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ON QUASI-VARIATIONAL INCLUSIONS AND ASYMPTOTICALLY STRICT PSEUDO-CONTRACTIONS

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Abstract. In this paper, quasi-variational inclusions and fixed point problems are considered. A general iterative process is introduced for finding a common element in the zero set of the sum of maximal monotone operators and inverse strongly-monotone mappings and the fixed point set of asymptotically strict pseudo-contractions. Further, weak convergence theorems for common elements in two sets mentioned above are established in real Hilbert spaces.

1. Introduction and preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a mapping and $F(S)$ denote the fixed point set of S. Recall the following definitions:

(1) S is said to be *nonexpansive* if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C.$$

(2) S is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|S^n x - S^n y\| \le k_n \|x - y\|, \quad \forall x, y \in C, \ n \ge 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [7] in 1972. It is known that, if C is a nonempty bounded closed convex subset of a Hilbert space space H, then every asymptotically nonexpansive self-mapping has a fixed point. Further, the set $F(S)$ of fixed points of S is closed and convex. Since 1972, a host of authors have studied the weak and strong convergence problems of iterative processes for such a class of mappings.

(3) S is said to be *strictly pseudo-contractive* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \kappa \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

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For such a case, S is also said to be *κ-strict pseudo-contraction*. The class of strict pseudo-contractions was introduced by Browder and Petryshyn [3] in 1967. It is clear that every nonexpansive mapping is a 0-strict pseudo-contraction.

(4) S is called an *asymptotically strict pseudo-contraction* if there exist a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and a constant $\kappa \in [0, 1)$ such that

$$\|S^n x - S^n y\|^2 \leq k_n \|x - y\|^2 + \kappa \|(I - S^n)x - (I - S^n)y\|^2, \quad \forall x, y \in C, \ n \geq 1.$$

For such a case, S is also called an *asymptotically κ-strict pseudo-contraction*. The class of asymptotically strict pseudo-contractions is introduced by Qihou [21] in 1996. It is clear that every asymptotically nonexpansive mapping is an asymptotical 0-strict pseudo-contraction.

(5) Let $A : C \to H$ be a mapping. A is said to be *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

(6) A is said to be *inverse strongly-monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be *α-inverse strongly monotone*.

Let $M : H \to 2^H$ be a set-valued mapping. The set $D(M)$ defined by $D(M) = \{x \in H : Mx \neq \emptyset\}$ is called the *domain* of M. The set $R(M)$ defined by $R(M) = \cup_{x \in H} Mx$ is called the *range* of M. The set $G(M)$ defined by $G(M) = \{(x, y) \in H \times H : x \in D(M), y \in R(M)\}$ is called the *graph* of M.

(7) M is said to be *monotone* if

$$\langle x - y, f - g \rangle > 0, \quad \forall (x, f), (y, g) \in G(M).$$

(8) M is said to be *maximal monotone* if it is not properly contained in any other monotone operator. Equivalently, M is maximal monotone if $R(I + rM) = H$ for all $r > 0$.

The class of monotone mappings is one of the most important classes of mappings among nonlinear mappings. For a maximal monotone operator M on H and $r > 0$, we may define the single-valued resolvent $J_r = (I + rM)^{-1} : H \to D(M)$. It is known that J_r is firmly nonexpansive and $M^{-1}(0) = F(J_r)$, where $F(J_r)$ denotes the fixed point set of J_r.

On the other hand, recall that the *classical variational inequality problem* is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

Denote by $VI(C, A)$ the solution set of the problem (1.1). It is known that $x \in C$ is a solution of the problem (1.1) if and only if x is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping.

Recently, many authors have considered the weak convergence of the iterative sequences for the variational inequality (1.1) and fixed point problems of nonlinear mappings (see, for example, [1, 5, 8-11, 13, 14, 16-20, 25-27]).

