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ON A HYERS-ULAM-AOKI-RASSIAS TYPE STABILITY AND A FIXED POINT THEOREM

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Abstract. Let X be a set with a binary operation $\circ$ and (Y, d) a complete metric space with a binary operation $\diamond$. Take a nonnegative function ε on $X \times X$, a nonnegative function δ on X and two mappings $f, g : X \to Y$. With the aid of Banach's fixed point theorem, we establish two general settings on which the following holds: If $d\big(f(x \circ x'), g(x) \diamond g(x')\big) \le \varepsilon(x, x')$ and $d\big(f(x), g(x)\big) \le \delta(x)$ for all $x, x' \in X$, then there exists a unique mapping $f_\infty : X \to Y$ such that $f_\infty(x \circ x') = f_\infty(x) \diamond f_\infty(x')$, $d\big(f(x), f_\infty(x)\big) \le A\,\varepsilon(x, x) + B\,\delta(x)$ and $d\big(g(x), f_\infty(x)\big) \le A\,\varepsilon(x, x) + C\,\delta(x)$ for all $x, x' \in X$ and some finite constants A, B and C. Moreover, we describe various concrete settings to which the above results are applicable. Some of them are the known results.

1. Introduction

It is natural to ask the following stability question:

(1.1) Given an approximate solution, can we find the strict solution near it?

This paper is motivated by this question.

In 1940, S. M. Ulam posed the following problem (cf. [16, 17]):

> For what metric group G, is it true that for any approximate automorphism f of G, there exists a strict automorphism of G near f?

Next year, D. H. Hyers [8] gave an affirmative answer to this problem as follows:

> Let X and Y be Banach spaces and $\varepsilon > 0$. Then for any (approximately additive) mapping $f : X \to Y$ satisfying $\|f(x + x') - f(x) - f(x')\| \le \varepsilon$ for all $x, x' \in X$, there exists a unique additive mapping $f_\infty : X \to Y$ such that $\|f(x) - f_\infty(x)\| \le \varepsilon$ for all $x \in X$.

In fact, Hyers got the solution f_∞ by putting $f_\infty(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ $(x \in X)$. Some years later, T. Aoki [2] and Th. M. Rassias [15] independently generalize Hyers' result as follows (cf. [11, 12]):

> Let X and Y be Banach spaces, $\lambda > 0$ and $0 \le p < 1$. Then for any mapping $f : X \to Y$ satisfying $\|f(x+x') - f(x) - f(x')\| \le \lambda\left(\|x\|^p + \|x'\|^p\right)$ for all $x, x' \in X$, there exists a unique additive mapping $f_\infty : X \to Y$ such that $\|f(x) - f_\infty(x)\| \le \left[\lambda/(1 - 2^{p-1})\right] \|x\|^p$ for all $x \in X$.

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Behind their proof, we can find Picard's method of successive approximation, and we learn the close relation between a perturbation and a fixed point. This point of view may be found in many recent paper [1, 3, 9, 10, 13]. For instance, Kim, Jun and Rassias [10] used the Diaz-Margolis fixed point theorem [4] to prove the Hyers-Ulam-Rassias stability of the Euler-Lagrange functional equation $f(ax+by)+f(ax-by)+2af(-x)=0$.

When we study mathematics, we often encounter the concept "*commutativity*", which is fundamental and important. For example, the additive mapping in Hyers' result [8] is regarded as the mapping which *commutes* the additive operation. Also, the functional equation $f(ax+by)+f(ax-by)+2af(-x)=0$ in [10] means the *commutativity* of the mapping f and some binary operation. As a generalization of such commutativity, we take up the following two commutative diagrams:

$$\text{(A)}: \quad \begin{array}{ccc} X\times X & \overset{\circ}{\rightarrow} & X \\ {\scriptstyle g\times g}\downarrow & & \downarrow{\scriptstyle f} \\ Y\times Y & \underset{\diamond}{\rightarrow} & Y \end{array} \qquad\qquad \text{(B)}: \quad \begin{array}{ccc} X & \overset{\sigma}{\rightarrow} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \underset{\tau}{\rightarrow} & Y \end{array}$$

If Y is a metric space, then we can define an approximate commutative diagram, and we can reformulate the problem (1.1) as follows:

> Can we find a strict commutative diagram near a given approximate commutative diagram?

The referred results in [8] and [10] are, of course, related with this problem for the commutative diagram of the form (A). While G.-L. Forti [5, 6] investigates such problem. In this paper, we also consider the above problem.

In Section 1, we show one consequence of Banach's fixed point theorem. Using it, we give the affirmative answer to the above stability problem on several settings. Section 2 is the main part of this paper, where we prove the Hyers-Ulam-Aoki-Rassias type stability for the commutative diagram of the form (A). The rest of the paper consists of its applications. Section 3 is about the commutative diagram of the form (B). In Sections 4–6, we describe various concrete settings to which the results in Section 2 are applicable. Some of them are the known results.

2. One consequence of Banach's fixed point theorem

Let X be a set and (Y,d) a complete metric space. Fix a mapping f of X into Y and a nonnegative function φ on X. By $\Delta_{f,\varphi}$, we denote the set of all mappings $u: X\to Y$ with the property that there exists a finite constant K_u satisfying

$$d\big(u(x),f(x)\big)\le K_u\,\varphi(x) \qquad (x\in X).$$

For any $u,v\in\Delta_{f,\varphi}$, we have

$$d\big(u(x),v(x)\big)\le d\big(u(x),f(x)\big)+d\big(f(x),v(x)\big)\le (K_u+K_v)\,\varphi(x) \quad (x\in X)$$

and hence we can define the distance

$$\rho_{f,\varphi}(u,v)=\inf\big\{K\ge 0: d\big(u(x),v(x)\big)\le K\varphi(x)\ (x\in X)\big\}.$$

We easily see that $(\Delta_{f,\varphi},\rho_{f,\varphi})$ is a complete metric space which contains f.

