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CONVERGENCE OF INFINITE PRODUCTS OF NONEXPANSIVE OPERATORS IN HILBERT SPACE

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ABSTRACT. Using angles between subspaces, we establish several convergence theorems regarding infinite products of orthogonal projections and nonexpansive operators in Hilbert space.

1. INTRODUCTION

Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive operators on a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. These operators need not be different from each other. Our goal in this paper is to find conditions which imply the convergence of the sequence $\{\mathcal{P}_n\}_{n=1}^{\infty}$, defined by $\mathcal{P}_n = A_n A_{n-1} \cdots A_1$, $n = 1, 2, \ldots$, in either the strong or uniform sense. Note that if all $A_i = A$, where A is a fixed nonexpansive operator, then $\mathcal{P}_n = A^n$ for all natural numbers n.

If all the operators A_i are linear, then weak convergence is known to hold under rather mild conditions [2], [1], [6]. At the same time, there are examples which show that strong convergence may fail even if all the operators A_i are contractive in the sense that $\|A_i x - A_i y\| < \|x - y\|$, $x \neq y$. Here is one such example.

Example. Let $\{\alpha_i\}_{i=1}^{\infty}$ and $\{\beta_i\}_{i=1}^{\infty}$ be two sequences of positive real numbers such that all $\alpha_i,\ \beta_i < 1$, and

$$\prod_{i=1}^{\infty} \alpha_i = a > 0, \qquad \prod_{i=1}^{\infty} \beta_i = 0.$$

Denoting by $\mathbf{x} = (x_1, x_2, \ldots)$ an arbitrary element in l^2, we define the following two linear operators $A, B : l^2 \to l^2$:

$$A\mathbf{x} = (0, \alpha_1 x_1, \beta_1 x_2, \ldots, \alpha_i x_{2i-1}, \beta_i x_{2i}, \ldots),\ \mathbf{x} \in l^2,$$

$$B\mathbf{x} = (0, \beta_1 x_1, \alpha_1 x_2, \ldots, \beta_i x_{2i-1}, \alpha_i x_{2i}, \ldots),\ \mathbf{x} \in l^2.$$

It is easy to see that $\|A\mathbf{x}\| < \|\mathbf{x}\|,\ \|B\mathbf{x}\| < \|\mathbf{x}\|$ for $\mathbf{x} \neq 0$ and that

$$\lim_{n\to\infty} \|A^n \mathbf{x}\| = \lim_{n\to\infty} \|B^n \mathbf{x}\| = 0 \qquad \text{for any } \ \mathbf{x} \in l_2.$$

At the same time, for $\mathbf{e}_1 = (1, 0, 0, \ldots)$ and $\mathbf{e}_2 = (0, 1, 0, \ldots)$, one gets

$$\|(BA)^n \mathbf{e}_1\| = \prod_{i=1}^{n} \alpha_i^2 \longrightarrow a^2 \neq 0$$

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and $\|(AB)^n\mathbf{e}_2\| \to a^2/\alpha_1 \neq 0$ too. Hence the sequences $\{(BA)^n\mathbf{e}_1\}$ and $\{(AB)^n\mathbf{e}_2\}$ have no strong limits in l^2.

The situation changes when some of operators A_i are orthogonal projections P_{S_i} onto some closed linear subspaces $S_i \subset H$. Any orthogonal projection P is selfadjoint and idempotent; moreover, these two properties are sufficient for a given linear operator on H to be an orthogonal projection onto some closed linear subspace of H. The projection operators are not contractive in general, but they do have the property that $\|Px\| < \|x\|$ whenever $Px \neq x$. Many properties and applications of orthogonal projections and of their infinite products can be found in the monograph [9] and in Chapter 9 of the monograph [4]. In what follows all subspaces are closed linear subspaces of H.

It turns out that projection operators behave well when they are applied immediately one after another and their compositions are repeated cyclically in the given infinite product. One of the first (and apparently the strongest) results of this kind is the result obtained by I. Halperin [12], which states that, for an arbitrary finite set of subspaces $S_1, S_2, \ldots, S_k$ with intersection S and for any $x \in H$, one has

$$\lim_{n\to\infty} \|(P_{S_k}P_{S_{k-1}}\cdots P_{S_1})^n x - P_S x\| = 0 \tag{1.1}$$

(the case $k = 2$ was proved much earlier by J. von Neumann [16]). Although Halperin's proof admits some extensions (e.g., to positive selfadjoint nonexpansive operators; see [4, p. 234]), the repeated order of operators is essential to the proof. Even a single change of the prescribed order or the inclusion of nonexpansive operators of other kinds can destroy the proof, making strong convergence either unproved or nonexistent [7, p. 104]. We remark in passing that a recent elementary geometric proof of von Neumann's classical theorem can be found in [15].

In the present paper we show that this drawback can in some cases be overcome by using some stronger relations between the adjacent projections, described by the "angles" between the corresponding subspaces. The concept of *angles between subspaces* has a long history and many different definitions. Various applications to correlation theory, computed tomography and mathematical statistics do not use just one angle, but the set of *principal angles* for any separate pair of subspaces (see, e.g., [3] and [13]). The same situation occurs in multidimensional geometry [19]. There exists a spectral approach (via spectra of the operators P_SP_T; see [14]), where the set of all angles between two given subspaces is infinite. Even in the definition of a single angle, one can use either the maximal or the minimal one, or an angle which is optimal in some other sense (see, e.g., [5]).

