of 1-set contractive and 1-ball contractive maps (e.g. operators of contractive type with compact or completely continuous perturbations, operators of semicontractive type, pseudo-contractive maps etc.) are derived. Moreover, an integral equation is solved as an application of our main result.

1. Introduction

In 1958, M. A. Krasnosel′skii [15] proved a fixed point theorem which is an important supplement to both the Schauder fixed point theorem and the Banach contraction principle. Krasnosel′skii fixed point theorem has a wide range of applications to nonlinear integral equations of mixed type. It has also been extensively used in differential and functional differential equations. Krasnosel′skii fixed point theorem has been generalized in many directions, see ([3, 6, 21]) and the references therein. Fixed point theorems for monotone operators in ordered Banach spaces are widely investigated and have found various applications in differential and integral equations (see [7, 8] and references therein). Ambrosetti [1] was the first to use the measure of noncompactness to prove existence results for differential equations in Banach spaces. Recently, Dhage [8] proved some common fixed point theorems for two condensing and weakly isotone mappings in ordered Banach spaces and obtained existence theorems for common solutions of two nonlinear differential equations in Banach spaces. Dhage et al. [9] further improved the results in [8] for common fixed points. Using a result from [9], we prove a common fixed point theorem for weakly isotone mappings where one of the mappings is 1-set contractive (or 1-ball contractive) on an ordered Banach space. We then apply our main result to establish some Krasnosel′skii type fixed point theorems, iterative approximation of common fixed points and Ky Fan type best approximation theorems for various types of 1-set contractive, 1-ball contractive and pseudo-contractive maps. Moreover, an integral equation is solved as an application of our main result.

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2. Preliminaries

Let M be a nonempty subset of a normed space X and A be a nonempty bounded subset of M, and let $\alpha(.)$ be the set measure of noncompactness, that is, $\alpha(A) = \inf\{ c > 0 : A$ can be covered by a finite number of sets of diameter $\leq c\}$ and $\chi(.)$ the ball measure of noncompactness, that is, $\chi(A) = \inf\{ c > 0 : A$ can be covered by a finite number of balls with centers in M and radius $c\}$. It is well known that the measures $\alpha(.)$ and $\chi(.)$ are different, although, they have a good deal in common (see for details [2, 17]). Let $T : M \to X$ be a mapping. If $T(M)$ is bounded and for every nonempty bounded subset A of M with $\alpha(A) > 0$, we have, $\alpha(T(A)) < \alpha(A)$, then T is called set-condensing. If there exists k, $0 \leq k \leq 1$, such that $T(M)$ is bounded and for each nonempty bounded subset A of M, we have $\alpha(T(A)) \leq k\alpha(A)$, then T is called k-set contractive. Clearly, every k-set contractive map with $k < 1$ (known as strict-set contraction in [8]) is set-condensing and that every set-condensing map is 1-set contractive. As in the case of $\alpha(.)$, for $\chi(.)$, we have ball-condensing and k-ball contractive maps. We shall denote by $\overline{M}$ (respectively ∂M, $int(M)$)the closure (respectively, the boundary, the interior) of M. Let $T : M \to X$ be a mapping. Then T is called: (1) nonlinear contraction if there exists a continuous and nondecreasing function $\phi_T : [0, \infty) \to [0, \infty)$ such that $\|Tx - Ty\| \leq \phi_T(\|x - y\|)$ for all $x, y \in M$, where $\phi_T(r) < r$ for $r > 0$. In particular, if $\phi_T(r) = kr$, $0 \leq k < 1$, then T is called a contraction mapping; (2) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in M$; (3) generalized contraction if, for each $x \in M$ there exists a number $k(x) < 1$ with $\|Tx - Ty\| \leq k(x)\|x - y\|$ for each $y \in M$; (4) completely continuous if it maps weakly convergent sequences into strongly convergent sequences; (5) compact if $\overline{T(A)}$ is compact when $A \subset M$ is bounded; (6) uniformly strictly contractive on M relative to X if the map $T : X \to X$ has the property that, for each $x \in X$ there exists a number $k(x) < 1$ such that $\|Tx - Ty\| \leq k(x)\|x - y\|$ for each $y \in M$; (7) $LANE$ (locally almost nonexpansive) if, for each $u \in M$ and $\varepsilon > 0$, there exists a weak neighborhood N_u of u in M (depending also on ε) such that $\|Tx - Ty\| \leq \|x - y\| + \varepsilon$ for each $x, y \in N_u$; (8) pseudo-contractive if $\|x - y\| \leq \|(1 + r)(x - y) - r(Tx - Ty)\|$ for each $x, y \in M$ and $r > 0$, or equivalently, $(1 - \lambda)\|x - y\| \leq \|(I - \lambda T)x - (I - \lambda T)y\|$ for each $x, y \in M$ and $\lambda \in (0, 1)$ (with I denoting the identity mapping); (9) k-dissipative with $k \in \mathbb{R}$, if $(Tx - Ty, f) \leq k\|x - y\|^2$ for all $f \in J(x - y)$ and $x, y \in M$ where $J : X \to 2^{X^*}$ is a normalized duality mapping defined by $J(x) = \{f \in X^* : (x, f) = \|f\|\|x\|$ and $\|f\| = \|x\|\}$.

Let M be a nonempty closed bounded subset of X and $T : M \to X$ a continuous map.