Let σ be a selfmap of X and τ a selfmap of Y. For any mapping $u : X \to Y$, we define the mapping $T_{\sigma,\tau}u : X \to Y$ by

$$(T_{\sigma,\tau}u)(x) = \tau(u(\sigma x)) \qquad (x \in X).$$

Moreover, if X, (Y, d), φ and σ, τ are as above, and if ε is a nonnegative function on $X \times X$, then we can consider three quantities:

$$\begin{aligned}\alpha_{\sigma,\varepsilon} &= \inf\bigl\{\, K \geq 0 \,:\, \varepsilon(\sigma x, \sigma x) \leq K\, \varepsilon(x, x)\ (x \in X) \,\bigr\},\\ \beta_{\sigma,\varphi} &= \inf\bigl\{\, K \geq 0 \,:\, \varphi(\sigma x) \leq K\, \varphi(x)\ (x \in X) \,\bigr\},\\ \gamma_\tau &= \inf\bigl\{\, K \geq 0 \,:\, d(\tau y, \tau y') \leq K\, d(y, y')\ (y, y' \in Y) \,\bigr\}.\end{aligned}$$

If one of these quantities is determined as a nonnegative real number, then we write $\alpha_{\sigma,\varepsilon} < \infty$, $\beta_{\sigma,\varphi} < \infty$ and $\gamma_\tau < \infty$ respectively. In each case, we have

$$\begin{aligned}\varepsilon(\sigma x, \sigma x) &\leq \alpha_{\sigma,\varepsilon}\, \varepsilon(x, x) && (x \in X),\\ \varphi(\sigma x) &\leq \beta_{\sigma,\varphi}\, \varphi(x) && (x \in X),\\ d(\tau y, \tau y') &\leq \gamma_\tau\, d(y, y') && (y, y' \in Y).\end{aligned}$$

We will use these symbols and these inequalities throughout this paper.

We here state our fixed point theorem, which is an easy consequence of Banach's fixed point theorem (the contraction principle):

Proposition 2.1. *Let X be a set and (Y, d) a complete metric space. Take a mapping $f : X \to Y$ and a nonnegative function φ on X. Let σ and τ be selfmaps of X and Y respectively. Suppose that*

$$T_{\sigma,\tau}f \in \Delta_{f,\varphi}, \quad \beta_{\sigma,\varphi} < \infty, \quad \gamma_\tau < \infty \quad \textit{and} \quad \beta_{\sigma,\varphi}\gamma_\tau < 1.$$

Then $T_{\sigma,\tau}(\Delta_{f,\varphi}) \subset \Delta_{f,\varphi}$ and $T_{\sigma,\tau}$ has a unique fixed point f_∞ in $\Delta_{f,\varphi}$. Moreover,

$$\lim_{n\to\infty} d\bigl((T^n_{\sigma,\tau}f)(x), f_\infty(x)\bigr) = 0 \quad \textit{and} \quad d\bigl(f(x), f_\infty(x)\bigr) = \frac{\rho_{f,\varphi}(T_{\sigma,\tau}f, f)}{1 - \beta_{\sigma,\varphi}\gamma_\tau}\varphi(x)$$

for all $x \in X$.

Proof. For the sake of simplicity, we write $\Delta = \Delta_{f,\varphi}$, $\rho = \rho_{f,\varphi}$, $T = T_{\sigma,\tau}$, $\beta = \beta_{\sigma,\varphi}$ and $\gamma = \gamma_\tau$.

We first observe that $T(\Delta) \subset \Delta$. Take $u \in \Delta$ arbitrarily. Then $\rho(u, f)$ is definite. Since $\beta, \gamma < \infty$, we have

$$\begin{aligned}d\bigl((Tu)(x), (Tf)(x)\bigr) &= d\bigl(\tau(u(\sigma x)), \tau(f(\sigma x))\bigr) \leq \gamma\, d\bigl(u(\sigma x), f(\sigma x)\bigr)\\ &\leq \gamma\, \rho(u, f)\, \varphi(\sigma x) \leq \gamma\, \rho(u, f)\, \beta\, \varphi(x)\end{aligned}$$

for all $x \in X$. Note that $Tf \in \Delta$ implies $\rho(Tf, f) < \infty$. Then we have

$$\begin{aligned}d\bigl((Tu)(x), f(x)\bigr) &\leq d\bigl((Tu)(x), (Tf)(x)\bigr) + d\bigl((Tf)(x), f(x)\bigr)\\ &\leq \beta\gamma\, \rho(u, f)\varphi(x) + \rho(Tf, f)\varphi(x) = \bigl[\beta\gamma\, \rho(u, f) + \rho(Tf, f)\bigr]\varphi(x).\end{aligned}$$

for all $x \in X$. Hence $Tu \in \Delta$. We get $T(\Delta) \subset \Delta$.

Let us verify that T is a contraction of Δ. For any $u, v \in \Delta$, we have

$$\begin{aligned}d\bigl((Tu)(x), (Tv)(x)\bigr) &= d\bigl(\tau(u(\sigma x)), \tau(v(\sigma x))\bigr) \leq \gamma\, d(u(\sigma x), v(\sigma x))\\ &\leq \gamma\, \rho(u, v)\varphi(\sigma x) \leq \gamma\, \rho(u, v)\, \beta\, \varphi(x) \qquad (x \in X),\end{aligned}$$

and so

$$\rho(Tu, Tv) \leq \beta\gamma\, \rho(u, v). \tag{2.1}$$

Hence the hypothesis $\beta\gamma < 1$ implies that T is a contraction of Δ.