In our considerations we adopt the definition of angles between subspaces given by K. Friedrichs in [8], which turns out to be the most useful in the study of projections. A rather full theory of such angles is given in [4], using various properties of products of projection operators and the methods of Functional Analysis. For the reader's convenience, we give shorter and more elementary proofs of some needed facts, using methods of three-dimensional geometry (not only for illustration but for the complete proof); our approach is based on Lemma 2.3 below. Another key point of our approach is that we do not prescribe any special order or the character of the operators in the whole product; we are only concerned with some special segments of this product. Consequently, we cannot state the rate of convergence of the given infinite product, which is the main rationale and the main application of the angles between subspaces in [4] and many related works. Instead, we are only interested in convergence. Of course, some estimates of the rate of convergence could be derived from our results in the presence of sufficient information on all the other operators participating in the infinite product, but this is outside the scope of the present paper.

2. Angles between subspaces of Hilbert space

Let H be a real Hilbert space. As usual, for any $x, y \in H$, we define the angle $\theta(x, y) \in [0, \pi]$ between x and y by

$$\cos\theta(x, y) = \frac{\langle x, y\rangle}{\|x\| \cdot \|y\|},$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in H. Let S be a closed linear subspace of H and let P_S denote the orthogonal projection of H onto S. Then, for any $x \in H$ which is not orthogonal to S, we define the angle $\theta(x, S)$ between x and S as $\theta(x, P_S x)$. When $x \perp S$ we set $\theta(x, S) = \frac{\pi}{2}$ (we also agree that $\theta(x, 0) = \frac{\pi}{2}$ for any $x \neq 0$).

The following three simple lemmata (principles) will be widely used in our arguments below.

Lemma 2.1 (principle of minimality)**.** *For every element $x \in H$ and any subspace $S \subset H$, we have $\theta(x, P_S x) \leq \theta(x, y)$ for any $y \in S$; moreover, $\theta(x, S) \leq \pi/2$ and $\theta(x, S) = 0$ if and only if $x \in S$.*

Proof. This assertion follows immediately from the inequality $\|x - P_S x\| \leq \|x - y\|$, $y \in S$. □

Lemma 2.2 (lemma on three perpendiculars)**.** *An element $x \in H$ is orthogonal to some $z \in S$ if and only if $P_S x \perp z$.*

Proof. Using the fact that every orthogonal projection is self-adjoint, we obtain $\langle P_S x, z\rangle = \langle x, P_S z\rangle = \langle x, z\rangle$. Hence $\langle P_S x, z\rangle$ and $\langle x, z\rangle$ have to vanish together. □

Lemma 2.3 (principle of geometric treatment)**.** *Given three vectors $x, y, z \in H$, their properties (including lengths and angles) may be studied as if these vectors were in $\mathbb{R}^3$, that is, using the standard geometric pictures and methods.*

Proof. Indeed, the vectors x, y, z may be considered to belong to some three-dimensional space L, where we can choose three orthogonal unit vectors e_1, e_2, e_3. This basis can be extended to a complete orthonormal basis of H, yielding a subspace M so that $H = L \oplus M$. Since the inner product of vectors from L is obviously independent of M, all properties involving vector lengths and angles may be studied just in L. □

Now we can move on to the main topic of this section.

Definition 2.4. Let S and T be two subspaces of H such that none of them coincides with $S \cap T$. The angle $\theta(T, S)$ between these subspaces is defined to be $\inf \theta(x, S)$, where the infimum is taken over all $x \in T$ such that $x \perp (S \cap T)$.

Alternatively, if at least one of subspaces S, T contains the other one, then we set $\theta(T, S) = 0$.

According to this definition, the angle between abstract subspaces is a natural generalization of the standard geometric angle between either two lines or two planes, and even between a line and a plane in three-dimensional geometry. In spite of the non-symmetric form of the definition, the angle, as defined above, is completely symmetric with respect to S and T. Indeed, by Lemma 2.2, the condition $x \perp (S \cap T)$ implies that $P_S x \perp (S \cap T)$ as well. Hence

$$\theta(T, S) = \inf\{\theta(x, y) :\ x \in T,\ y \in S,\ x, y \perp (S \cap T)\} = \theta(S, T).$$

Consequently, $\theta(x, P_S x) \geq \theta(S, T)$ for any $x \in T$ such that $x \perp (S \cap T)$ and $\theta(y, P_T y) \geq \theta(S, T)$ for any $y \in S$ such that $y \perp (S \cap T)$.

In the rest of this section we always assume that $S, T \neq S \cap T$. We call the vectors $x \in T$ and $y \in S$ *admissible* if they are orthogonal to $S \cap T$; only such vectors are needed for the definition of $\theta(S, T)$. Define

$$S = S^\circ \oplus (S \cap T) \qquad \text{and} \qquad T = T^\circ \oplus (S \cap T),$$

where $\oplus$ means an orthogonal sum. Now the definition of $\theta(S, T)$ may be rewritten as

$$\theta(S, T) = \inf\{\theta(x, y) :\ x \in T^\circ,\ y \in S^\circ\}.$$

The concept of the angle between vectors has its own useful properties, e.g., the "triangle inequality" $\theta(x, y) \leq \theta(x, z) + \theta(z, y)$ (see, e.g., [10, p. 151]). However, in practical computations it is more convenient to use $\cos\theta(x, y)$. This leads to the relation

$$\cos\theta(S, T) = \sup\{\langle x, y\rangle :\ x \in T^\circ,\ y \in S^\circ,\ \|x\| \leq 1, \|y\| \leq 1\}.$$

The main problem in the following applications is to check that $\theta(S, T) > 0$ for two given subspaces S, T and thus $\cos\theta(S, T) < 1$. The next example shows that this property may not be so simple to ascertain.