(1) Suppose there exists a continuous mapping $V : X \times X \to X$ such that $Tx = V(x, x)$ for $x \in M$. Then
   - (i) T is strictly semicontractive if, for each $x \in M$, $V(., x)$ is contraction and $V(x, .)$ is compact;
   - (ii) T is weakly semicontractive if, for each $x \in M$, $V(., x)$ is nonexpansive and $V(x, .)$ is compact;
   - (iii) T is semicontractive if, for each $x \in M$, $V(., x)$ is nonexpansive and $V(x, .)$ is completely continuous.

(2) Suppose there exists a continuous mapping $V : M \times M \to X$ such that $Tx = V(x, x)$ for $x \in M$. Then
   (iv) T is weakly semicontractive type if, for each $x \in M$, $V(., x)$ is a nonexpansive map of M into X and $x \to V(x, .)$ of M into the space of continuous mappings of M into X is compact;
   (v) T is semicontractive type if, for each $x \in M$, $V(., x)$ is a nonexpansive map of M into X and $V(x, .)$ is completely continuous from M to X, uniformly for $x \in M$. For more details of these mappings, we refer the reader to [20].

For $x \in X$, let $I_M(x) := \{x + a(u - x) : u \in M, a \geq 0\}$.

Let M be a convex subset of a Banach space X with $0 \in int(M)$. We define the Minkowski functional $p : X \to [0, \infty)$ of M as

$$p(x) = \inf\{r > 0 : x \in rM\}, \quad x \in X.$$

The following properties are well known;

(i) p is continuous;

(ii) $p(x + y) \leq p(x) + p(y)$ for $x, y \in X$;

(iii) $p(\lambda x) = \lambda p(x), \lambda \geq 0, x \in X$;

(iv) $0 \leq p(x) < 1$ for $x \in int(M)$;

(v) $p(x) > 1$ for $x \notin \overline{M}$;

(vi) $p(x) = 1$ for $x \in \partial M$.

We define for any $x \in X$, $d_p(x, M) = \inf\{p(x - y) : y \in M\}$.

Let $S, T : M \to X$ be mappings. A point $x \in M$ is a common fixed point of S and T if $x = Sx = Tx$. The set of fixed points of S is denoted by $F(S)$. A mapping $T : M \to X$ is called weakly continuous if $\{x_n\}$ converges weakly to x implies $\{Tx_n\}$ converges weakly to Tx. If M is convex, then T is called affine if $T((1 - k)x + ky) = (1 - k)Tx + kTy$ for all $x, y \in M$ and $k \in [0, 1]$. A mapping $T : M \to X$ is said to be demiclosed at y if for every sequence $\{x_n\} \subset M$ such that $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges strongly to y, we have $y = Tx$.

Let X be a Banach space and K be a cone in X (i.e. K is closed subset such that $K + K \subseteq K$, $tK \subseteq K$ for all $t \geq 0$, $K \cap (-K) = \{0\}$). We define an order relation $\preceq$ in X with the help of the cone K as follows: for $x, y \in X$, $x \preceq y$ iff. $y - x \in K$. By an ordered Banach space X, we mean the Banach space X equipped with a partial ordering $\preceq$ induced by K. A mapping $T : M \to M$ is said to be isotone increasing if $x, y \in M$ with $x \preceq y$, then $Tx \preceq Ty$. Two mappings $S, T : M \to M$ are said to be weakly isotone increasing if $Sx \preceq TSx$ and $Tx \preceq STx$ hold for all $x \in M$. Similarly, the mappings $S, T : M \to M$ are said to be weakly isotone decreasing if $Sx \succeq TSx$ and $Tx \succeq STx$ hold for all $x \in M$. We say that two mappings $S, T : M \to M$ are weakly isotone if they are either weakly isotone increasing or weakly isotone decreasing on M.

The following results will be needed.

Theorem 2.1 ([8, 9])**.** *Let M be a nonempty closed subset of an ordered Banach space X and $S, T : M \to M$ be two continuous set-condensing (or ball-condensing) mappings. If S and T are weakly isotone decreasing, then $F(S) \cap F(T) \neq \emptyset$.*

Theorem 2.2 ([8])**.** *Let M be a nonempty closed convex and bounded subset of a Banach space X, and let $T : M \to M$ be a nonlinear contraction. Then T has a unique fixed point x_0 and the sequence of successive iterations $\{T^n x\}$ converges to x_0 for each $x \in M$.*

3. Main results

Theorem 3.1. *Let M be a nonempty closed convex subset of an ordered reflexive Banach space X and $S : M \to M$ be an affine continuous and set-condensing (or ball-condensing) map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to M$ is continuous 1-set contractive (or 1-ball contractive) map. Suppose that the following two conditions are satisfied:*

(a) *$(I - T)$ is demiclosed at 0,*

(b) *S and T are weakly isotone decreasing.*

Then $F(S) \cap F(T) \neq \emptyset$.