Once T is a contraction of a complete metric space (Δ, ρ), Banach's fixed point theorem tells us that T has a unique fixed point f_∞ in Δ. At the same time, it shows that $\lim_{n\to\infty} \rho(T^n f, f_\infty) = 0$. Noting that $d\big((T^n f)(x), f_\infty(x)\big) \leq \rho(T^n f, f_\infty)\, \varphi(x)$, we get

$$\lim_{n\to\infty} d\big((T^n f)(x), f_\infty(x)\big) = 0 \qquad (x \in X).$$

Moreover, the repeated use of (2.1) yields

$$\rho(T^k f, T^{k-1} f) \leq (\beta\gamma)^{k-1}\, \rho(Tf, f)$$

for $k = 1, 2, \ldots$. Noting that $\beta\gamma < 1$, we have

$$\rho(T^n f, f) \leq \sum_{k=1}^{n} \rho(T^k f, T^{k-1} f) \leq \sum_{k=1}^{n} (\beta\gamma)^{k-1} \rho(Tf, f) < \frac{\rho(Tf, f)}{1 - \beta\gamma}.$$

Since $\lim_{n\to\infty} \rho(T^n f, f_\infty) = 0$, the continuity of the metric ρ shows that $\rho(f_\infty, f) \leq \rho(Tf, f)/(1 - \beta\gamma)$, and hence

$$d\big(f(x), f_\infty(x)\big) \leq \rho(f, f_\infty)\, \varphi(x) \leq \frac{\rho(Tf, f)}{1 - \beta\gamma} \varphi(x) \qquad (x \in X).$$

□

In many papers on the Hyers-Ulam stability problem, we often find the applications of the fixed point theorem by J. B. Diaz and B. Margolis ([4]). We note that the above proposition is based on Banach's fixed point theorem only.

3. A Stability problem for commutative diagram (A)

In this section, we establish two general settings, on which we can show the Hyers-Ulam-Aoki-Rassias type stability for the commutative diagram (A). These settings are joined in a property such as duality, and each of them works as a complement of the other to be applicable to many cases. The first setting is described in the following theorem:

Theorem 3.1. *Let X be a set with a binary operation $\circ$, and suppose that the square operator $\hat{\sigma} : x \mapsto x \circ x$ is an automorphism of X with the inverse $\hat{\sigma}^{-1}$. Let (Y, d) be a complete metric space with a continuous binary operation $\diamond$, and suppose that the square operator $\hat{\tau} : y \mapsto y \diamond y$ is an endomorphism of Y. Fix a nonnegative function ε on $X \times X$ and a nonnegative function δ on X. Suppose that*

$$\alpha_{\hat{\sigma}^{-1},\varepsilon} < \infty, \quad \beta_{\hat{\sigma}^{-1},\delta} < \infty, \quad \gamma_{\hat{\tau}} < \infty \quad \text{and} \quad \gamma_{\hat{\tau}} \max\{\alpha_{\hat{\sigma}^{-1},\varepsilon}, \beta_{\hat{\sigma}^{-1},\delta}\} < 1.$$

If two mappings $f, g : X \to Y$ satisfy

$$d\big(f(x \circ x'),\, g(x) \diamond g(x')\big) \leq \varepsilon(x, x') \qquad (x, x' \in X), \tag{3.1}$$

$$d\big(f(x), g(x)\big) \leq \delta(x) \qquad (x \in X), \tag{3.2}$$

then there exists a unique mapping $f_\infty : X \to Y$ *such that*

$$f_\infty(x \circ x') = f_\infty(x) \diamond f_\infty(x') \qquad (x, x' \in X), \tag{3.3}$$

$$d\big(f(x), f_\infty(x)\big) \le \frac{\alpha_{\hat\sigma^{-1},\varepsilon}\,\varepsilon(x,x) + \beta_{\hat\sigma^{-1},\delta}\gamma_{\hat\tau}\,\delta(x)}{1 - \gamma_{\hat\tau}\max\{\alpha_{\hat\sigma^{-1},\varepsilon}, \beta_{\hat\sigma^{-1},\delta}\}} \qquad (x \in X), \tag{3.4}$$

$$d\big(g(x), f_\infty(x)\big) \le \frac{\alpha_{\hat\sigma^{-1},\varepsilon}\,\varepsilon(x,x) + \delta(x)}{1 - \gamma_{\hat\tau}\max\{\alpha_{\hat\sigma^{-1},\varepsilon}, \beta_{\hat\sigma^{-1},\delta}\}} \qquad (x \in X). \tag{3.5}$$

Proof. For simplicity, we write $\sigma = \hat\sigma^{-1}$ and $\tau = \hat\tau$. It is obvious that σ is an automorphism of X. We adopt the abbreviations: $\alpha = \alpha_{\sigma,\varepsilon} = \alpha_{\hat\sigma^{-1},\varepsilon}$, $\beta = \beta_{\sigma,\delta} = \beta_{\hat\sigma^{-1},\delta}$, $\gamma = \gamma_\tau = \gamma_{\hat\tau}$ and $T = T_{\sigma,\tau}$. We break the proof into five steps.

[Step 1] Put $\varphi(x) = \alpha\,\varepsilon(x,x) + \beta\gamma\,\delta(x)$ for all $x \in X$. To f and φ, we apply the argument in Section 1. We first observe that $Tf \in \Delta_{f,\varphi}$. Take $x \in X$. Replacing x and x' in (3.1) by σx, we get

$$d\big(f(\sigma x \circ \sigma x),\ g(\sigma x) \diamond g(\sigma x)\big) \le \varepsilon(\sigma x, \sigma x).$$

Since $\sigma x \circ \sigma x = \hat\sigma(\sigma x) = x$, $g(\sigma x) \diamond g(\sigma x) = \tau(g(\sigma x)) = (Tg)(x)$ and $\varepsilon(\sigma x, \sigma x) \le \alpha\,\varepsilon(x,x)$, it follows that

$$d\big(f(x), (Tg)(x)\big) \le \alpha\,\varepsilon(x,x). \tag{3.6}$$

While we make use of (3.2) to see

$$\begin{aligned} d\big((Tf)(x), (Tg)(x)\big) &= d\big(\tau(f(\sigma x)), \tau(g(\sigma x)\big) \\ &\le \gamma\, d\big(f(\sigma x), g(\sigma x)\big) \le \gamma\,\delta(\sigma x) \le \gamma\beta\,\delta(x). \end{aligned} \tag{3.7}$$

Combining these inequalities, we have

$$\begin{aligned} d\big((Tf)(x), f(x)\big) &\le d\big((Tf)(x), (Tg)(x)\big) + d\big((Tg)(x), f(x)\big) \\ &\le \beta\gamma\,\delta(x) + \alpha\,\varepsilon(x,x) \\ &= \varphi(x). \end{aligned}$$

Hence $Tf \in \Delta_{f,\varphi}$ and $\rho_{f,\varphi}(Tf, f) \le 1$.