For finding a common element in the solution set of the variational inequality (1.1) and the fixed point set of nonexpansive mappings, Takahashi and Toyoda [26] proved the following weak convergence theorem:

Theorem 1.1. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H and S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n), \quad \forall n \geq 0, \end{cases}$$

where $\lambda_n \in [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\alpha_n \in [c, d]$ for some $c, d \in (0, 1)$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in F(S) \cap VI(C, A)$, where

$$z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n.$$

Let $S : C \to C$ be an asymptotically strict pseudo-contraction, $A : C \to H$ be an α-inverse strongly-monotone mapping, $M : H \to 2^H$ be a maximal monotone operator such that $D(M) \subset C$, where $D(M)$ is the domain of M, $B : C \to H$ be a β-inverse strongly-monotone mapping and $W : H \to 2^H$ be a maximal monotone operator such that $D(W) \subset C$, where $D(W)$ is the domain of W.

In this paper, motivated by Theorem 1.1, we consider the problem of finding a common element in the following set:

$$F(S) \cap (A + M)^{-1}(0) \cap (B + W)^{-1}(0),$$

where $(A + M)^{-1}(0)$ is the zero point set of $A + M$ and $(B + W)^{-1}(0)$ is the zero point set of $B + W$, prove some weak convergence theorems of common elements are established in real Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by Takahashi and Toyoda [26] and others.

In order to prove our main results, we also need the following lemmas:

Lemma 1.2 ([12])**.** *Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ be an asymptotically κ-strict pseudo-contraction. Then*

(1) *S is Lipschitz continuous.*
(2) *$I - S$ is demi-closed, this is, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Sx_n \to 0$, then $x \in F(S)$.*

Lemma 1.3 ([24])**.** *Let H be a Hilbert space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in H such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$$

for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.4 ([2]). *Let C be a nonempty closed convex subset of a Hilbert space H and $A : C \to H$ be a mapping and $M : H \to 2^H$ be a maximal monotone mapping. Then*

$$F(J_r(I - rA)) = (A + M)^{-1}(0), \quad \forall r > 0.$$

Lemma 1.5. *In a real Hilbert space H, the following inequality holds:*

$$\|ax + (1-a)y\|^2 = a\|x\|^2 + (1-a)\|y\|^2 - a(1-a)\|x-y\|^2, \quad \forall a \in [0,1],\ x, y \in H.$$

Lemma 1.6 ([28]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \le (1 + b_n)a_n + c_n, \quad \forall n \ge n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

2. Main results

Now, we give our main results in this paper.

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, $S : C \to C$ be an asymptotically κ-strict pseudo-contraction with the sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $A : C \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be a β-inverse strongly monotone mapping. Let $M : H \to 2^H$ and $W : H \to 2^H$ be maximal monotone operators such that $D(M) \subset C$ and $D(W) \subset C$, respectively. Assume that $\mathcal{F} := F(S) \cap (A+M)^{-1}(0) \cap (B + W)^{-1}(0) \ne \emptyset$. Let $\{x_n\}$ be the sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ z_n = J_{s_n}(x_n - s_n B x_n), \\ y_n = J_{r_n}(z_n - r_n A z_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\big(\beta_n y_n + (1-\beta_n) S^n y_n\big), \quad \forall n \ge 0, \end{cases}$$

where $J_{r_n} = (I + r_n M)^{-1}$, $J_{s_n} = (I + r_n W)^{-1}$, $\{r_n\}$ is a sequence in $(0, 2\alpha)$, $\{s_n\}$ is a sequence in $(0, 2\beta)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Assume that the following restrictions are satisfied:

(a) $0 < a \le \alpha_n \le b < 1$, *where $a, b \in \mathbb{R}$ are two constants;*
(b) $0 \le \kappa \le \beta_n \le c < 1$, *where $c \in \mathbb{R}$ is a constant;*
(c) $d \le r_n \le e$ *and* $f \le s_n \le g$, *where $0 < d < e < 2\alpha$ and $0 < f < g < 2\beta$ are four constants.*

Then the sequence $\{x_n\}$ converges weakly to a point $\bar{x} \in \mathcal{F}$.