Proof. Without loss of generality, we may assume that $0 \in M$. Define $T_n : M \to M$ by

$$T_n x = \Big(1 - \frac{1}{n+1}\Big)Tx \tag{3.1}$$

for all $x \in M$. Then each T_n is a continuous set-condensing (or ball-condensing) map. We show that for each n, the pair $\{S, T_n\}$ is weakly isotone decreasing. As S and T are weakly isotone decreasing, S is affine and $Sx \succeq 0$, so for any $x \in M$,

$$\begin{aligned} T_n x &= \Big(1 - \frac{1}{n+1}\Big)Tx \\ &\succeq \Big(1 - \frac{1}{n+1}\Big)STx \\ &= S\Big(\Big(1 - \frac{1}{n+1}\Big)Tx\Big) \\ &= ST_n x \end{aligned}$$

and

$$T_n Sx = \Big(1 - \frac{1}{n+1}\Big)TSx \preceq \Big(1 - \frac{1}{n+1}\Big)Sx \preceq Sx.$$

Thus by Theorem 2.1, there is $x_n \in M$ such that $x_n = Sx_n = T_n x_n$ for all n. This implies that $x_n - Tx_n = (\frac{1}{n+1})Tx_n$. Since $T(M)$ is bounded, it follows that $x_n - Tx_n \to 0$ as $n \to \infty$. Since X is reflexive and $\{x_n\}$ a bounded sequence, we may assume (eventually considering a subsequence) that $\{x_n\}$ is weakly convergent to an element $x_0 \in M$. As $(I - T)$ is demiclosed at 0, we have that $Tx_0 = x_0$. The map S is continuous and affine, so it is weakly continuous and hence $Sx_0 = x_0$ as desired. ☐

Note that Theorem 3.1 remains valid if S is positively homogeneous (or, in particular, $S(kx) = kS(x)$ where $0 < k \leq 1$) and weakly continuous instead of being affine. Such operators are well known in the study of nonlinear economic systems [14]. *Further, if $S\ (M) \subseteq K$, the condition "$Sx \succeq 0$ for each $x \in M$" is automatically satisfied. Thus we obtain the following;*

Corollary 3.2. *Let K be a convex cone in an ordered reflexive Banach space X and $S : K \to K$ be a positively homogeneous, continuous in weak and strong topologies of X and set-condensing (or ball-condensing) map. Assume that $T : K \to K$ is continuous 1-set contractive (or 1-ball contractive) map and the conditions (a) and (b) of Theorem 3.1 are satisfied. Then $F(S) \cap F(T) \neq \emptyset$.*

Theorem 3.3. *Let M be a nonempty closed bounded convex subset of an ordered reflexive Banach space X and $S : M \to M$ be an affine continuous and condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $f : M \to M$ is nonexpansive and $g : M \to M$ is continuous compact map. If $T = f + g$ and S satisfy the conditions (a) and (b) of Theorem 3.1, then $F(S) \cap F(T) \neq \emptyset$.*

Proof. Clearly, T is 1-set contractive. The result now follows from Theorem 3.1. □

It is well known [5] that if M is a nonempty closed convex subset of a uniformly convex Banach space and $T : M \to X$ is nonexpansive, then $(I - T)$ is demiclosed and T is 1-set contractive. Consequently, we have:

Corollary 3.4. *Let M be a nonempty closed convex subset of an ordered uniformly convex Banach space X and $S : M \to M$ be an affine continuous and set-condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to M$ is nonexpansive and the condition (b) of Theorem 3.1 is satisfied. Then $F(S) \cap F(T) \neq \emptyset$.*

Theorem 3.5. *Let M be a nonempty closed bounded convex subset of an ordered reflexive Banach space X and $S : M \to M$ be an affine continuous and set-condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $f : M \to M$ is a generalized contraction and $g : M \to M$ completely continuous. If $T = f + g$ and S satisfy the condition (b) of Theorem 3.1, then $F(S) \cap F(T) \neq \emptyset$.*

Proof. Since $g : M \to M$ is a completely continuous map, $\alpha(g(A)) = 0$ for each subset A of M. Hence $T = f + g$ is 1-set contractive map. We show that $(I - T)$ is demiclosed at 0. Suppose $\{x_n\} \subseteq M$ be such that $x_n \longrightarrow x_0$ weakly and $x_n - Tx_n \longrightarrow 0$ strongly as $n \to \infty$. Since g is completely continuous, it follows that $gx_n \longrightarrow gx_0$ strongly as $n \to \infty$. Now $x_n - fx_n = x_n - Tx_n + gx_n \to gx_0$ strongly as $n \to \infty$. As in the proof of Lemma 2.1 [20], $\{x_n\}$ is a cauchy sequence which necessarily converges strongly to x_0 . Consequently, we have $x_0 - fx_0 = gx_0$ and so $x_0 - Tx_0 = 0$. Thus $(I - T)$ is demiclosed at 0. Now Theorem 3.1 guarantees that $F(S) \cap F(T) \neq \emptyset$. □

Theorem 3.6. *Let M be a nonempty closed bounded convex subset of an ordered reflexive Banach space X and $S : M \to M$ be an affine continuous and set-condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $f : M \to X$ is a continuous compact map and $g : M \to X$ a uniformly strictly contractive map on M relative to X. If $fx + gy \in M$ whenever $x, y \in M$ and $T = f + g$ and S satisfy the condition (b) of Theorem 3.1, then $F(S) \cap F(T) \neq \emptyset$.*

Proof. Since $T : M \to M$ is 1-set contractive map; in view of Theorem 3.1, it suffices to show that $(I - T)$ is demiclosed at 0. Suppose that $\{x_n\} \subseteq M$ is such that $x_n \longrightarrow x_0$ weakly and $x_n - Tx_n \longrightarrow 0$ strongly as $n \to \infty$. Since $\{x_n\}$ is bounded and f is compact, it follows that $fx_n \longrightarrow y$ strongly in X as $n \to \infty$. Now $x_n - gx_n = x_n - Tx_n + fx_n \to y$ strongly as $n \to \infty$. This implies that $x_n - hx_n \to 0$ as $n \to \infty$, where $h : M \to X$ is uniformly strictly contractive map defined by $hx = gx + y$. As in [13], $\{x_n\}$ is a Cauchy sequence which necessarily converges strongly to x_0. This and the continuity of T imply that $x_0 - Tx_0 = 0$. Thus $(I - T)$ is demiclosed at 0. Now Theorem 3.1 guarantees that $F(S) \cap F(T) \neq \emptyset$. □