Example. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis in a Hilbert space H. We consider two infinite dimensional subspaces S and T with the bases $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$, respectively, defined for $k = 0, 1, \dots$ by

$$u_{3k+1} = \frac{1}{\sqrt{2}}(e_{4k+1} - e_{4k+3}),\ u_{3k+2} = \frac{1}{\sqrt{2}}(e_{4k+1} + e_{4k+3}),\ u_{3k+3} = e_{4k+4},$$

$$v_{2k+1} = (\cos\frac{1}{k})\, e_{4k+1} + (\sin\frac{1}{k})\, e_{4k+2}, \quad v_{2k+2} = e_{4k+4}.$$

It is a simple task to check that both bases $\{u_n\}$ and $\{v_n\}$ are orthonormal and that $S \cap T$ is the span of $\{e_{4k+4}\}$. Thus S° is spanned by the basis $\{u_{3k+1}, u_{3k+2}\}$ and T° is spanned by the basis $\{v_{2k+1}\}$. Note in addition that, for any of the basis vectors $u_m \in S^\circ$ and $v_n \in T^\circ$ thus obtained, their angles $\theta(u_m, v_n) > \frac{\pi}{4}$, since $\langle u_m, v_n\rangle < \frac{1}{\sqrt{2}}$ for all combinations of m, n. At the same time, the vectors $e_{4k+1} = \frac{1}{\sqrt{2}}(u_{3k+1} + u_{3k+2}) \in S^\circ$ for all k and $\langle e_{4k+1}, v_{2k+1}\rangle = \cos\frac{1}{k} \to 1$ as $k \to \infty$; hence $\theta(S, T) = 0$. Note, in conclusion, that this fact has not been obtained by considering just the basis vectors, making the problem of deciding whether $\theta(S,T) > 0$ rather difficult in general. At this point, one should mention the remarkable result of F. Deutsch (see [4, p. 222]), stating that $\theta(S,T) > 0$ if and only if the subspace $S + T$ is closed in H. This fact has many useful theoretical consequences, but its practical verification is not easier than the initial problem.

The infinite dimensionality of all the spaces in this example is essential as shown by the following assertion which was first mentioned in [8].

Theorem 2.5. *If at least one of the spaces S° and T° is finite dimensional, then the angle between S and T is positive.*

Proof. It is not difficult to see that $\theta(x,S) = \theta(x,P_S x)$ is continuous as a function of x. Moreover, this function is independent of $\|x\|$ and may be considered only on the unit sphere of the subspace T°. If T° is finite dimensional, then this sphere is a compact set and the continuous function $\theta(x,S)$ attains its minimal value at some point x_0 of this sphere, that is, $\theta(T,S) = \theta(x_0,S)$. Thus we obtain that if $\theta(T,S) = 0$, then $\theta(x_0,S) = 0$ for some $x_0 \in T^\circ$ with $\|x_0\| = 1$. But from Lemma 2.1 we know that this is possible only if $x_0 \in S$ and thus $x_0 \in S \cap T^\circ = \{0\}$, which contradicts the property $x_0 \neq 0$ and proves our theorem. □

Of course, the hypothesis of this theorem holds if one of the spaces S or T is finite dimensional.

The following assertion allows us to relate the properties of the angles between subspaces with those between their orthogonal complements. It is mentioned in [4, p. 224] without proof, but with the remark that "an elementary proof of this fact seems fairly lengthy". Since our proof is both short and elementary, we may hope that it is new.

Theorem 2.6. $\theta(S,T) = \theta(S^\perp,T^\perp)$.

Proof. Note that we may only consider the case where both the angles $\theta(S,T)$ and $\theta(S^\perp,T^\perp)$ are positive, referring to Theorem 9.35 from the monograph [4], which asserts that these angles must vanish together. Proceeding to the proof, we first of all ascertain which of the vectors $y_1 \in S^\perp$ and $y_2 \in T^\perp$ are admissible for computing $\theta(S^\perp,T^\perp)$. The condition $y \perp (S^\perp \cap T^\perp)$ means that

$$y \in (S^\perp \cap T^\perp)^\perp = \overline{(S^\perp)^\perp + (T^\perp)^\perp} = \overline{S+T},$$

that is, a vector y is admissible for the couple $S^\perp$, $T^\perp$ if and only if it is orthogonal to one of spaces S or T and belongs to the closure of the sum $S + T$.

Case 1. Let $\theta(S,T) = \frac{\pi}{2}$. This means that $\theta(u,P_T u) = \theta(v,P_S v) = \frac{\pi}{2}$ for any $u \in S^\circ$ and $v \in T^\circ$, i.e., $S^\circ \perp T^\circ$. Consequently, $\overline{S+T} = S^\circ \oplus T = S \oplus T^\circ$ and a vector y_1 from $\overline{S+T}$ is orthogonal to S if and only if $y_1 \in T^\circ$. Analogously, a vector y_2 from $\overline{S+T}$ is orthogonal to T if and only if $y_2 \in S^\circ$. We obtain that $y_1 \perp y_2$ for any admissible vectors $y_1 \in S^\perp$ and $y_2 \in T^\perp$, and thus $\theta(S^\perp,T^\perp) = \frac{\pi}{2}$, as required.