Proof. First, we show that $\{x_n\}$ is bounded. In fact, note that $(I - r_n A)$ and $(I - s_n B)$ are nonexpansive for each fixed $n \ge 0$. Indeed, it follows from the restriction (c) that

$$\begin{aligned} \|(I - r_n A)x - (I - r_n A)y\|^2 &= \|x-y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\le \|x - y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2 \\ &\le \|x-y\|^2, \quad \forall x, y \in C. \end{aligned}$$

This shows that $(I - r_nA)$ is nonexpansive for each fixed $n \geq 0$ and so is $(I - s_nB)$. Put

$$S_nx = \beta_nx + (1 - \beta_n)S^nx, \quad \forall x \in C.$$

Fix $p \in \mathcal{F}$. In view of Lemma 1.5, it follows from the restriction (b) that

$$\begin{aligned}
&\|S_ny_n - p\|^2 \\
&= \beta_n\|y_n - p\|^2 + (1 - \beta_n)\|S^ny_n - p\|^2 - \beta_n(1 - \beta_n)\|S^ny_n - y_n\|^2 \\
&\leq \beta_n\|y_n - p\|^2 + (1 - \beta_n)k_n\|y_n - p\|^2 + (1 - \beta_n)(\kappa - \beta_n)\|S^ny_n - y_n\| \\
&\leq k_n\|y_n - p\|^2.
\end{aligned} \tag{2.1}$$

Since J_{r_n}, J_{s_n}, $I - r_nA$ and $I - s_nB$ are nonexpansive, we see from (2.1) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|S_ny_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n\|y_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n\|z_n - p\|^2 \\
&\leq \big(1 + (k_n - 1)\big)\|x_n - p\|^2.
\end{aligned}$$

In view of Lemma 1.6, we see that the limit of the sequence $\{\|x_n - p\|\}$ exists. This shows that the sequence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{z_n\}$. Without loss of generality, we may assume that $\lim_{n\to\infty} \|x_n - p\| = d > 0$. Notice from (2.1) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|S_ny_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n\|(z_n - r_nAz_n) - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n(\|z_n - p\|^2 - r_n(2\alpha - r_n)\|Az_n - Ap\|^2) \\
&\leq k_n\|x_n - p\|^2 - (1 - \alpha_n)k_nr_n(2\alpha - r_n)\|Az_n - Ap\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1 - \alpha_n)k_nr_n(2\alpha - r_n)\|Az_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (k_n - 1)\|x_n - p\|^2.
\end{aligned}$$

In view of the restrictions (a) and (c), we see that

$$\lim_{n\to\infty} \|Az_n - Ap\| = 0. \tag{2.2}$$

Notice from (2.1) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|S_ny_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n\|y_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n\|z_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n\|(x_n - s_nBx_n) - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n(\|x_n - p\|^2 - s_n(2\beta - s_n)\|Bx_n - Bp\|^2) \\
&\leq k_n\|x_n - p\|^2 - (1 - \alpha_n)k_ns_n(2\beta - s_n)\|Bx_n - Bp\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}&(1-\alpha_n)k_n s_n(2\beta - s_n)\|Bx_n - Bp\|^2\\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (k_n - 1)\|x_n - p\|^2.\end{aligned}$$

In view of the restrictions (a) and (c), we see that

$$\lim_{n\to\infty} \|Bx_n - Bp\| = 0. \tag{2.3}$$

Since J_{r_n} is firmly nonexpansive, we obtain

$$\begin{aligned}\|y_n - p\|^2 &= \|J_{r_n}(z_n - r_nAz_n) - J_{r_n}(p - r_nAp)\|^2\\ &\le \langle y_n - p, (z_n - r_nAz_n) - (p - r_nAp)\rangle\\ &= \frac{1}{2}\big(\|y_n - p\|^2 + \|(z_n - r_nAz_n) - (p - r_nAp)\|^2\\ &\quad - \|(y_n - p) - \big((z_n - r_nAz_n) - (p - r_nAp)\big)\|^2\big)\\ &\le \frac{1}{2}\big(\|y_n - p\|^2 + \|z_n - p\|^2 - \|y_n - z_n + r_n(Az_n - Ap)\|^2\big)\\ &= \frac{1}{2}\big(\|y_n - p\|^2 + \|z_n - p\|^2 - \|y_n - z_n\|^2 - r_n^2\|Az_n - Ap\|^2\\ &\quad - 2r_n\langle y_n - z_n, Az_n - Ap\rangle\big)\\ &\le \frac{1}{2}\big(\|y_n - p\|^2 + \|x_n - p\|^2 - \|y_n - z_n\|^2 + 2r_n\|y_n - z_n\|\|Az_n - Ap\|\big),\end{aligned}$$

which in turn implies that

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|y_n - z_n\|^2 + 2r_n\|y_n - z_n\|\|Az_n - Ap\|. \tag{2.4}$$