Corollary 3.7. *Let M be a nonempty closed bounded convex subset of an ordered uniformly convex Banach space X and $S : M \to M$ be an affine continuous and set-condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to M$ is a continuous $LANE$ map and T and S satisfy the condition (b) of Theorem 3.1. Then $F(S) \cap F(T) \neq \emptyset$.*

Proof. By Nussbaum [19], T is 1-set contractive and $(I - T)$ is demiclosed at 0. The result now follows from Theorem 3.1. □

Corollary 3.8. *Let M be a nonempty closed bounded convex subset of an ordered uniformly convex Banach space X and $S : M \to M$ be an affine continuous and set-condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $f : M \to M$ is continuous $LANE$ map and $g : M \to M$ is completely continuous. If $T = f + g$ and S satisfy the condition (b) of Theorem 3.1, then $F(S) \cap F(T) \neq \emptyset$.*

Proof. By Remark 3.7 in [20], T is a $LANE$ map. From Corollary 3.7, $F(S) \cap F(T) \neq \emptyset$. □

Corollary 3.9. *Let M be a nonempty closed bounded convex subset of an ordered reflexive Banach space X and $S : M \to M$ be an affine continuous and ball-condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to M$ is a continuous weakly semicontractive map and T and S satisfy the conditions (a) and (b) of Theorem 3.1. Then $F(S) \cap F(T) \neq \emptyset$.*

Proof. By Petryshyn [[20], Lemma 3.1], T is 1-ball contractive and hence the proof from Theorem 3.1. □

Corollary 3.10. *Let M be a nonempty closed bounded convex subset of an ordered reflexive Banach space X and $S : M \to M$ be an affine continuous and set-condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to M$ is a continuous map of weakly semicontractive type and T and S satisfy the conditions (a) and (b) of Theorem 3.1. Then $F(S) \cap F(T) \neq \emptyset$.*

Proof. By Petryshyn [[20], Lemma 3.2], T is 1-set contractive. Now Theorem 3.1 guarantees that $F(S) \cap F(T) \neq \emptyset$. □

Corollary 3.11. *Let M be a nonempty closed bounded convex subset of an ordered uniformly convex Banach space X and $S : M \to M$ be an affine continuous and condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to M$ is a continuous map of semicontractive type and T and S satisfy the condition (b) of Theorem 3.1. Then $F(S) \cap F(T) \neq \emptyset$.*

Proof. By Browder [5], $(I-T)$ is demiclosed on M. Since every mapping of semicontractive type is also of weakly semicontractive type, the result follows from Corollary 3.10. □

Theorem 3.12. *Let X be an ordered Banach space. Let M be a nonempty closed bounded convex subset of X with $0 \in int(M)$ and $S : M \to M$ be an affine continuous and set-condensing (ball-condensing) map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to M$ is a continuous map and the condition (b) of Theorem 3.1 is satisfied. Suppose further that one of the following conditions holds:*

(i) *X is reflexive space and T is a strictly semicontractive map;*
(ii) *X is reflexive space which admits a weakly continuous duality mapping from X into X^* and T is a semicontractive map;*
(iii) *X and X^* are uniformly convex and T is both 1-set contractive and pseudocontractive map;*
(iv) *X is reflexive space and T is both 1-set contractive and k-dissipative with $k < 1$.*

Then $F(S) \cap F(T) \neq \emptyset$.

Proof. (i) By Lemma 3.1 in [20], T is k-ball contractive with $k < 1$ and hence T is 1-ball contractive. We show that $(I-T)$ is demiclosed at 0. Suppose that $\{x_n\} \subseteq M$ is such that $x_n \longrightarrow x_0$ weakly and $x_n - Tx_n \longrightarrow 0$ strongly as $n \to \infty$. Then the set $\{x_n\}$ is precompact as in the proof of Theorem 3.1 in [20]. Without loss of generality, we may assume that $\{x_n\}$ itself converges to some $x \in M$. Clearly, $x = x_0$. Further the continuity of T implies that $x_0 - Tx_0 = 0$. Thus $(I-T)$ is demiclosed at 0. Now Theorem 3.1 guarantees that $F(S) \cap F(T) \neq \emptyset$.

(ii) Let $x \in M$. Since the map $V(x, .)$ is completely continuous and X is reflexive, it follows that $V(x, .)$ is also compact. Therefore, T is weakly semicontractive. By Lemma 3.1 in [20], T is 1-ball contractive. We show that $(I-T)$ is demiclosed at 0. Suppose that $\{x_n\} \subseteq M$ is such that $x_n \longrightarrow x_0$ weakly and $x_n - Tx_n \longrightarrow 0$ strongly as $n \to \infty$. Then, as in the proof of Theorem 3.2 in [20], $x_0 - Tx_0 = 0$. Thus $(I-T)$ is demiclosed at 0. Now Theorem 3.1 guarantees that $F(S) \cap F(T) \neq \emptyset$.

(iii) By Browder [5], $(I-T)$ is demiclosed on M. Theorem 3.1 implies that $F(S) \cap F(T) \neq \emptyset$.