Case 2. Suppose now that $\theta(S,T) < \frac{\pi}{2}$. This implies the existence of a vector $u_1 \in T^\circ$ such that $\theta(u_1,P_S u_1) = \alpha < \frac{\pi}{2}$ as well. Set $P_S u_1 = u_2$ and $P_T u_2 = u_3$. Obviously, $u_2 \in S^\circ$, $u_3 \in T^\circ$ and $\theta(u_2,u_3) = \beta \leq \alpha$ (the last inequality follows from Lemma 2.1).

Consider now the two vectors $y_1 = u_1 - u_2$ and $y_2 = u_3 - u_2$ which obviously belong to $S + T$. By the definition of projections, $y_1 \perp S$ and $y_2 \perp T$; hence these vectors are admissible for computing the angle between $S^\perp$ and $T^\perp$. We obtain that

$$\cos\theta(y_1, y_2) = \frac{\langle u_1 - u_2, u_3 - u_2\rangle}{\|u_1 - u_2\|\,\|u_3 - u_2\|} = \frac{\langle u_2, u_2 - u_3\rangle}{\|u_1\|\sin\alpha\;\|u_2\|\sin\beta},$$

because $(u_3 - u_2) \perp u_1$. Calculations yield

$$\langle u_2, u_2 - u_3\rangle = \|u_2\|^2 - \|u_2\|\,\|u_3\|\cos\beta = \|u_2\|^2(1 - \cos^2\beta).$$

Hence

$$\cos\theta(y_1, y_2) = \frac{\|u_2\|\sin\beta}{\|u_1\|\sin\alpha} = \cos\alpha\,\frac{\sin\beta}{\sin\alpha}.$$

Therefore

$$\cos\theta(S^\perp, T^\perp) \geq \cos\alpha\,\frac{\sin\beta}{\sin\alpha} \tag{2.1}$$

for any admissible u_1 and the angles α, β as defined above.

By definition of the angle $\theta(S,T)$, there is a sequence of vectors $\{u_1^{(n)}\}$ such that the corresponding angles $\alpha^{(n)} \to \theta(S,T)$. Since $\alpha^{(n)} \geq \beta^{(n)} \geq \theta(S,T)$, we also obtain that $\beta^{(n)} \to \theta(S,T)$ and, passing to the limit on the right-hand side of (2.1) (recall that $\theta(S,T) \neq 0$), we see that

$$\cos\theta(S^\perp, T^\perp) \geq \cos\theta(S,T).$$

By the symmetry between S, T and $S^\perp, T^\perp$, we obtain the reverse inequality and thus the required equality. ☐

Corollary 2.7. *If at least one of the subspaces S or T is of finite codimension, then $\theta(S,T) > 0$.*

Proof. The hypothesis of the corollary means that at least one of the subspaces $S^\perp$ or $T^\perp$ is finite dimensional, a fact which by Theorem 2.5 implies that $\theta(S^\perp, T^\perp) > 0$. Theorem 2.6 now gives the same positive value for $\theta(S,T)$. ☐

3. Uniform convergence

Consider now an infinite product of nonexpansive (possibly nonlinear) operators $\{A_i\}$, acting on a Hilbert space H, which are not necessarily different from each other. The problem is to study the behavior of the partial products $B_n = A_n A_{n-1} \cdots A_1$ when $n \to \infty$. In this section we give some conditions which imply the uniform convergence of B_n on (bounded subsets of) the space H. The main assumption will be that some of the operators A_i (in fact, infinitely many) are orthogonal projections onto given subspaces of H. To illustrate our methods, we consider the case where the number of these subspaces is rather small.

Theorem 3.1. *Let S and T be two subspaces of H with intersection $F = S \cap T$, and let the angle $\theta(T,S)$ be positive. Let their intersection $F = S \cap T$ be invariant under all the operators participating in a given infinite product $\prod_{i=1}^{\infty} A_i$ of possibly nonlinear nonexpansive operators and let at least one of compositions $P_S P_T$ or $P_T P_S$ be present in this product infinitely many times. Then, for any initial point $x_0 \in H$, the corresponding partial products $x_n = B_n x_0$ form a sequence uniformly* *approaching F, that is, the distance $\rho(x_n, F) = \|x_n - P_F x_n\| \to 0$, uniformly over any bounded set of initial points x_0.*

Proof. Let the given infinite product have infinitely many compositions of the form $P_S P_T$. We are interested in the set of all natural numbers k such that

$$A_{k+1} = P_T \text{ and } A_{k+2} = P_S.$$

It is clear that this set is a strictly increasing sequence of natural numbers which we denote by $\{k_n\}_{n=1}^{\infty}$. Suppose that an iteration x_{k_n} has already been reached so that the next iterations are $x_{k_n+1} = P_T x_{k_n}$ and $x_{k_n+2} = P_S x_{k_n+1}$. We have

$$\begin{aligned} \rho(x_{k_n+1}, F) &= \|x_{k_n+1} - P_F x_{k_n+1}\| = \|P_T x_{k_n} - P_F P_T x_{k_n}\| \\ &\le \|x_{k_n} - P_F x_{k_n}\| = \rho(x_{k_n}, F), \end{aligned}$$

since $P_F P_T = P_T P_F$, and analogously,

$$\rho(x_{k_n+2}, F) = \|x_{k_n+2} - P_F x_{k_n+2}\| = \|P_S\big(x_{k_n+1} - P_F x_{k_n+1}\big)\|.$$

Now we observe that $x_{k_n+1} - P_F x_{k_n+1} = P_T(x_{k_n} - P_F x_{k_n}) \in T$; moreover, this vector is orthogonal to $F = S \cap T$. Therefore

$$\|P_S\big(x_{k_n+1} - P_F x_{k_n+1}\big)\| = \|x_{k_n+1} - P_F x_{k_n+1}\| \cos\alpha,$$

where α is the angle between the vector $x_{k_n+1} - P_F x_{k_n+1}$ and its projection onto S. By definition, $\alpha \ge \theta(S, T)$ and thus $\cos\alpha \le \cos\theta(S, T) = q < 1$, which yields $\rho(x_{k_n+2}, F) \le q\rho(x_{k_n+1}, F)$. Since the distance between the iterates and the set F decreases, this implies that $\rho(x_{k_n+2}, F) \le q\rho(x_{k_n}, F)$.