Next, we estimate the quantity $\beta_{\sigma,\varphi}$. For any $x \in X$, we have

$$\begin{aligned} \varphi(\sigma x) &= \alpha\,\varepsilon(\sigma x, \sigma x) + \beta\gamma\,\delta(\sigma x) \le \alpha^2\varepsilon(x,x) + \beta^2\gamma\,\delta(x) \\ &\le \max\{\alpha,\beta\}\,\big(\alpha\,\varepsilon(x,x) + \beta\gamma\,\delta(x)\big) = \max\{\alpha,\beta\}\,\varphi(x). \end{aligned}$$

Hence $\beta_{\sigma,\varphi} \le \max\{\alpha,\beta\} < \infty$ and $\beta_{\sigma,\varphi}\gamma_\tau \le \gamma\max\{\alpha,\beta\} < 1$.

Now, let us apply Proposition 2.1. Then T has a unique fixed point $f_\infty \in \Delta_{f,\varphi}$ and

$$\lim_{n\to\infty} d\big((T^n f)(x), f_\infty(x)\big) = 0, \tag{3.8}$$

$$d\big(f(x), f_\infty(x)\big) \le \frac{\rho_{f,\varphi}(Tf,f)}{1 - \beta_{\sigma,\varphi}\gamma_\tau}\varphi(x) \le \frac{\alpha\,\varepsilon(x,x) + \beta\gamma\,\delta(x)}{1 - \gamma\max\{\alpha,\beta\}}$$

for all $x \in X$. The last inequality is (3.4).

[Step 2] We put $\psi(x) = \alpha\,\varepsilon(x,x) + \delta(x)$ for all $x \in X$. Let us discuss g and ψ similarly. By (3.6) and (3.2), we obtain

$$d\big((Tg)(x), g(x)\big) \le d\big((Tg)(x), f(x)\big) + d\big(f(x), g(x)\big) \le \alpha\,\varepsilon(x,x) + \delta(x) = \psi(x)$$

for all $x \in X$. Hence $Tg \in \Delta_{g,\psi}$ and $\rho_{g,\psi}(Tg, g) \leq 1$. Also, we have

$$\begin{aligned}\psi(\sigma x) &= \alpha\,\varepsilon(\sigma x, \sigma x) + \delta(\sigma x) \leq \alpha^2\varepsilon(x, x) + \beta\,\delta(x)\\ &\leq \max\{\alpha, \beta\}\,\big(\alpha\,\varepsilon(x, x) + \delta(x)\big) = \max\{\alpha, \beta\}\,\psi(x)\end{aligned}$$

for all $x \in X$. Hence $\beta_{\sigma,\psi} \leq \max\{\alpha, \beta\} < \infty$ and $\beta_{\sigma,\psi}\gamma_\tau \leq \gamma \max\{\alpha, \beta\} < 1$. Thus we can apply Proposition 2.1 and see that T has a unique fixed point $g_\infty \in \Delta_{g,\psi}$ and

$$\lim_{n\to\infty} d\big((T^n g)(x), g_\infty(x)\big) = 0, \tag{3.9}$$

$$d\big(g(x), g_\infty(x)\big) \leq \frac{\rho_{g,\psi}(Tg, g)}{1 - \beta_{\sigma,\psi}\gamma_\tau}\psi(x) \leq \frac{\alpha\,\varepsilon(x, x) + \delta(x)}{1 - \gamma \max\{\alpha, \beta\}} \tag{3.10}$$

for all $x \in X$.

[Step 3] Let us show that

$$f_\infty(x \circ x') = g_\infty(x) \diamond g_\infty(x') \qquad (x, x' \in X). \tag{3.11}$$

For any $x, x' \in X$, we have

$$\begin{aligned}d\big(f_\infty(x \circ x'),\ g_\infty(x) \diamond g_\infty(x')\big) \leq{}& d\big(f_\infty(x \circ x'),\ (T^n f)(x \circ x')\big)\\ &+ d\big((T^n f)(x \circ x'),\ (T^n g)(x) \diamond (T^n g)(x')\big)\\ &+ d\big((T^n g)(x) \diamond (T^n g)(x'),\ g_\infty(x) \diamond g_\infty(x')\big).\end{aligned}$$

Let $n \to \infty$. Then the first and third terms in the right hand side tend to 0, by (3.8), (3.9) and the continuity of $\diamond$. Also, the second term tends to 0, because $\gamma\alpha < 1$ and (3.1) yields

$$\begin{aligned}&d\big((T^n f)(x \circ x'),\ (T^n g)(x) \diamond (T^n g)(x')\big)\\ &\quad= d\big(\tau^n(f(\sigma^n(x \circ x'))),\ \tau^n(g(\sigma^n x)) \diamond \tau^n(g(\sigma^n x'))\big)\\ &\quad= d\big(\tau^n(f(\sigma^n x \circ \sigma^n x')),\ \tau^n(g(\sigma^n x) \diamond g(\sigma^n x'))\big)\\ &\quad\leq \gamma^n\, d\big(f(\sigma^n x \circ \sigma^n x'),\ g(\sigma^n x) \diamond g(\sigma^n x')\big)\\ &\quad\leq \gamma^n\, \varepsilon(\sigma^n x, \sigma^n x')\\ &\quad\leq \gamma^n \alpha^n\, \varepsilon(x, x'),\end{aligned}$$

where τ^n and σ^n denote the n-fold compositions of endomorphisms τ and σ, respectively. These facts force that $d\big(f_\infty(x \circ x'),\ g_\infty(x) \diamond g_\infty(x')\big) = 0$, which is nothing but (3.11).