In a similar way, we can obtain

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|z_n - x_n\|^2 + 2s_n\|z_n - x_n\|\|Bx_n - Bp\|. \tag{2.5}$$

Combining (2.1) with (2.4) yields that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|S_ny_n - p\|^2\\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n\|y_n - p\|^2\\ &\le k_n\|x_n - p\|^2 - (1-\alpha_n)k_n\|y_n - z_n\|^2\\ &\quad + 2(1-\alpha_n)k_nr_n\|y_n - z_n\|\|Az_n - Ap\|,\end{aligned}$$

which in turn implies that

$$\begin{aligned}(1-\alpha_n)k_n\|y_n - z_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (k_n - 1)\|x_n - p\|^2\\ &\quad + 2(1-\alpha_n)k_nr_n\|y_n - z_n\|\|Az_n - Ap\|.\end{aligned}$$

In view of the restrictions (a) and (c), it follows from (2.2) that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{2.6}$$

Combining (2.1) with (2.5) yields that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|S_ny_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)k_n\|y_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)k_n\|z_n-p\|^2\\ &\le k_n\|x_n-p\|^2-(1-\alpha_n)k_n\|z_n-x_n\|^2\\ &\quad+2(1-\alpha_n)k_ns_n\|z_n-x_n\|\|Bx_n-Bp\|,\end{aligned}$$

which in turn implies that

$$\begin{aligned}(1-\alpha_n)k_n\|z_n-x_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(k_n-1)\|x_n-p\|^2\\ &\quad+2(1-\alpha_n)k_ns_n\|z_n-x_n\|\|Bx_n-Bp\|.\end{aligned}$$

In view of the restrictions (a) and (c), it follows from (2.3) that

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{2.7}$$

Combining (2.6) with (2.7) yields that

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{2.8}$$

Notice that

$$\|S_ny_n-p\|\le\sqrt{k_n}\|y_n-p\|\le\sqrt{k_n}\|x_n-p\|.$$

This implies that

$$\limsup_{n\to\infty}\|S_ny_n-p\|\le d.$$

On the other hand, we have

$$\lim_{n\to\infty}\|\alpha_n(x_n-p)+(1-\alpha_n)(S_ny_n-p)\|=d.$$

In view of Lemma 1.3, we obtain

$$\lim_{n\to\infty}\|S_ny_n-x_n\|=0. \tag{2.9}$$

Note that

$$S^ny_n-x_n=\frac{S_ny_n-x_n}{1-\beta_n}+\frac{\beta_n(x_n-y_n)}{1-\beta_n}.$$

From (2.8), (2.9) and the restriction (b), it follows that

$$\lim_{n\to\infty}\|S^ny_n-x_n\|=0. \tag{2.10}$$

On the other hand, it follows from Lemma 1.2 that

$$\begin{aligned}\|S^nx_n-x_n\| &\le \|S^nx_n-S^ny_n\|+\|S^ny_n-x_n\|\\ &\le L\|x_n-y_n\|+\|S^ny_n-x_n\|,\end{aligned}$$

where L denotes the Lipschitz constant, and so, from (2.8) and (2.10),

$$\lim_{n\to\infty}\|S^nx_n-x_n\|=0.$$

Since S is Lipschitz continuous, we see that

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0. \tag{2.11}$$

Since $\{x_n\}$ is bounded, we see that there exits a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a point $\bar{x}$. By virtue of Lemma 1.2, it follows that $\bar{x} \in F(S)$.

Next, we show that $\bar{x} \in (A+M)^{-1}(0)$. In fact, notice that

$$z_n - r_n A z_n \in y_n + r_n M y_n.$$

Let $\mu \in M\nu$. Since M is monotone, we have

$$\langle \frac{z_n - y_n}{r_n} - Az_n - \mu, y_n - \nu \rangle \geq 0.$$

In view of the restriction (c), it follows from (2.6) that

$$\langle -A\bar{x} - \mu, \bar{x} - \nu \rangle \geq 0.$$

This implies that $-A\bar{x} \in M\bar{x}$, that is, $\bar{x} \in (A+M)^{-1}(0)$.