Now let Q denote the product of all intermediate operators in the given infinite product up to the next appearance of the composition $P_S P_T$, that is, $Q = A_{k_{n+1}} A_{k_{n+1}-1} \cdots A_{k_n+3}$. By the hypotheses of the theorem, it is a nonexpansive operator and $F = S \cap T$ is an invariant subspace of Q. Using our notations, we obtain that $x_{k_{n+1}} = Q x_{k_n+2}$ and then

$$\rho(x_{k_{n+1}}, F) \le \|x_{k_{n+1}} - Q P_F x_{k_n+2}\|,$$

since $Q P_F x_{k_n+2}$ is a point of F. This implies that

$$\begin{aligned} \rho(x_{k_{n+1}}, F) &\le \|Q x_{k_n+2} - Q P_F x_{k_n+2}\| \\ &\le \|x_{k_n+2} - P_F x_{k_n+2}\| = \rho(x_{k_n+2}, F) \le q\rho(x_{k_n}, F). \end{aligned}$$

Thus the theorem is proved, because any next appearance of $P_S P_T$ adds one more factor $q < 1$, independent of x_0. □

Now we prove an extension of Theorem 3.1 to the case of three subspaces S, T and U.

Theorem 3.2. *Let the subspace $F = S \cap T \cap U$ be invariant under all nonexpansive operators acting on a Hilbert space H and participating in a given infinite product. Let the composition $P_U P_T P_S$ be present in this product infinitely many times. Finally, assume that the angles $\theta(S, T) = \alpha$ and $\theta(S \cap T, U) = \beta$ are positive. Then, for any initial point $x_0 \in H$, the corresponding partial products form a sequence $\{x_n\}$ such that $\lim_{n\to\infty} \rho(x_n, F) = 0$, uniformly over any bounded set of initial points x_0.*

Proof. As before, we consider the strictly increasing sequence $\{k_n\}_{n=1}^{\infty}$ of all numbers k_n such that

$$A_{k_n+1} = P_S, \quad A_{k_n+2} = P_T, \quad A_{k_n+3} = P_U.$$

Consider an iteration x_{k_n} reached just before an application of the composition $P_U P_T P_S$. Denote $a_n = \rho(x_{k_n}, F) = \|x_{k_n} - P_F x_{k_n}\|$. The next iteration is $x_{k_n+1} = P_S x_{k_n}$ and then $a_{n+1} := \rho(x_{k_n+1}, F) = \|x_{k_n+1} - P_F x_{k_n+1}\|$. But $P_F x_{k_n+1} = P_F P_S x_{k_n} = P_F x_{k_n}$, since $F \subset S$, that is, the point $O = P_F x_{k_n}$ is the nearest point in F to the point $A = x_{k_n+1}$ as well as to the point x_{k_n}. Proceeding further, we obtain analogously that the same point in F is the nearest one to the points $B = x_{k_n+2} = P_T x_{k_n+1}$ and $C = x_{k_n+3} = P_U x_{k_n+2}$. Passing to geometric language, we need to compare the lengths $a_{n+1} = |\overrightarrow{OA}|$, $a_{n+2} = |\overrightarrow{OB}|$ and $a_{n+3} = |\overrightarrow{OC}|$ with a_n.

At the first step we readily obtain that

$$a_{n+1} = \|P_S(x_{k_n} - P_F x_{k_n})\| \le \|x_{k_n} - P_F x_{k_n}\| = a_n.$$

Thereafter we define the point $D = P_{S\cap T} x_{k_n+1}$, obtaining the vector $\overrightarrow{OD} \in S \cap T$, and (due to Lemma 2.3) restrict ourselves to the Euclidean space $\mathbb{R}^3$ containing the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OD}$. In fact, we have to consider the faces of the tetrahedron $OABD$. By construction, we get the following values of the angles:

$$\measuredangle ODA = \measuredangle ODB = \measuredangle OBA = \measuredangle ABD = \frac{\pi}{2}, \quad \measuredangle ADB = \gamma \ge \theta(S,T) = \alpha.$$

From the triangle $\triangle ODB$ we get $OB^2 = OD^2 + DB^2$, from $\triangle ADB$ we get $DB = AD\cos\gamma$ and from $\triangle ODA$ we get $AD^2 = OA^2 - OD^2$. Hence $OB^2 = OA^2\cos^2\gamma + OD^2\sin^2\gamma$. Setting $OD = b_{n+1}$, we obtain the recursion formula

$$a_{n+2}^2 = a_{n+1}^2\cos^2\gamma + b_{n+1}^2\sin^2\gamma.$$

Since $b_{n+1} = OD \le OB = a_{n+1}$, it follows that $a_{n+2} \le a_{n+1} \le a_n$. Unfortunately, this is, of course, not yet sufficient to conclude that $a_n \to 0$.