[Step 4] Pick $x \in X$. Replacing x and x' in (3.11) by σx, we have

$$f_\infty(\sigma x \circ \sigma x) = g_\infty(\sigma x) \diamond g_\infty(\sigma x).$$

Since $\sigma x \circ \sigma x = \hat{\sigma}(\sigma x) = x$ and $g_\infty(\sigma x) \diamond g_\infty(\sigma x) = \tau(g_\infty(\sigma x)) = (Tg_\infty)(x) = g_\infty(x)$, it follows that $f_\infty(x) = g_\infty(x)$. Hence (3.11) and (3.10) become (3.3) and (3.5), respectively.

[Step 5] Finally we check the uniqueness of f_∞. Suppose that the mapping $f_* : X \to Y$ has the same properties (3.3)–(3.5) as f_∞. The property (3.4) implies that

$$d\big(f_*(x), f(x)\big) \leq \frac{\alpha\,\varepsilon(x,x) + \beta\gamma\,\delta(x)}{1-\gamma\max\{\alpha,\beta\}} = \frac{1}{1-\gamma\max\{\alpha,\beta\}}\,\varphi(x) \qquad (x \in X),$$

and so $f_* \in \Delta_{f,\varphi}$. By the property (3.3): $f_*(x \circ x') = f_*(x) \diamond f_*(x')$ $(x, x' \in X)$, we get

$$(Tf_*)(x) = \tau(f_*(\sigma x)) = f_*(\sigma x) \diamond f_*(\sigma x) = f_*(\sigma x \circ \sigma x) = f_*(\hat{\sigma}(\sigma x)) = f_*(x)$$

for all $x \in X$. Hence f_* is a fixed point of T in $\Delta_{f,\varphi}$. Thus Step 1 shows that $f_* = f_\infty$. □

Consider the case $\delta = 0$ or the case $\varepsilon = 0$ in Theorem 3.1. Then we obtain the following two corollaries immediately:

Corollary 3.2. *Let X, $\circ$, $\hat{\sigma}$ and (Y,d), $\diamond$, $\hat{\tau}$ be as in Theorem 3.1. Take a nonnegative function ε on $X \times X$. Suppose that $\alpha_{\hat{\sigma}^{-1},\varepsilon} < \infty$, $\gamma_{\hat{\tau}} < \infty$ and $\alpha_{\hat{\sigma}^{-1},\varepsilon}\gamma_{\hat{\tau}} < 1$. If a mapping $f : X \to Y$ satisfies $d\big(f(x \circ x'),\, f(x) \diamond f(x')\big) \leq \varepsilon(x,x')$ for all $x, x' \in X$, then there exists a unique mapping $f_\infty : X \to Y$ such that*

$$f_\infty(x \circ x') = f_\infty(x) \diamond f_\infty(x') \quad \text{and} \quad d\big(f(x), f_\infty(x)\big) \leq \frac{\alpha_{\hat{\sigma}^{-1},\varepsilon}}{1-\alpha_{\hat{\sigma}^{-1},\varepsilon}\gamma_{\hat{\tau}}}\varepsilon(x,x)$$

for all $x, x' \in X$.

Corollary 3.3. *Let X, $\circ$, $\hat{\sigma}$ and (Y,d), $\diamond$, $\hat{\tau}$ be as in Theorem 3.1. Take a nonnegative function δ on X. Suppose that $\beta_{\hat{\sigma}^{-1},\delta} < \infty$, $\gamma_{\hat{\tau}} < \infty$ and $\beta_{\hat{\sigma}^{-1},\delta}\gamma_{\hat{\tau}} < 1$. If two mappings $f, g : X \to Y$ satisfy $f(x \circ x') = g(x) \diamond g(x')$ and $d\big(f(x), g(x)\big) \leq \delta(x)$ for all $x, x' \in X$, then there exists a unique mapping $f_\infty : X \to Y$ such that*

$$f_\infty(x \circ x') = f_\infty(x) \diamond f_\infty(x'),$$

$$d\big(f(x), f_\infty(x)\big) \leq \frac{\beta_{\hat{\sigma}^{-1},\delta}\gamma_{\hat{\tau}}}{1-\beta_{\hat{\sigma}^{-1},\delta}\gamma_{\hat{\tau}}}\,\delta(x) \quad \text{and} \quad d\big(g(x), f_\infty(x)\big) \leq \frac{1}{1-\beta_{\hat{\sigma}^{-1},\delta}\gamma_{\hat{\tau}}}\,\delta(x)$$

for all $x, x' \in X$.

The next theorem explains the second setting, on which we can show the Hyers-Ulam-Aoki-Rassias type stability for the commutative diagram (A).

Theorem 3.4. *Let X be a set with a binary operation $\circ$, and suppose that the square operator $\hat{\sigma} : x \mapsto x \circ x$ is an endomorphism of X. Let (Y,d) be a complete metric space with a continuous binary operation $\diamond$, and suppose that the square operator $\hat{\tau} : y \mapsto y \diamond y$ is an automorphisn of Y with the inverse $\hat{\tau}^{-1}$. Fix a nonnegative function ε on $X \times X$ and a nonnegative function δ on X. Suppose that*

$$\alpha_{\hat{\sigma},\varepsilon} < \infty, \quad \beta_{\hat{\sigma},\delta} < \infty, \quad \gamma_{\hat{\tau}^{-1}} < \infty \quad \text{and} \quad \gamma_{\hat{\tau}^{-1}}\max\{\alpha_{\hat{\sigma},\varepsilon}, \beta_{\hat{\sigma},\delta}\} < 1.$$

