Journal of Nonlinear and Convex Analysis
Volume 11, Number 3, 2010, 455–460

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ANOTHER NOTE ON THE VON NEUMANN ALTERNATING PROJECTIONS ALGORITHM

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Abstract. We present another elementary geometric proof of von Neumann's classical convergence theorem regarding alternating orthogonal projections in Hilbert space. In contrast with previous proofs, this time our argument is based on the two-dimensional case.

1. Introduction

A few years ago we presented [6] an elementary geometric proof of von Neumann's classical convergence theorem regarding alternating orthogonal projections in Hilbert space. The main purpose of the present note is to present another elementary geometric proof of this seminal result. In contrast with previous proofs, this time our argument is based on a reduction to the two-dimensional case.

Let S_1 and S_2 be two closed subspaces of a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$, and let $P_1 : H \mapsto S_1$ and $P_2 : H \mapsto S_2$ be the corresponding orthogonal projections of H onto S_1 and S_2, respectively. Denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of nonnegative integers. Let x_0 be an arbitrary point in H, and define the sequence $\{x_n : n \in \mathbb{N}\}$ of alternating projections by

$$x_{2n+1} = P_1 x_{2n} \quad \text{and} \quad x_{2n+2} = P_2 x_{2n+1}, \tag{1.1}$$

where $n \in \mathbb{N}$.

Theorem 1.1. *The sequence $\{x_n : n \in \mathbb{N}\}$ defined by (1.1) converges in norm as $n \to \infty$ to $P_S x_0$, where $P_S : H \mapsto S$ is the orthogonal projection of H onto the intersection $S = S_1 \cap S_2$.*

This is von Neumann's classical theorem [8, p. 475]. It was rediscovered by several other authors; see, for example, [1], [7] and [10]. More information regarding this theorem and its manifold applications can be found in [3] and the references mentioned therein. Other proofs of Theorem 1.1 can be found, for instance, in [4], [9], [2] and [6]. We take this opportunity to note that on line 6 of [6, p. 383], the set on the right-hand side of the equality should be intersected with D.

We begin the next section of our paper with an orthogonal decomposition lemma [5] and then continue with two simple lemmata regarding alternating projections between two lines. Section 3 is devoted to several lemmata concerning alternating

2010 *Mathematics Subject Classification.* 46C05, 41A65, 90C25.

Key words and phrases. Alternating orthogonal projections, Hilbert space, orthogonal decomposition.

The first author was supported by Grant FWF-P19643-N18. The second author was partially supported by the Israel Science Foundation (Grant 647/07), the Fund for the Promotion of Research at the Technion (Grant 2001893), and by the Technion President's Research Fund (Grant 2007842).

projections between two finite-dimensional subspaces. These results lead in Section 4 to a key estimate which, in its turn, yields a proof of Theorem 1.1 itself.

2. Lines and planes

We first present an orthogonal decomposition lemma [5]. It shows that any two finite-dimensional subspaces $X, Y \subset H$ with $1 \le \dim X = m \le n = \dim Y$ possess orthonormal bases $\{e_i\}_{i=1}^m$ and $\{f_i\}_{i=1}^n$, respectively, so that $X+Y$ can be written as a sum of the following pairwise orthogonal, at most two-dimensional, subspaces defined by the basis vectors:

$$\operatorname{span}\{e_1, f_1\} \oplus \cdots \oplus \operatorname{span}\{e_m, f_m\} \oplus \operatorname{span}\{f_{m+1}\} \oplus \cdots \oplus \operatorname{span}\{f_n\}. \tag{2.1}$$

Lemma 2.1. *Let X and Y be two subspaces of H with $1 \le \dim X = m \le n = \dim Y$. Then there exist orthonormal bases $\{e_i\}_{i=1}^m$ and $\{f_i\}_{i=1}^n$ of X and Y, respectively, and $0 \le k \le m$ so that*

(i) *$e_i = f_i$ if and only if $i \le k$;*
(ii) *the at most two-dimensional spaces $\operatorname{span}\{e_i\}$, $i \le k$, $\operatorname{span}\{e_i, f_i\}$, $k < i \le m$, and $\operatorname{span}\{f_i\}$, $m < i \le n$, are all pairwise orthogonal.*

We continue with two simple lemmata concerning alternating projections between two lines.