Recall that for any three vectors x, y, z, we have the "triangle inequality for angles" $\theta(x,y) \le \theta(x,z) + \theta(z,y)$ [10, p. 151]. For instance,

$$\measuredangle DOC \le \measuredangle DOB + \measuredangle BOC.$$

But $\measuredangle DOC \ge \theta(S\cap T, U) = \beta$, since $\overrightarrow{DO} \perp (S\cap T)\cap U$. Hence at least one of the angles $\measuredangle DOB$ or $\measuredangle BOC$ is no smaller than $\beta/2$. If $\measuredangle BOC \ge \beta/2$, then from $\triangle BOC$ with the right angle $\angle BCO$ we get that $OC \le OB\cos(\beta/2)$, that is,

$$a_{n+3} \le q_1 a_{n+2} \le q_1 a_n \qquad \text{with} \quad q_1 = \cos(\beta/2) < 1. \tag{3.1}$$

Assume now that $\measuredangle DOB \ge \beta/2$. Going back to $\triangle ODB$ of the tetrahedron $OABD$, we get that $DB \ge OB\sin(\beta/2)$. Then from $\triangle ADB$ we obtain that $AB = DB\tan\gamma \ge OB\sin(\beta/2)\tan\alpha$. Finally, we use the relation $AB^2 = AO^2 - OB^2$ from $\triangle ABO$, which yields the inequality

$$AO^2 \ge OB^2(1 + \sin^2(\beta/2)\tan^2\alpha).$$

Passing to our initial notations, we obtain

$$a_{n+2} \le q_2 a_{n+1}, \qquad \text{where} \quad q_2 = (1 + \sin^2(\beta/2)\tan^2\alpha)^{-1/2}. \tag{3.2}$$

Note that $a_{n+3} \le a_{n+2}$ in any case as a leg and the hypotenuse of $\triangle BOC$. Thus, setting $q = \max(q_1, q_2)$, we obtain from (3.1) and (3.2) that in all situations $a_{n+3} \le qa_n$ with $q < 1$. In the initial (algebraic) notations this means that $\rho(x_{k_n+3}, F) \le q\rho(x_{k_n}, F)$.

Since the last part of the proof is almost the same as the last part of the proof of Theorem 3.1, it is omitted.

In some particular (extreme) cases the geometric picture may be slightly different from the one described above. For example, the vector $\overrightarrow{AD}$ could be orthogonal to T so that $D = B$ and the triangle $\triangle ADB$ degenerates. But in this case $OD = b_{n+1} = a_{n+2}$ and from the hypotheses of the theorem we get immediately that $a_{n+3} \le qa_{n+2}$ with $q = \cos\beta < 1$, as needed. The reader can easily modify the proof in other similar cases. □

Remark 3.3. The conditions on the projection operators in Theorem 3.2 are fulfilled if at least one of the subspaces S or T is finite-dimensional. Indeed, the subspace $S \cap T$ in this case is also finite-dimensional and both angles $\theta(S, T)$ and $\theta(S \cap T, U)$ are positive by Theorem 2.5.

Remark 3.4. If all the operators of the given infinite product commute with P_F and, for all of them, the subspace F is not only invariant, but consists exclusively of common fixed points (e.g., all operators are projections), then, for any initial point x_0, the sequence of the corresponding iterations $\{x_n\}_{n=0}^{\infty}$ converges to the best approximation $P_F x_0$ of x_0. This is true for Theorem 3.1 as well.

4. Strong convergence

The results of the previous section show that, in the case where the products of projections are uniformly convergent, the insertion of additional nonexpansive operators into these products does not interfere with this property, and the new infinite products continue to converge uniformly (over bounded subsets of H). The situation changes if the convergence of projection products is not uniform. This indeed happens in the setting of the theorems of von Neumann and Halperin when not all relevant angles are positive. For example, Halperin's theorem implies that the sequence of operators $B_n = \big(P_U(P_S P_T)^k\big)^n$ is strongly convergent for any fixed k as $n \to \infty$, even without any knowledge about the angle $\theta(S \cap T, U)$. But this theorem does not apply if k changes from factor to factor and/or other non-projection operators are incorporated into the definition of B_n. In this section we consider some cases where the infinite product remains strongly convergent, even with these changes present.

For our considerations, we need formula (1.1) for the case $k = 2$ in the form

$$\lim_{n\to\infty} \|(P_U P_V)^n x - P_{U\cap V} x\| = 0 \qquad \text{for any } \; x \in H, \tag{4.1}$$

where U and V are arbitrary subspaces of H. We need also the following assertion from [17]:

Proposition 4.1. *Let $A:\ H \to H$ be a nonexpansive operator and let $F \subset H$ be a closed set such that, for any given $x \in H$, $\rho(A^n x, F) \to 0$. Let a sequence $\{x_n\} \subset H$*

be such that, for each $n = 1, 2, \ldots$,

$$\|x_{n+1} - Ax_n\| \leq \gamma_n, \qquad \sum_{n=1}^{\infty} \gamma_n < \infty.$$

Then $\rho(x_n, F) \to 0$. Moreover, if the sequence $\{A^n x\}$ is strongly convergent for each $x \in H$, then $\|x_n - x^*\| \to 0$ for some point $x^* \in F$ (dependent on the sequence $\{x_n\}$).

Using this result, we prove, first of all, an auxiliary assertion.