If two mappings $f, g : X \to Y$ satisfy

$$d\big(f(x \circ x'),\, g(x) \diamond g(x')\big) \leq \varepsilon(x,x') \qquad (x, x' \in X), \tag{3.12}$$

$$d\big(f(x), g(x)\big) \leq \delta(x) \qquad (x \in X), \tag{3.13}$$

then there exists a unique mapping $f_\infty : X \to Y$ *such that*

$$f_\infty(x \circ x') = f_\infty(x) \diamond f_\infty(x') \qquad (x, x' \in X), \tag{3.14}$$

$$d\big(f(x), f_\infty(x)\big) \le \frac{\gamma_{\hat\tau^{-1}}\, \varepsilon(x,x) + \delta(x)}{1 - \gamma_{\hat\tau^{-1}} \max\{\alpha_{\hat\sigma,\varepsilon}, \beta_{\hat\sigma,\delta}\}} \qquad (x \in X), \tag{3.15}$$

$$d\big(g(x), f_\infty(x)\big) \le \frac{\gamma_{\hat\tau^{-1}}\big(\varepsilon(x,x) + \beta_{\hat\sigma,\delta}\, \delta(x)\big)}{1 - \gamma_{\hat\tau^{-1}} \max\{\alpha_{\hat\sigma,\varepsilon}, \beta_{\hat\sigma,\delta}\}} \qquad (x \in X). \tag{3.16}$$

Proof. The proof is similar to that of Theorem 3.1. This time, we put $\sigma = \hat\sigma$ and $\tau = \hat\tau^{-1}$. Also, we adopt the abbreviations: $\alpha = \alpha_{\sigma,\varepsilon} = \alpha_{\hat\sigma,\varepsilon}$, $\beta = \beta_{\sigma,\delta} = \beta_{\hat\sigma,\delta}$, $\gamma = \gamma_\tau = \gamma_{\hat\tau^{-1}}$ and $T = T_{\sigma,\tau}$.

[Step 1] Put $\psi(x) = \gamma\big(\varepsilon(x,x) + \beta\, \delta(x)\big)$ for all $x \in X$. We first observe that $Tg \in \Delta_{g,\psi}$. Take $x \in X$. Since $\tau(g(x) \diamond g(x)) = \tau(\hat\tau(g(x))) = g(x)$, we use (3.12) to see

$$\begin{aligned} d\big((Tf)(x), g(x)\big) &= d\big(\tau(f(\sigma x)),\ g(x)\big) = d\big(\tau(f(x \circ x)),\ \tau(g(x) \diamond g(x))\big) \\ &\le \gamma\ \big(f(x \circ x),\ g(x) \diamond g(x)\big) \le \gamma\, \varepsilon(x,x). \end{aligned} \tag{3.17}$$

While the computation (3.7) using (3.13) implies $d\big((Tf)(x), (Tg)(x)\big) \le \gamma\beta\, \delta(x)$. Hence

$$d\big((Tg)(x), g(x)\big) \le d\big((Tg)(x), (Tf)(x)\big) + d\big((Tf)(x), g(x)\big) \le \gamma\beta\, \delta(x) + \gamma\, \varepsilon(x,x) = \psi(x).$$

Therefore, $Tg \in \Delta_{g,\psi}$ and $\rho_{g,\psi}(Tg, g) \le 1$.

Next, we estimate the quantity $\beta_{\sigma,\psi}$. For any $x \in X$, we have

$$\begin{aligned} \psi(\sigma x) = \gamma\big(\varepsilon(\sigma x, \sigma x) + \beta\, \delta(\sigma x)\big) &\le \gamma\big(\alpha\, \varepsilon(x,x) + \beta^2\, \delta(x)\big) \\ &\le \gamma\ \max\{\alpha, \beta\}\ \big(\varepsilon(x,x) + \beta\, \delta(x)\big) = \max\{\alpha,\beta\}\ \psi(x). \end{aligned}$$

Hence $\beta_{\sigma,\psi} \le \max\{\alpha, \beta\} < \infty$ and $\beta_{\sigma,\psi}\gamma_\tau \le \gamma \max\{\alpha,\beta\} < 1$.

Thus we can apply Proposition 2.1. Consequently, T has a unique fixed point $g_\infty \in \Delta_{g,\psi}$ and

$$\lim_{n\to\infty} d\big((T^n g)(x), g_\infty(x)\big) = 0, \tag{3.18}$$

$$d\big(g(x), g_\infty(x)\big) \le \frac{\rho_{g,\psi}(Tg, g)}{1 - \beta_{\sigma,\psi}\gamma_\tau}\, \psi(x) \le \frac{\gamma\big(\varepsilon(x,x) + \beta\, \delta(x)\big)}{1 - \gamma \max\{\alpha, \beta\}} \tag{3.19}$$

for all $x \in X$.

[Step 2] Put $\varphi(x) = \gamma\, \varepsilon(x,x) + \delta(x)$ for all $x \in X$. By (3.17) and (3.13), we have

$$d\big((Tf)(x), f(x)\big) \le d\big((Tf)(x), g(x)\big) + d\big(g(x), f(x)\big) \le \gamma\, \varepsilon(x,x) + \delta(x) = \varphi(x)$$

for all $x \in X$. Hence $Tf \in \Delta_{f,\varphi}$ and $\rho_{f,\varphi}(Tf, f) \le 1$. Also, we can easily see that $\varphi(\sigma x) \le \max\{\alpha,\beta\}\, \varphi(x)$ for all $x \in X$. Therefore, $\beta_{\sigma,\varphi} \le \max\{\alpha,\beta\} < \infty$ and $\beta_{\sigma,\varphi}\gamma_\tau \le \gamma \max\{\alpha,\beta\} < 1$. Thus we can apply Proposition 2.1 and see that T has a unique fixed point $f_\infty \in \Delta_{f,\varphi}$ and

$$\lim_{n\to\infty} d\big((T^n f)(x), f_\infty(x)\big) = 0, \tag{3.20}$$

$$d\big(f(x), f_\infty(x)\big) \le \frac{\rho_{f,\varphi}(Tf, f)}{1 - \beta_{\sigma,\varphi}\gamma_\tau}\, \varphi(x) \le \frac{\gamma\, \varepsilon(x,x) + \delta(x)}{1 - \gamma \max\{\alpha, \beta\}}$$

for all $x \in X$. The last inequality is (3.15).