(iv) By Lemma 4.1 in [16], $(I-T)$ is demiclosed on M and so the result from Theorem 3.1. □

Remark 3.13. All results of this paper (3.1- 3.12) remain valid if we replace "S is set-(ball-)condensing map" by either "S is strict-set contraction" or "S is nonlinear-set contraction" or "S is nonlinear contraction [4]"(cf. Remarks 2.1-2.2[8]). Further, we note that the new results obtained, provide iterative convergence scheme for finding unique common fixed point of S and T; we state and prove only one result below; other results can be obtained similarly.

Theorem 3.14. *Let M be a nonempty closed convex and bounded subset of an ordered reflexive Banach space X and $S : M \to M$ be an affine and nonlinear contraction map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to M$ is continuous 1-set contractive (1-ball contractive) map and T and S satisfy the conditions (a) and* *(b) of Theorem 3.1. Then* $F(S) \cap F(T) = \{x_0\}$ *and for each* $x \in M$*, the sequence of Picard iterates* $\{S^n x\}$ *of* S *converges to* x_0.

Proof. Since S is a nonlinear contraction, it is continuous and set-condensing (ball-condensing). Thus by Theorem 3.1, S and T have a common fixed point x_0. Since S has a unique fixed point, so S and T have a unique common fixed point x_0. Further, by Theorem 2.2, for each $x \in M$, the sequence of Picard iterates $\{S^n x\}$ of S converges to x_0 which is the unique common fixed point of S and T. □

4. Applications

As an application of the results established in previous section, we develop here the Ky Fan type approximation theorems which generalize results in [11, 12, 17, 18] and many others. For any closed convex subset M of a Banach space X with $0 \in int(M)$, we define $R : X \to M$ by(see [17, 18])

$$Rx = \begin{cases} x & \text{if} \quad x \in M \\ \frac{x}{p(x)} & \text{if} \quad x \notin M \end{cases}$$

where p is the Minkowski functional of M.

Theorem 4.1. *Let* M *be a closed convex subset of an ordered reflexive Banach space* X *and* $S : M \to M$ *be an affine continuous and set-condensing (or ball-condensing) map with* $Sx \succeq 0$ *for each* $x \in M$*. Assume that* $T : M \to X$ *is continuous* 1*-set contractive (or* 1*-ball contractive) map such that* $(I - RT)$ *is demiclosed at* 0 *where* R *is a mapping defined above. Suppose that* S *and* T *are weakly isotone decreasing whenever* ST *is defined and the following holds;*

(c) $RT(x) \succeq SRT(x)$, *if* $Tx \notin M$.

Then there exists $x_0 \in M$ *such that*

$$p(x_0 - Tx_0) = p(Sx_0 - Tx_0) = d_p(Tx_0, M) = d_p(Tx_0, \overline{I_M(x_0)}).$$

More precisely, either

(1) S *and* T *have a common fixed point* $x_0 \in M$*, or*
(2) *there exists* $x_0 \in \partial M$ *with*

$$0 < p(Tx_0 - Sx_0) = p(Tx_0 - x_0) = d_p(Tx_0, M) = d_p(Tx_0, \overline{I_M(x_0)}).$$

Proof. Clearly, R is continuous and $R(A) \subseteq \overline{con}(\{0\} \cup A)$ for any subset A of M where "$con(B)$" denotes convex hull of the set B. As a result, R is a 1-set contractive (1-ball contractive) map. Define $f : M \to M$ by $fx = RoT(x) = RT(x)$. It is easy to see that f is a continuous 1-set contractive (1-ball contractive) map. Now we prove that f and S are weakly isotone. For any $x \in M$,

$$fSx = RTSx = \begin{cases} TSx \preceq Sx & \text{if} \quad TSx \in M \\ \frac{TSx}{p(TSx)} \preceq \frac{Sx}{p(TSx)} \preceq Sx & \text{if} \quad TSx \notin M \end{cases}$$

Also,

$$fx = RTx = Tx \succeq STx = Sfx \quad \text{if} \quad Tx \in M$$

and by (c) $\quad fx \succeq Sfx \quad$ if $\quad Tx \notin M$

Thus f and S are weakly isotone. By Theorem 3.1, there exists $x_0 \in M$ such that $x_0 = fx_0 = Sx_0$. The proof is broken up into two cases.

(1) Suppose that $Tx_0 \in M$. Then $x_0 = fx_0 = RTx_0 = Tx_0$. As a result of this, we get:
$p(x_0 - Tx_0) = p(Sx_0 - Tx_0) = 0 = d_p(Tx_0, M)$ and x_0 is a common fixed point of S and T.

(2) Suppose that $Tx_0 \notin M$. Then

$$x_0 = fx_0 = RTx_0 = \frac{Tx_0}{p(Tx_0)}.$$

Thus for any $x \in M$, we have

$$\begin{aligned} p(Tx_0 - Sx_0) &= p(Tx_0 - x_0) = p\Big(Tx_0 - \frac{Tx_0}{p(Tx_0)}\Big) \\ &= p\Big(\frac{p(Tx_0)Tx_0 - Tx_0}{p(Tx_0)}\Big) \\ &= \Big(\frac{p(Tx_0) - 1}{p(Tx_0)}\Big)p(Tx_0) \\ &= p(Tx_0) - 1 \leq p(Tx_0) - p(x) \\ &= p((Tx_0 - x) + x) - p(x) \\ &\leq p(Tx_0 - x) \\ &\leq \inf\{p(Tx_0 - z) : z \in M\} = d_p(Tx_0, M). \end{aligned}$$

Consequently, $p(Tx_0 - Sx_0) = p(Tx_0 - x_0) = d_p(Tx_0, M)$ and $p(Tx_0 - x_0) > 0$ since $p(Tx_0 - x_0) = p(Tx_0) - 1$.
It remains to show that

$$p(Tx_0 - x_0) = d_p(Tx_0, \overline{I_M(x_0)}).$$

For this, let $z \in I_M(x_0)\backslash M$. Then there exist $y \in M$ and $a > 1$ with $z = x_0 + a(y - x_0)$(note if $0 \leq a \leq 1$, then $z = (1-a)x_0 + ay \in M$). Assume that $p(Tx_0 - z) < p(Tx_0 - x_0)$.