Lemma 4.2. *Let U and V be two subspaces of a Hilbert space H, and let Q_n, $n = 1, 2, \ldots$, be a sequence of nonexpansive operators on H such that, for all $x \in H$,*

$$\|Q_n x - P_V x\| \leq \gamma_n \|x\|, \qquad \sum_{n=1}^{\infty} \gamma_n < \infty.$$

Then, for any $x \in H$, there exists $x^* \in U \cap V$ such that

$$\lim_{n\to\infty} \|P_U Q_n P_U Q_{n-1} \cdots P_U Q_1 \, x - x^*\| = 0. \tag{4.2}$$

Proof. Let $A = P_U P_V$. Then formula (4.1) means that $\rho(A^n x, U \cap V) \to 0$ and the sequence $\{A^n x\}$ is strongly convergent for any $x \in H$. Fix such an x and define a sequence $\{x_n\}$ recursively by

$$x_1 = x, \quad x_{n+1} = P_U Q_n \, x_n, \quad n = 1, 2, \ldots.$$

Then

$$\|x_{n+1} - Ax_n\| = \|P_U(Q_n x_n - P_V x_n)\| \leq \|Q_n x_n - P_V x_n\| \leq \gamma_n \|x_n\|.$$

But the hypotheses of the lemma imply that $Q_n(0) = 0$ for any n and thus $\|Q_n x\| \leq \|x\|$ for any $x \in H$. Therefore $\|x_{n+1}\| \leq \|x_n\|$ and

$$\|x_{n+1} - Ax_n\| \leq \gamma_n \|x_1\| \qquad \text{for all } \; n = 1, 2, \ldots.$$

Consequently, Proposition 4.1 yields a point $x^* \in U \cap V$ such that $\|x_{n+1} - x^*\| \to 0$. By definition of the sequence $\{x_n\}$, this coincides with (4.2). □

The next results are based on Lemma 4.2. They are obtained by using particular realizations of the operators Q_n.

Theorem 4.3. *Let S, T and U be three subspaces of a Hilbert space H such that the angle $\theta(S,T)$ is positive. Let the nonexpansive operators A_n, $n = 1, 2, \ldots$, be such that all elements of the subspace $V = S \cap T$ are fixed points for each A_n. Let a sequence of natural numbers $\{k_n\}$ be such that*

$$\sum_{n=1}^{\infty} q^{k_n} < \infty, \qquad \text{where} \quad q = \cos\theta(S,T). \tag{4.3}$$

Define the operators

$$Q_n = A_n (P_S P_T)^{k_n}, \qquad n = 1, 2, \ldots.$$

Then, for any $x \in H$, there exists $x^* \in S \cap T \cap U$ such that

$$\lim_{n\to\infty} \|P_U A_n (P_S P_T)^{k_n} P_U A_{n-1} (P_S P_T)^{k_{n-1}} \cdots P_U A_1 (P_S P_T)^{k_1} x - x^*\| = 0.$$

Proof. By the hypotheses of the theorem any $x \in V$ is a fixed point of A_n. Hence $A_n P_V x = P_V x$ for any $x \in H$ and thus

$$\|Q_n x - P_V x\| = \|A_n (P_S P_T)^{k_n} x - A_n P_V x\| \leq \|(P_S P_T)^{k_n} x - P_V x\|,$$

since A_n is a nonexpansive operator. The next arguments are similar to those in the proof of Theorem 3.1. Namely, $P_V = P_V P_S P_T$, and thus

$$\|(P_S P_T)^{k_n} x - P_V x\| = \|(P_S P_T)^{k_n} x - P_V (P_S P_T)^{k_n - 1} x\| = \|P_S P_T y - P_V y\|,$$

where we set $y = (P_S P_T)^{k_n - 1} x$. The vector $P_T(y - P_V y)$ belongs to T and is orthogonal to $V = S \cap T$. Hence

$$\|P_S P_T y - P_V y\| = \|P_S P_T (y - P_V y)\| \leq \|P_T (y - P_V y)\| \cos \theta(S, T) \leq q \|y - P_V y\|.$$

Proceeding inductively, we obtain that

$$\|(P_S P_T)^{k_n} x - P_V x\| \leq q^{k_n} \|x - P_V x\| \leq q^{k_n} \|x\|.$$

By condition (4.3), this inequality yields all the hypotheses of Lemma 4.2 and therefore proves the theorem. □

Theorem 4.3 admits an interesting new application to Numerical Analysis. Suppose we are interested in finding the point $P_{S \cap T \cap U} x_0$ for some given $x_0 \in H$. Due to Halperin's theorem we may use the iterations $x_n = (P_U P_S P_T)^n x_0$ which converge to the needed point. Suppose the subspace U is such that any computation of the projection P_U is much harder in comparison with the other two projections. Omitting all A_n, we see that, in the case where $\theta(S, T) > 0$, Theorem 4.3 provides us with another iteration process, namely,

$$x_N = P_U (P_S P_T)^{k_n} P_U (P_S P_T)^{k_{n-1}} \cdots P_U (P_S P_T)^{k_1} x_0, \quad N = n + \sum_{i=1}^{n} k_i\,,$$

with arbitrarily quickly increasing k_n and, correspondingly, arbitrarily rare computations of P_U. Indeed, in the absence of all non-projection operators, the point x^* from Theorem 4.3 obviously coincides with $P_{S \cap T \cap U} x_0$.

Using the methods of Theorem 3.2, we can generalize the result of Theorem 4.3 to the intersection of four subspaces.