[Step 3] Repeating the argument in the same part of the proof of Theorem 3.1, we can show from (3.20), (3.18) and (3.12) that

$$f_\infty(x \circ x') = g_\infty(x) \diamond g_\infty(x') \qquad (x, x' \in X). \tag{3.21}$$

[Step 4] Using (3.21), we see that

$$\begin{aligned} f_\infty(x) &= (Tf_\infty)(x) = \tau(f_\infty(\sigma x)) \\ &= \tau(f_\infty(x \circ x)) = \tau(g_\infty(x) \diamond g_\infty(x)) \\ &= \tau(\hat\tau(g_\infty(x))) = g_\infty(x) \end{aligned}$$

for all $x \in X$. Hence (3.21) and (3.19) become (3.14) and (3.16), respectively.

[Step 5] We can check the uniqueness of f_∞ in the same way as in Step 5 of the proof of Theorem 3.1. □

Considering the case $\delta = 0$ or the case $\varepsilon = 0$ in Theorem 3.4, we obtain the following two corollaries:

Corollary 3.5. *Let X, $\circ$, $\hat\sigma$ and (Y, d), $\diamond$, $\hat\tau$ be as in Theorem 3.4. Take a nonnegative function ε on $X \times X$. Suppose that $\alpha_{\hat\sigma,\varepsilon} < \infty$, $\gamma_{\hat\tau^{-1}} < \infty$ and $\alpha_{\hat\sigma,\varepsilon}\gamma_{\hat\tau^{-1}} < 1$. If a mapping $f : X \to Y$ satisfies $d\big(f(x \circ x'),\, f(x) \diamond f(x')\big) \le \varepsilon(x, x')$ for all $x, x' \in X$, then there exists a unique mapping $f_\infty : X \to Y$ such that*

$$f_\infty(x \circ x') = f_\infty(x) \diamond f_\infty(x') \quad \text{and} \quad d\big(f(x), f_\infty(x)\big) \le \frac{\gamma_{\hat\tau^{-1}}}{1 - \alpha_{\hat\sigma,\varepsilon}\gamma_{\hat\tau^{-1}}}\varepsilon(x, x).$$

for all $x, x' \in X$.

Corollary 3.6. *Let X, $\circ$, $\hat\sigma$ and (Y, d), $\diamond$, $\hat\tau$ be as in Theorem 3.4. Take a nonnegative function δ on X. Suppose that $\beta_{\hat\sigma,\delta} < \infty$, $\gamma_{\hat\tau^{-1}} < \infty$ and $\beta_{\hat\sigma,\delta}\gamma_{\hat\tau^{-1}} < 1$. If two mappings $f, g : X \to Y$ satisfy $f(x \circ x') = g(x) \diamond g(x')$ and $d\big(f(x), g(x)\big) \le \delta(x)$ for all $x, x' \in X$, then there exists a unique mapping $f_\infty : X \to Y$ such that*

$$f_\infty(x \circ x') = f_\infty(x) \diamond f_\infty(x'),$$

$$d\big(f(x), f_\infty(x)\big) \le \frac{1}{1 - \beta_{\hat\sigma,\delta}\gamma_{\hat\tau^{-1}}}\,\delta(x) \quad \text{and} \quad d\big(g(x), f_\infty(x)\big) \le \frac{\beta_{\hat\sigma,\delta}\gamma_{\hat\tau^{-1}}}{1 - \beta_{\hat\sigma,\delta}\gamma_{\hat\tau^{-1}}}\,\delta(x)$$

for all $x, x' \in X$.

4. A stability problem for commutative diagram (B)

We turn our attention to the commutative diagram of the from (B). Its Hyers-Ulam-Aoki-Rassias stability is obtained as an application of the preceding section.

Theorem 4.1. *Let X be a set and σ a bijective selfmap of X. Let (Y, d) be a complete metric space and τ be a continuous selfmap of Y. Fix two nonnegative functions ε, δ on X. Suppose that*

$$\beta_{\sigma^{-1},\varepsilon} < \infty, \quad \beta_{\sigma^{-1},\delta} < \infty, \quad \gamma_\tau < \infty \quad \text{and} \quad \gamma_\tau \max\{\beta_{\sigma^{-1},\varepsilon}, \beta_{\sigma^{-1},\delta}\} < 1.$$

If two mappings $f, g : X \to Y$ satisfy

$$d\big(f(\sigma x),\, \tau(g(x))\big) \le \varepsilon(x) \quad \text{and} \quad d\big(f(x), g(x)\big) \le \delta(x) \qquad (x \in X),$$

then there exists a unique mapping $f_\infty : X \to Y$ *such that*

$$\left.\begin{aligned} f_\infty(\sigma x) &= \tau(f_\infty(x)) \\ d\big(f(x), f_\infty(x)\big) &\le \frac{\beta_{\sigma^{-1},\varepsilon}\,\varepsilon(x) + \beta_{\sigma^{-1},\delta}\gamma_\tau\,\delta(x)}{1-\gamma_\tau \max\{\beta_{\sigma^{-1},\varepsilon}, \beta_{\sigma^{-1},\delta}\}} \\ d\big(g(x), f_\infty(x)\big) &\le \frac{\beta_{\sigma^{-1},\varepsilon}\,\varepsilon(x) + \delta(x)}{1-\gamma_\tau \max\{\beta_{\sigma^{-1},\varepsilon}, \beta_{\sigma^{-1},\delta}\}} \end{aligned}\right. \qquad (x \in X).$$