Clearly, $\frac{1}{a}z + (1 - \frac{1}{a})x_0 = y \in M$, so we have

$$\begin{aligned} p(Tx_0 - y) &= p\Big[\frac{1}{a}(Tx_0 - z) + \Big(1 - \frac{1}{a}\Big)(Tx_0 - x_0)\Big] \\ &\leq \frac{1}{a}p(Tx_0 - z) + \Big(1 - \frac{1}{a}\Big)p(Tx_0 - x_0) \\ &< p(Tx_0 - x_0), \end{aligned}$$

which contradicts the fact that $p(Tx_0 - x_0) = d_p(Tx_0, M)$. Thus $p(Tx_0 - x_0) \leq p(Tx_0 - z)$ for all $z \in I_M(x_0)$. Further, note that p is continuous, so we have $p(Tx_0 - x_0) \leq p(Tx_0 - z)$ for all $z \in \overline{I_M(x_0)}$. This implies that $p(Tx_0 - x_0) \leq d_p(Tx_0, \overline{I_M(x_0)})$. Also we have equality since $x_0 \in \overline{I_M(x_0)}$. Hence

$$0 < p(Tx_0 - Sx_0) = p(Tx_0 - x_0) = d_p(Tx_0, M) = d_p(Tx_0, \overline{I_M(x_0)}).$$

It is well known that $\overline{I_M(x_0)} = X$ provided $x_0 \in int(M)$, so $d_p(Tx_0, \overline{I_M(x_0)}) = 0$. Thus $x_0 \in \partial M$ as desired. □

Corollary 4.2. *Let M be a closed ball with center at origin and radius r in an ordered reflexive Banach space X and $S : M \to M$ be an affine continuous and set-condensing (or ball-condensing) map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to X$ is continuous 1-set contractive (or 1-ball contractive) map such that $(I - RT)$ is demiclosed at 0. Suppose that S and T are weakly isotone decreasing whenever ST is defined and*

(d) *$RT(x) \succeq SRT(x)$, if $\|Tx\| > r$ holds.*

Then there exists $x_0 \in M$ such that

$$\|x_0 - Tx_0\| = \|Sx_0 - Tx_0\| = d(Tx_0, M) = d(Tx_0, \overline{I_M(x_0)}).$$

More precisely, the conclusion (1) or (2) of Theorem 4.1 holds with norm "$\|.\|$" instead of Minkowski functional "p".

Proof. It is clear that $p(x) = \frac{\|x\|}{r}$ is the Minkowski functional of M [19] and the mapping R becomes

$$Rx = \begin{cases} x & \text{if} \quad \|x\| \le r \\ \frac{rx}{\|x\|} & \text{if} \quad \|x\| > r \end{cases}.$$

This is well known retraction map of X onto M [18]. Now apply Theorem 4.1 to obtain the result. □

Corollary 4.3. *Let M be a closed ball with center at origin and radius r in an ordered reflexive Banach space X and $S : M \to M$ be an affine continuous and set-condensing (or ball-condensing) map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to X$ is continuous 1-set contractive (or 1-ball contractive) map such that $(I - RT)$ is demiclosed at 0 where R is a retraction of X onto M. Suppose that S and RT are weakly isotone decreasing on M. Then there exists $x_0 \in M$ such that*

$$\|x_0 - Tx_0\| = \|Sx_0 - Tx_0\| = d(Tx_0, M) = d(Tx_0, \overline{I_M(x_0)}).$$

More precisely, the conclusion (1) or (2) of Theorem 4.1 holds with norm "$\|.\|$" instead of Minkowski functional "p".

Theorem 4.4. *Let M, X, S , T and R be as in Theorem 4.1. Suppose that $(I - RT)$ is demiclosed at 0. Suppose that S and T are weakly isotone decreasing whenever ST is defined and the condition (c) holds.*

Moreover T satisfies one of the following conditions for each $x \in \partial M$, with $x \neq Tx$:

(i) *$p(Tx - y) < p(Tx - x)$ for some y in $\overline{I_M(x)}$;*
(ii) *There is a λ such that $|\lambda| < 1$ and $\lambda x + (1 - \lambda)Tx \in \overline{I_M(x)}$;*
(iii) *$Tx \in \overline{I_M(x)}$;*
(iv) *For each $\lambda \in (0, 1), x \neq \lambda Tx$;*
(v) *There exists $\alpha \in (1, \infty)$ such that, $p^\alpha(Tx - x) \ge p^\alpha(Tx) - 1$;*
(vi) *There exists $\beta \in (0, 1)$ such that, $p^\beta(Tx - x) \le p^\beta(Tx) - 1$.*

Then $F(S) \cap F(T) \neq \emptyset$.