Theorem 4.4. *Let S, T, U and W be four subspaces of a Hilbert space H such that the angles $\theta(S, T) = \alpha > 0$ and $\theta(S \cap T, U) = \beta > 0$. Let the nonexpansive operators A_n, $n = 1, 2, \ldots$, be such that all elements of the subspace $V = S \cap T \cap U$ are fixed points of each A_n. Let a sequence of natural numbers $\{k_n\}$ be such that*

$$\text{(4.4)} \qquad \sum_{n=1}^{\infty} q^{k_n} < \infty\,, \quad \textit{where} \quad q = \max\{\cos(\beta/2),\ (1 + \sin^2(\beta/2) \tan^2 \alpha)^{-1/2}\}.$$

Define the operators

$$Q_n = A_n (P_U P_S P_T)^{k_n}, \qquad n = 1, 2, \ldots.$$

Then, for any $x \in H$, there exists $x^* \in S \cap T \cap U \cap W$ such that

$$\lim_{n\to\infty} \|P_W A_n (P_U P_S P_T)^{k_n} P_W A_{n-1} (P_U P_S P_T)^{k_{n-1}} \cdots P_W A_1 (P_U P_S P_T)^{k_1} x - x^*\| = 0.$$

After computing q for (4.4) in accordance with the constructions in the proof of Theorem 3.2, the remaining part of the proof of this theorem is very similar to that of Theorem 4.3 and therefore we omit it.

Proposition 4.1 (and the arguments in [18]) also prove to be useful in the case of uniform convergence. They lead to an improvement of Theorems 3.1 and 3.2 other than the one mentioned in Remark 3.4, omitting the requirement that the operators A_n and P_F commute. As before, we obtain that the sequence of iterations $\{x_n\}$ not only approaches the intersection F, but converges to some point $x^* \in F$. Unfortunately, we cannot now assert that x^* coincides with the best approximation $P_F x_0$ of the initial point x_0.

The following discussion does not depend on the number of subspaces and on the particular estimate of the factor q. In order to unify the proof, we denote by $\mathcal{P}$ the product $P_S P_T$ in the case of Theorem 3.1 with q defined as in (4.3). In the case of Theorem 3.2 the same $\mathcal{P}$ will mean the product $P_S P_T P_U$ with q defined as in (4.4). The main inequalities obtained in the course of the proofs of Theorems 3.1 and 3.2 can now be written in the same form:

$$\|\mathcal{P}x - P_F x\| \le q\|x - P_F x\| \qquad \text{for any } \; x \in H. \tag{4.5}$$

Theorem 4.5. *Let $x_0 \in H$ be an arbitrary initial point for the sequence of iterations*

$$x_n = \prod_{i=1}^{n} (A_i \mathcal{P}) x_0, \qquad n = 1, 2, \ldots,$$

where all A_i are nonexpansive operators such that all points of the subspace F are their fixed points. Let inequality (4.5) be satisfied with some $q < 1$. Then there exists $x^* \in F$ such that $\lim_{n\to\infty} \|x_n - x^*\| = 0$, uniformly over any bounded set of initial points x_0.

Proof. Take the operator $\mathcal{P}$ as the operator A in Proposition 4.1. Since $P_F \mathcal{P} x = P_F x$ for any $x \in H$, we readily obtain from (4.5) that

$$\|\mathcal{P}^n x - P_F x\| \le q\|\mathcal{P}^{n-1} x - P_F x\| \le \cdots \le q^n \|x - P_F x\| \longrightarrow 0$$

as $n \to \infty$. Thus the sequence $\{\mathcal{P}^n x\}$ is strongly convergent for each $x \in H$ (uniformly over any bounded set of initial points), and so the conditions imposed on the operator A in Proposition 4.1 are satisfied. We claim that the sequence of iterations $\{x_n\}$ also satisfies all needed conditions. Indeed, since $A_{n+1} P_F = P_F$, we have

$$\begin{aligned} \|x_{n+1} - \mathcal{P} x_n\| &= \|A_{n+1} \mathcal{P} x_n - \mathcal{P} x_n\| \\ &\le \|A_{n+1} \mathcal{P} x_n - A_{n+1} P_F x_n\| + \|\mathcal{P} x_n - A_{n+1} P_F x_n\| \\ &\le 2\|\mathcal{P} x_n - P_F x_n\| \le 2q\|x_n - P_F x_n\|. \end{aligned}$$

The properties of projections imply that

$$\|x_n - P_F x_n\| \le \|x_n - P_F x_{n-1}\| = \|A_n \mathcal{P} x_{n-1} - A_n P_F x_{n-1}\| \le q\|x_{n-1} - P_F x_{n-1}\|,$$

and proceeding by induction we arrive at the final estimate

$$\|x_{n+1} - \mathcal{P} x_n\| \le 2q^{n+1}\|x_0 - P_F x_0\| = \gamma_n,$$

which provides all needed properties of the sequence $\{x_n\}$. The claim concerning uniform convergence over any bounded set of initial points follows as in [18]. $\square$

Before finishing the paper, we would like to make a few comments regarding conditions (4.3) and (4.4), which were imposed on the sequence $\{k_i\}_{i=1}^{\infty}$. If all k_i are different from each other, both conditions are obviously fulfilled. If at least one of the values of k_i is repeated infinitely many times, then both series in (4.3) and (4.4) are divergent. Suppose now that every value $n \in \mathbb{N}$ is repeated among all $\{k_i\}$ exactly m_n times (some m_n could be zero). Then $\sum q^{k_i} = \sum m_n q^n$, that is, we obtain a convergence problem for a power series and it is enough to require that $\limsup \sqrt[n]{m_n} < 1/q$. But algorithms for the computation of angles between given subspaces (and thus the computation of q) have not yet been developed (in the case of finite dimensional subspaces some explicit formulas can be found in [11]). At the same time, the mere verification of positivity of the needed angles might be much simpler and the inequality $\limsup \sqrt[n]{m_n} \le 1$ covers all these cases.

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