In similar way, we can obtain that $\bar{x} \in (B+W)^{-1}(0)$. This proves that $\bar{x} \in \mathcal{F}$.

Assume that there exits another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to a point x'. By the above proof, we also have $x' \in \mathcal{F}$.

If $\bar{x} \neq x'$, it follows from Opial's condition ([15]) that

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \bar{x}\| &= \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\| < \liminf_{i\to\infty} \|x_{n_i} - x'\| \\
&= \lim_{n\to\infty} \|x_n - x'\| = \liminf_{j\to\infty} \|x_{n_j} - x'\| \\
&< \liminf_{j\to\infty} \|x_{n_j} - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\|,
\end{aligned}$$

which is a contradiction. Hence we have $\bar{x} = x'$. This implies that $x_n \rightharpoonup \bar{x} \in \mathcal{F}$. This completes the proof. □

If S is asymptotically nonexpansive in Theorem 2.1, then we have the following:

Corollary 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, $S : C \to C$ be an asymptotically nonexpansive mapping with the sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $A : C \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be a β-inverse strongly monotone mapping. Let $M : H \to 2^H$ and $W : H \to 2^H$ be maximal monotone operators such that $D(M) \subset C$ and $D(W) \subset C$, respectively. Assume that $\mathcal{F} := F(S) \cap (A+M)^{-1}(0) \cap (B+W)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ z_n = J_{s_n}(x_n - s_n B x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S^n J_{r_n}(z_n - r_n A z_n), \quad \forall n \geq 0, \end{cases}$$

where $J_{r_n} = (I + r_n M)^{-1}$, $J_{s_n} = (I + r_n W)^{-1}$, $\{r_n\}$ is a sequence in $(0, 2\alpha)$, $\{s_n\}$ is a sequence in $(0, 2\beta)$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Assume that the following restrictions are satisfied:

(a) $0 < a \leq \alpha_n \leq b < 1$, *where $a, b \in \mathbb{R}$ are two constants;*
(b) $d \leq r_n \leq e$ *and* $f \leq s_n \leq g$, *where $0 < d < e < 2\alpha$ and $0 < f < g < 2\beta$ are four constants.*

Then the sequence $\{x_n\}$ converges weakly to a point $\bar{x} \in \mathcal{F}$.

3. Applications

Let H be a Hilbert space and $f : H \to (-\infty, +\infty]$ be a proper convex lower semi-continuous function. Then the *subdifferential* ∂f of f is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z - x, y\rangle, \ \forall z \in H\}, \quad \forall x \in H.$$

From Rockafellar [22, 23], we know that ∂f is maximal monotone. It is easy to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y \in H} f(y)$. Let I_C be the indicator function of C, i.e.,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{3.1}$$

Since I_C is a proper lower semi-continuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator.

Lemma 3.1 ([27]). *Let C be a nonempty closed convex subset of a real Hilbert space H, P_C be the metric projection from H onto C and ∂I_C be the subdifferential of I_C, where I_C is as defined in* (3.1) *and $J_r = (I + r\partial I_C)^{-1}$. Then*

$$y = J_r x \iff y = P_C x, \quad \forall x \in H, y \in C.$$

Now, we consider the existence of solutions of the variation inequality (1.1).

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, $S : C \to C$ be an asymptotically κ-strict pseudo-contraction with the sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $A : C \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be a β-inverse strongly monotone mapping. Assume that $\mathcal{F} := F(S) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ z_n = P_C(x_n - s_n B x_n), \\ y_n = P_C(z_n - r_n A z_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(\beta_n y_n + (1 - \beta_n) S^n y_n\big), \quad \forall n \geq 0, \end{cases}$$

where $\{r_n\}$ is a sequence in $(0, 2\alpha)$, $\{s_n\}$ is a sequence in $(0, 2\beta)$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Assume that the following restrictions are satisfied:

(a) *$0 < a \leq \alpha_n \leq b < 1$, where $a, b \in \mathbb{R}$ are two constants;*
(b) *$0 \leq \kappa \leq \beta_n \leq c < 1$, where $c \in \mathbb{R}$ is a constant;*
(c) *$d \leq r_n \leq e$ and $f \leq s_n \leq g$, where $0 < d < e < 2\alpha$ and $0 < f < g < 2\beta$ are four constants.*

Then the sequence $\{x_n\}$ converges weakly to a point $\bar{x} \in \mathcal{F}$.