Proof. Theorem 4.1, guarantees that either
(1) S and T have a common fixed point $x_0 \in M$, or
(2) there exists $x_0 \in \partial M$ with

$$0 < p(Tx_0 - Sx_0) = p(Tx_0 - x_0) = d_p(Tx_0, M) = d_p(Tx_0, \overline{I_M(x_0)}).$$

(i). Now suppose (2) holds. If $x_0 \neq Tx_0$, then by the condition (i) , there exists $y \in \overline{I_M(x)}$ such that $p(Tx_0 - y) \prec p(x_0 - Tx_0)$. This contradicts $p(Tx_0 - x_0) = d_p(Tx_0, \overline{I_M(x_0)})$.

(ii). Now suppose (2) holds. If $x_0 \neq Tx_0$. Then the condition (ii) implies that there is a λ such that $|\lambda| < 1$ and $\lambda x_0 + (1-\lambda)Tx_0 \in \overline{I_M(x)}$. By (2) we have

$$\begin{aligned} 0 < p(Tx_0 - x_0) &\leq p\left(Tx_0 - [\lambda x_0 + (1-\lambda)Tx_0\right) = p\left(\lambda(Tx_0 - x_0)\right) = \lambda p(Tx_0 - x_0) \\ &< p(Tx_0 - x_0), \end{aligned}$$

which is a contradiction.

If T satisfies the condition (iii), then T satisfies the condition (ii) by letting $\lambda = 0$.

Suppose that T satisfies the condition (iv). Now suppose (2) holds and $x_0 \neq Tx_0$. Note that

$$x_0 = fx_0 = RTx_0 = \frac{Tx_0}{p(Tx_0)}$$

and $p(Tx_0) > 1$ and this implies that
$x_0 = \lambda_0 x_0$ where $\lambda_0 = \frac{1}{p(Tx_0)} \in (0,1)$. This contradicts the condition (iv).

Suppose that T satisfies the condition (v). Now suppose (2) holds and $x_0 \neq Tx_0$. Then the condition (v) implies that there exists $\alpha \in (1, \infty)$ such that, $p^\alpha(Tx - x) \geq p^\alpha(Tx) - 1$. Let $\lambda_0 = \frac{1}{p(Tx_0)}$. Note that $\lambda_0 \in (0,1)$ and

$$\begin{aligned} \frac{(p(Tx_0) - 1)^\alpha}{p^\alpha(Tx_0)} &= (1-\lambda_0)^\alpha \\ &< 1 - (\lambda_0)^\alpha \\ &= \frac{p^\alpha(Tx_0) - 1}{p^\alpha(Tx_0)} \\ &\leq \frac{p^\alpha(Tx_0 - x_0)}{p^\alpha(Tx_0)}. \end{aligned}$$

This implies that $p(Tx_0 - x_0) > p(Tx_0) - 1$, and this contradicts $p(Tx_0 - x_0) = p(Tx_0) - 1$. Further,
$x_0 = \lambda_0 x_0$ where $\lambda_0 = \frac{1}{p(Tx_0)} \in (0,1)$. Hence $Tx_0 \in M$ and $Sx_0 = x_0 = Tx_0$ as desired.

Suppose that T satisfies the condition (vi). Using an argument similar to the one employed for the condition (v), we obtain the desired result □

Corollary 4.5. *Let M, X, S , T, R and (d) be as in Corollary 4.2. Suppose that $(I - RT)$ is demiclosed at* 0 *and T satisfies one of the following conditions for each $x \in \partial M$, with $x \neq Tx$:*

(i) $\|Tx - y\| < \|Tx - x\|$ *for some y in $\overline{I_M(x)}$* ;
(ii) *There is a λ such that $|\lambda| < 1$ and $\lambda x + (1-\lambda)Tx \in \overline{I_M(x)}$;*

(iii) $Tx \in \overline{I_M(x)}$;
(iv) *For each* $\lambda \in (0,1), x \neq \lambda Tx$;
(v) *There exists* $\alpha \in (1,\infty)$ *such that,* $\|Tx - x\|^\alpha \geq \|Tx\|^\alpha - r^\alpha$;
(vi) *There exists* $\beta \in (0,1)$ *such that,* $\|Tx - x\|^\alpha \leq \|Tx\|^\beta - r^\beta$.

Then $F(S) \cap F(T) \neq \emptyset$.

Following the above ideas, it is possible to obtain approximation and fixed point theorems in Hilbert spaces (here the mapping R is replaced by the proximity map P).

Theorem 4.6. *Let M be a nonempty closed convex subset of an ordered Hilbert space H and $S : M \to M$ be an affine continuous and set-condensing (or ball-condensing) map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to H$ is continuous 1-set contractive (or 1-ball contractive) map such that $(I - PT)$ is demi-closed at 0 where P is the proximity map on M. If S and PT are weakly isotone, then there exists $x_0 \in M$ such that*

$$\|x_0 - Tx_0\| = \|Sx_0 - Tx_0\| = d(Tx_0, M) = d(Tx_0, \overline{I_M(x_0)}).$$

More precisely, either

(1) *S and T have a common fixed point $x_0 \in M$, or* (2) *there exists $x_0 \in \partial M$ with*

$$0 < \|Tx_0 - Sx_0\| = \|Tx_0 - x_0\| = d(Tx_0, M) = d(Tx_0, \overline{I_M(x_0)}).$$

Proof. Let P be the proximity map on M; that is for each $x \in H$, we have, $\|Px - x\| = d(x, M)$. It is well known that P is nonexpansive in H. Thus $PT : M \to M$ is 1-set contractive (1-ball contractive) map. By Theorem 3.1, there exists $x_0 \in M$ such that $x_0 = Sx_0 = PTx_0$. Thus we obtain, as in Theorem 4.1, the desired conclusion. □

Corollary 4.7. *Let M be a nonempty closed convex subset in an ordered Hilbert space H and $S : M \to M$ be an affine continuous and condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to H$ is nonexpansive map. If S and PT are weakly isotone, then there exists $x_0 \in M$ such that*

$$\|x_0 - Tx_0\| = \|Sx_0 - Tx_0\| = d(Tx_0, M) = d(Tx_0, \overline{I_M(x_0)}).$$

More precisely, the conclusion (1) or (2) of Theorem 4.6 holds.