Lemma 2.2. *Let e and f be two unit vectors in a Hilbert space H, and let P_e and P_f be the two orthogonal projections of H onto the lines spanned by e and f, respectively. Let $u \in \operatorname{span}\{e\}$ and $u_k = (P_f P_e)^k u$, $k \in \mathbb{N}$. Then $|u - u_k|^2 \le |u|^2 - |u_k|^2$ for all $k \in \mathbb{N}$.*

Proof. We have $\langle u - u_1, f\rangle = 0$ and $u_k = t_k f$ with $\operatorname{sign} t_k = \operatorname{sign} t_1$ and $|t_k| \le |t_1|$ for all $k \ge 1$. Consequently, we also have $|u|^2 = |u-u_k|^2 + 2\langle u - u_k, u_k\rangle + |u_k|^2 = |u-u_k|^2 + 2\langle u - t_k f, t_k f\rangle + |u_k|^2 = |u-u_k|^2 + 2\langle u_1 - t_k f, t_k f\rangle + |u_k|^2 = |u-u_k|^2 + 2(t_1 - t_k)t_k + |u_k|^2 \ge |u-u_k|^2 + |u_k|^2$ for all $k \ge 1$, as asserted. □

Lemma 2.3. *Let e and f be two unit vectors in a Hilbert space H, and let P_e and P_f be the orthogonal projections of H onto the lines spanned by e and f, respectively. Let $v \in H$ and $v_k = (P_f P_e)^k v$, $k \in \mathbb{N}$. Then $|v - v_k|^2 \le 2(|v|^2 - |v_k|^2)$ for all $k \in \mathbb{N}$.*

Proof. Let $u = P_e v$ and $u_k = (P_f P_e)^k u$, $k \in \mathbb{N}$. Then $u_k = v_k$, $k \ge 1$, and it follows from the triangle inequality and Lemma 2.2 that $|v - v_k|^2 \le 2(|v-u|^2 + |u - v_k|^2) = 2(|v-u|^2 + |u-u_k|^2) = 2(|v|^2 - |u|^2 + |u-u_k|^2) \le 2(|v|^2 - |u|^2 + |u|^2 - |u_k|^2) = 2(|v|^2 - |u_k|^2) = 2(|v|^2 - |v_k|^2)$ for all $k \ge 1$, as claimed. □

3. Alternating projections

In this section we collect several lemmata regarding alternating projections between two finite-dimensional subspaces. We first provide representations of the composition of two orthogonal projections and its iterates.

Lemma 3.1. *Let X and Y be two finite-dimensional subspaces of a Hilbert space H with $m = \dim X \le \dim Y = n$, and let*

$$X + Y = E_1 \oplus E_2 \oplus E_m \oplus \cdots \oplus E_n \tag{3.1}$$

be the decomposition obtained in Lemma 2.1. Let $Q_j : H \mapsto E_j$ be the orthogonal projection onto E_j, $1 \le j \le n$, $P_i = P_{f_i} P_{e_i}$, $1 \le i \le m$, and set $P = P_Y P_X$. Then

$$Pz = \sum_{i=1}^{m} P_i Q_i z \tag{3.2}$$

for all $z \in H$.

Proof. Fix a point $z \in H$. For each $1 \le j \le n$, let $Q_j z = \alpha_j e_j + \beta_j f_j$, $1 \le j \le n$, where $\alpha_j = 0$ for $m < j \le n$. We have $P_X(Q_j z) = \sum_{i=1}^{m} \langle Q_j z, e_i \rangle e_i = \sum_{i=1}^{m} \langle \alpha_j e_j + \beta_j f_j, e_i \rangle e_i = [\alpha_j \langle e_j, e_j \rangle + \beta_j \langle e_j, f_j \rangle] e_j = (\alpha_j + \beta_j \langle e_j, f_j \rangle) e_j$, $1 \le j \le m$, and hence $P_X z = \sum_{j=1}^{n} P_X(Q_j z) = \sum_{i=1}^{m} (\alpha_i + \beta_i \langle e_i, f_i \rangle) e_i$. Therefore

$$\begin{aligned} Pz &= P_Y P_X z \\ &= \sum_{j=1}^{n} \Big(\sum_{i=1}^{m} (\alpha_i + \beta_i \langle e_i, f_i \rangle) \langle e_i, f_j \rangle \Big) f_j \\ &= \sum_{j=1}^{m} (\alpha_j + \beta_j \langle e_j, f_j \rangle) \langle e_j, f_j \rangle f_j. \end{aligned}$$

On the other hand, for each $1 \le i \le m$,

$$\begin{aligned} P_i(Q_i z) &= P_{f_i} P_{e_i} (\alpha_i e_i + \beta_i f_i) \\ &= P_{f_i} (\alpha_i \langle e_i, e_i \rangle e_i + \beta_i \langle e_i, f_i \rangle e_i) \\ &= P_{f_i} (\alpha_i e_i + \beta_i \langle e_i, f_i \rangle e_i) \\ &= \alpha_i \langle e_i, f_i \rangle f_i + \beta_i \langle e_i, f_i \rangle \langle e_i, f_i \rangle f_i, \end{aligned}$$

and the result follows. □

In the next three lemmata and in their proofs we continue to use the notations of Lemma 3.1.