Proof. Put $M = W = \partial I_C$. Next, we show that $VI(C, A) = (A + \partial I_C)^{-1}(0)$ and $VI(C, B) = (B + \partial I_C)^{-1}(0)$, respectively. Notice that

$$\begin{aligned} x \in (A + \partial I_C)^{-1}(0) &\iff 0 \in Ax + \partial I_C x \\ &\iff -Ax \in \partial I_C x \\ &\iff \langle Ax, y - x\rangle \geq 0 \\ &\iff x \in VI(C, A). \end{aligned}$$

In the same way, we can obtain

$$x \in (B + \partial I_C)^{-1}(0) \Longleftrightarrow x \in VI(C, B).$$

From Lemma 3.1, we can conclude the desired conclusion immediately. This completes the proof. □

Putting $\beta_n = 0$ and $B = 0$ (: the zero mapping) in Theorem 3.2, we have the following:

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, $S : C \to C$ be an asymptotically nonexpansive mapping with the sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $A : C \to H$ be an α-inverse strongly monotone mapping. Assume that $\mathcal{F} := F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated in the following manner:*

$$x_0 \in C,\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S^n P_C(x_n - r_n A x_n), \quad \forall n \geq 0,$$

where $\{r_n\}$ is a sequence in $(0, 2\alpha)$ and $\{\alpha_n\}$ is a sequences in $[0, 1]$. Assume that the following restrictions are satisfied

(a) $0 < a \leq \alpha_n \leq b < 1$, *where $a, b \in \mathbb{R}$ are two constants;*
(b) $d \leq r_n \leq e$, *where $0 < d < e < 2\alpha$ are two constants.*

Then the sequence $\{x_n\}$ converges weakly to a point $\bar{x} \in \mathcal{F}$.

Remark 3.4. If the mapping S is nonexpansive in Corollary 3.3, then Corollary 3.3 is reduced to Theorem 1.1 in Section 1.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{3.2}$$

Forward, $EP(F)$ denotes the solution set of the equilibrium problem (3.2).

To study the equilibrium problems (3.2), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Putting $F(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$, we see that the equilibrium problem (3.2) is reduced to the variational inequality (1.1).

The following lemma can be found in [4] and [6]:

Lemma 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}, \quad \forall r > 0, x \in H. \tag{3.3}$$

Then we have the following:

(a) T_r *is single-valued;*
(b) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) $F(T_r) = EP(F)$;
(d) $EP(F)$ *is closed and convex.*

Lemma 3.6 ([27]). *Let C be a nonempty closed convex subset of a real Hilbert space H, F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4) and A_F be a set-valued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x,y) \geq \langle y - x, z\rangle,\ \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{3.4}$$

Then A_F is a maximal monotone operator with the domain $D(A_F) \subset C$, $EP(F) = A_F^{-1}(0)$ and

$$T_r x = (I + rA_F)^{-1}x, \quad \forall x \in H, r > 0,$$

where T_r is defined as in (3.3)

Theorem 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H, $S : C \to C$ be an asymptotically κ-strict pseudo-contraction with the sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let F_M and F_W be two bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Assume that $\mathcal{F} := F(S) \cap EP(F_M) \cap EP(F_M) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ z_n \in C \text{ such that } F_W(z_n, u) + \frac{1}{s_n}\langle v - z_n, z_n - x_n\rangle \geq 0, \quad \forall v \in C, \\ y_n \in C \text{ such that } F_M(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - z_n\rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(\beta_n y_n + (1 - \beta_n)S^n y_n\big), \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Assume that the following restrictions are satisfied:

(a) $0 < a \leq \alpha_n \leq b < 1$, *where $a, b \in \mathbb{R}$ are two constants;*
(b) $0 \leq \kappa \leq \beta_n \leq c < 1$, *where $c \in \mathbb{R}$ is a constant;*
(c) $0 < d \leq r_n \leq e < \infty$ *and* $0 < f \leq s_n \leq g < \infty$, *where $d, e, f, g \in \mathbb{R}$ are four constants.*

Then the sequence $\{x_n\}$ converges weakly to a point $\bar{x} \in \mathcal{F}$.

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