Proof. Let P be the proximity map of H onto M. Since P is nonexpansive, $PT : M \to M$ is also nonexpansive and hence 1-set contractive map. Now by Browder's result [5], $(I - PT)$ is demiclosed at 0. Hence the proof by Theorem 4.6. □

Theorem 4.8. *Let M be a nonempty closed bounded convex subset of an ordered Hilbert space H and $S : M \to M$ be an affine continuous and condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to H$ is a continuous semicontractive map. If S and PT are weakly isotone, then there exists $x_0 \in M$ such that*

$$\|x_0 - Tx_0\| = \|Sx_0 - Tx_0\| = d(Tx_0, M) = d(Tx_0, \overline{I_M(x_0)}).$$

More precisely, the conclusion (1) or (2) of Theorem 4.6 holds.

Proof. Let P be the proximity map of H onto M. Then $PT : M \to M$ is a continuous semicontractive map. Now by Browder's result [5], $(I - PT)$ is demiclosed at 0 (see also Theorem 3.12(ii)). Since T is 1-ball contractive map so Theorem 4.6 implies the conclusion. □

Theorem 4.9. *Let M be a nonempty closed bounded convex subset of an ordered Hilbert space H and $S : M \to M$ be an affine continuous and condensing map with $Sx \succeq 0$ for each $x \in M$. Assume that $T : M \to H$ is a LANE map. If S and PT are weakly isotone, then there exists $x_0 \in M$ such that*

$$\|x_0 - Tx_0\| = \|Sx_0 - Tx_0\| = d(Tx_0, M) = d(Tx_0, \overline{I_M(x_0)}).$$

More precisely, the conclusion (1) or (2) of Theorem 4.6 holds.

Proof. Let P be the proximity map of H onto M. Since T is a $LANE$ map and P is nonexpansive, therefore $PT : M \to M$ is also a $LANE$ map. As in Corollay 3.7, $(I - PT)$ is demiclosed at 0 and T is 1-set contractive map. The result now follows from Theorem 4.6. □

5. An example

Let's consider the implicit integral equation

$$p(t, x(t)) = \int_0^1 q(t, s, x(s))ds, \quad t \in [0, 1], \tag{5.1}$$

where $x \in L^p[0, 1]$, $1 < p < \infty$. Integral equations like (5.1) were introduced by Fečkan [10] and could occur in the study of nonlinear boundary value problems of ordinary differential equations.

Let M be a nonempty closed convex subset of $L^p[0, 1]$ partially ordered by the closed convex cone

$$K = \{x \in L^p[0, 1] : \quad x(t) \geq 0, \ \ a.e.\}.$$

Let α denote the Kuratowski measure of noncompactness on $L^p[0, 1]$ and $b = \sup\{\|x\| : \ x \in M\}$. We assume the following:

(p1) $p : \ [0, 1] \times M \to M$.
(p2) $\|p(t, x(t)) - p(t, y(t))\| \leq \|x - y\|$ for all $x, \ y \in M$.
(p3) $p(t, s) \leq s$ for all $t \in [0, 1]$ and $s \in [0, b]$.
(q1) $q(t, s, x(s)) \geq 0$ on $[0, 1] \times [0, 1] \times M$.
(q2) $q(t, s, kx(s)) \leq k \, q(t, s, x(s))$ for all $k \in (0, 1]$.
(q3) $\int_0^1 q(t, s, x(s)))ds \in M$ for all $t \in [0, 1]$ and $x \in M$.
(q4) $\alpha(\int_0^1 q(\cdot, \cdot, B(s))ds) < \alpha(B(\cdot))$ for all bounded subsets B of M.
(q5) $q(t, s, p(s, x(s))) \leq p(t, x(t))$ for all $s, \ t \in [0, 1]$.

Define

$$(Tx)(t) = p(t, x(t)) \quad \text{and} \quad (Sx)(t) = \int_0^1 q(t, s, x(s))ds.$$

The condition $(p2)$ implies that $T : M \to M$ is nonexpansive, so that $I - T$ is demiclosed on M ([22], Proposition 10.9) and T is 1-set contractive. By $(q4)$, S is condensing , and $Sx \succeq 0$ for all $x \in M$ by $(q1)$. Furthermore, for $k \in (0,1]$,

$$S(kx)(t) = \int_0^1 q(t,s,kx(s))ds \leq k \int_0^1 q(t,x(s))ds = k\ S(x)(t)$$

by $(q2)$.

Using $(p3)$ and $(q5)$, respectively, we get

$$\int_0^1 q(t,s,x(s)))ds \succeq p(t, \int_0^1 q(t,x(s))ds),$$

and

$$\int_0^1 q(t,p(s,x(s)))ds \preceq \int_0^1 p(t,x(t))ds = p(t,x(t)).$$

Hence $Sx \succeq TSx$ and $Tx \succeq STx$ imply that S and T are weakly isotone decreasing.

Now we can apply Theorem 3.1 to conclude that the integral equation has a solution in $L^p[0,1]$.

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