Lemma 3.2. *For each $k \ge 1$ and $z \in H$,*

$$P^k z = \sum_{i=1}^{m} P_i^k (Q_i z). \tag{3.3}$$

Proof. We use induction on k. For $k = 1$ this equality holds by Lemma 3.1. Assume it is true for a certain natural number k. Then

$$\begin{aligned} P^{k+1} z = P\Big(\sum_{i=1}^{m} P_i^k (Q_i z)\Big) &= \sum_{i=1}^{m} P_i \Big(Q_i \Big(\sum_{j=1}^{m} P_j^k (Q_j z)\Big)\Big) \\ = \sum_{i=1}^{m} P_i P_i^k (Q_i z) &= \sum_{i=1}^{m} P_i^{k+1} (Q_i z), \end{aligned}$$

as required. □

Lemma 3.3. *For each $k \geq 1, z \in H$ and $1 \leq i \leq m$,*

$$P_i^k Q_i z = Q_i P^k z. \tag{3.4}$$

Proof. By Lemma 3.2 we have $P^k z = \sum_{j=1}^{m} P_j^k(Q_j z)$ and therefore $Q_i P^k z = P_i^k Q_i z$, as claimed. ☐

Lemma 3.4. *For each $k \in \mathbb{N}$ and $z \in H$,*

$$|z - P^k z|^2 \leq 2(|z|^2 - |P^k z|^2). \tag{3.5}$$

Proof. Fix $z \in H$. Then $z = \sum_{i=1}^{m} Q_i z + y$, where the point $y \in H$ is orthogonal to $E_1 \oplus E_2 \oplus \cdots \oplus E_m$. Therefore

$$\begin{aligned}
|z - P^k z|^2 &= |\sum_{i=1}^{m} Q_i z + y - \sum_{i=1}^{m} P_i^k(Q_i z))|^2 = |\sum_{i=1}^{m}(Q_i z - P_i^k(Q_i z)) + y|^2 \\
&= \sum_{i=1}^{m} |Q_i z - P_i^k(Q_i z)|^2 + |y|^2 \leq 2\sum_{i=1}^{m}(|Q_i z|^2 - |P_i^k(Q_i z)|^2) + |y|^2 \\
&= 2\sum_{i=1}^{m}(|Q_i z|^2 - |Q_i P^k z|^2) + |y|^2 = 2(|z|^2 - |y|^2 - |P^k z|^2) + |y|^2 \\
&= 2(|z|^2 - |P^k z|^2) - |y|^2 \leq 2(|z|^2 - |P^k z|^2),
\end{aligned}$$

where we have used Lemmata 3.2, 2.3 and 3.3. ☐

Remark 3.5. In the setting of Lemma 3.1, let $R_i = P_{e_i} P_{f_i}$, $1 \leq i \leq m$, and set $R = P_X P_Y$. Then analogous computations to those used in the proofs of Lemmata 3.1, 3.2, 3.3 and 3.4 lead to the following facts:

$$Rz = \sum_{i=1}^{m} R_i Q_i z, \quad z \in H; \tag{3.6}$$

$$R^k z = \sum_{i=1}^{m} R_i^k(Q_i z), \quad k \geq 1, \quad z \in H; \tag{3.7}$$

$$R_i^k Q_i z = Q_i R^k z, \quad k \geq 1, \quad z \in H, \quad 1 \leq i \leq m; \tag{3.8}$$

$$|z - R^k z|^2 \leq 2(|z|^2 - |R^k z|^2), \quad k \geq 1, \quad z \in H. \tag{3.9}$$

4. Convergence

In this section we first prove a key estimate and then Theorem 1.1 itself.

Proposition 4.1. *Let the sequence $\{x_n : n \in \mathbb{N}\}$ be defined by (1.1), and let p and r belong to $\mathbb{N}$. Then*

$$|x_r - x_{r+p}|^2 \leq 4(|x_r|^2 - |x_{r+p}|^2). \tag{4.1}$$

Proof. We may assume without any loss of generality that $x_{r+1} \in S_1$. Assume first that $p = 2k$ for some $k \geq 1$. Let

$$X := \operatorname{span}\{x_{r+1}, x_{r+3}, \ldots, x_{r+2k+1}\} \subset S_1 \tag{4.2}$$

and

$$Y := \operatorname{span}\{x_{r+2}, x_{r+4}, \ldots, x_{r+2k}\} \subset S_2. \tag{4.3}$$

Then $x_{r+2k} = (P_2P_1)^k x_r = (P_Y P_X)^k x_r$. Hence

$$|x_r - x_{r+p}|^2 = |x_r - x_{r+2k}|^2 \leq 2(|x_r|^2 - |x_{r+2k}|^2) = 2(|x_r|^2 - |x_{r+p}|^2)$$

by Lemma 3.4 and Remark 3.5.

Assume now that $p = 2k + 1$ for some $k \geq 1$. Then

$$\begin{aligned} |x_r - x_{r+p}|^2 &= |x_r - x_{r+2k+1}|^2 \\ &\leq 2(|x_r - x_{r+2k}|^2 + |x_{r+2k} - x_{r+2k+1}|^2) \\ &= 2(|x_r - x_{r+2k}|^2 + |x_{r+2k}|^2 - |x_{r+2k+1}|^2) \\ &\leq 2(2(|x_r|^2 - |x_{r+2k}|^2) + |x_{r+2k}|^2 - |x_{r+2k+1}|^2), \end{aligned}$$

where we have used the previous case (that is, $p = 2k$). Hence

$$\begin{aligned} |x_r - x_{r+p}|^2 &\leq 2(2|x_r|^2 - |x_{r+2k}|^2 - |x_{r+2k+1}|^2) \\ &\leq 2(2|x_r|^2 - 2|x_{r+2k+1}|^2) \\ &= 4(|x_r|^2 - |x_{r+2k+1}|^2) \\ &= 4(|x_r|^2 - |x_{r+p}|^2), \end{aligned} \tag{4.4}$$

as asserted. □

Alternatively, when $p = 2k + 1$ we have

$$\begin{aligned} |x_r - x_{r+p}|^2 &= |x_r - x_{r+2k+1}|^2 \\ &\leq 2(|x_r - x_{r+1}|^2 + |x_{r+1} - x_{r+2k+1}|^2) \\ &= 2(|x_r|^2 - |x_{r+1}|^2 + |x_{r+1} - (P_1P_2)^k x_{r+1}|^2) \\ &= 2(|x_r|^2 - |x_{r+1}|^2 + |x_{r+1} - (P_X P_Y)^k x_{r+1}|^2) \\ &\leq 2(|x_r|^2 - |x_{r+1}|^2 + 2(|x_{r+1}|^2 - |x_{r+2k+1}|^2)) \\ &= 2(|x_r|^2 + |x_{r+1}|^2 - 2|x_{r+2k+1}|^2) \\ &\leq 2(2|x_r|^2 - 2|x_{r+2k+1}|^2) \\ &= 4(|x_r|^2 - |x_{r+2k+1}|^2) \\ &= 4(|x_r|^2 - |x_{r+p}|^2). \end{aligned}$$

Proof of Theorem 1.1. Since the numerical sequence $\{|x_n| : n \in \mathbb{N}\}$ decreases to its limit as $n \to \infty$, Proposition 4.1 shows that $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence which converges in norm as $n \to \infty$ to $P_S x_0$ by part (c) of [6, Lemma 2.1]. □

Alternatively, once we know that

$$|x_r - x_{r+2k}|^2 \leq 2(|x_r|^2 - |x_{r+2k}|^2),$$

we have

$$|x_{2m} - x_{2(m+k)}|^2 \leq 2(|x_{2m}|^2 - |x_{2(m+k)}|^2), \tag{4.5}$$

so $x_{2m} = (P_2P_1)^m x_0 \to z$, a fixed point of P_2P_1, as $m \to \infty$. This limit z clearly belongs to S_2. If z were not in S_1, then we would obtain $|P_2P_1z| \leq |P_1z| < |z|$, a contradiction. Thus $z \in S_1$, $x_{2m+1} = P_1x_{2m} \to P_1z = z$ as $m \to \infty$, and the whole sequence $\{x_n : n \in \mathbb{N}\}$ converges in norm as $n \to \infty$ to $z = P_Sx_0$, as claimed.

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Manuscript received July 19, 2010
revised July 26, 2010

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