Proof. Define a binary operation $\circ$ on X by $x \circ x' = \sigma x$ for all $x, x' \in X$. Then $x \circ x = \sigma x$ holds for all $x \in X$, and so σ is the square operator on X, Also, σ is an automorphism of X, because $\sigma(x \circ x') = \sigma(\sigma x) = (\sigma x) \circ (\sigma x')$ for all $x, x' \in X$. Next, we define a binary operation $\diamond$ on Y by $y \diamond y' = \tau y$ for all $y, y' \in Y$. Clearly, $\diamond$ is continuous. Since $y \diamond y = \tau y$ for all $y \in Y$, τ is the square operator on Y. Also, τ is an endomorphism of Y, because $\tau(y \diamond y') = \tau(\tau y) = (\tau y) \diamond (\tau y')$ for all $y, y' \in Y$. Finally, we set $\tilde{\varepsilon}(x, x') = \varepsilon(x)$ for all $x, x' \in X$. Then $\alpha_{\sigma^{-1},\tilde{\varepsilon}} = \beta_{\sigma^{-1},\varepsilon}$. Thus we can apply Theorem 3.1 with replacing $\hat{\sigma}$ by σ, $\hat{\tau}$ by τ and ε by $\tilde{\varepsilon}$. The theorem is obtained immediately. □

In particular, if $\delta = 0$, then we have the following corollary:

Corollary 4.2. *Let X, σ and (Y, d), τ be as in Theorem 4.1. Take a nonnegative function ε on X. Suppose that $\beta_{\sigma^{-1},\varepsilon} < \infty$, $\gamma_\tau < \infty$ and $\beta_{\sigma^{-1},\varepsilon}\gamma_\tau < 1$. If a mapping $f : X \to Y$ satisfies $d\big(f(\sigma x), \tau(f(x))\big) \le \varepsilon(x)$ for all $x \in X$, then there exists a unique mapping $f_\infty : X \to Y$ such that*

$$f_\infty(\sigma x) = \tau(f_\infty(x)) \quad \textit{and} \quad d\big(f(x), f_\infty(x)\big) \le \frac{\beta_{\sigma^{-1},\varepsilon}}{1-\beta_{\sigma^{-1},\varepsilon}\gamma_\tau}\varepsilon(x)$$

for all $x \in X$.

The next theorem is the consequence of Theorem 3.4.

Theorem 4.3. *Let X be a set and σ a selfmap of X. Let (Y, d) be a complete metric space and τ be a bijective continuous selfmap of Y. Fix two nonnegative functions ε, δ on X. Suppose that*

$$\beta_{\sigma,\varepsilon} < \infty, \quad \beta_{\sigma,\delta} < \infty, \quad \gamma_{\tau^{-1}} < \infty \quad \textit{and} \quad \gamma_{\tau^{-1}} \max\{\beta_{\sigma,\varepsilon}, \beta_{\sigma,\delta}\} < 1.$$

If two mappings $f, g : X \to Y$ satisfy

$$d\big(f(\sigma x), \tau(g(x))\big) \le \varepsilon(x) \quad \textit{and} \quad d\big(f(x), g(x)\big) \le \delta(x) \qquad (x \in X),$$

then there exists a unique mapping $f_\infty : X \to Y$ such that

$$\left.\begin{aligned} f_\infty(\sigma x) &= \tau(f_\infty(x)) \\ d\big(f(x), f_\infty(x)\big) &\le \frac{\gamma_{\tau^{-1}}\,\varepsilon(x) + \delta(x)}{1-\gamma_{\tau^{-1}} \max\{\beta_{\sigma,\varepsilon}, \beta_{\sigma,\delta}\}} \\ d\big(g(x), f_\infty(x)\big) &\le \frac{\gamma_{\tau^{-1}}\big(\varepsilon(x) + \beta_{\sigma,\delta}\,\delta(x)\big)}{1-\gamma_{\tau^{-1}} \max\{\beta_{\sigma,\varepsilon}, \beta_{\sigma,\delta}\}} \end{aligned}\right. \qquad (x \in X).$$

Proof. Define a binary operation $\circ$ on X by $x \circ x' = \sigma x$ for all $x, x' \in X$. As we saw in the proof of Theorem 4.1, σ is the square operator on X and an endomorphism of X. Next, we define a binary operation $\diamond$ on Y by $y \diamond y' = \tau y$ for all $y, y' \in Y$.

Similarly, we see that $\diamond$ is continuous and that τ is the square operator on Y and an automorphism of Y. Finally, we set $\tilde{\varepsilon}(x,x') = \varepsilon(x)$ for all $x, x' \in X$. Then $\alpha_{\sigma,\tilde{\varepsilon}} = \beta_{\sigma,\varepsilon}$. Thus we can apply Theorem 3.4, replacing $\hat{\sigma}$ by σ, $\hat{\tau}$ by τ and ε by $\tilde{\varepsilon}$. The theorem is obtained immediately. ☐

In case $\delta = 0$, we have the following:

Corollary 4.4. *Let X, σ and (Y,d), τ be as in Theorem 4.3. Take a nonnegative function ε on X. Suppose that $\beta_{\sigma,\varepsilon} < \infty$, $\gamma_{\tau^{-1}} < \infty$ and $\beta_{\sigma,\varepsilon}\gamma_{\tau^{-1}} < 1$. If a mapping $f : X \to Y$ satisfies $d\big(f(\sigma x), \tau(f(x))\big) \le \varepsilon(x)$ for all $x \in X$, then there exists a unique mapping $f_\infty : X \to Y$ such that*

$$f_\infty(\sigma x) = \tau(f_\infty(x)) \quad \text{and} \quad d\big(f(x), f_\infty(x)\big) \le \frac{\gamma_{\tau^{-1}}}{1-\beta_{\sigma,\varepsilon}\gamma_{\tau^{-1}}}\varepsilon(x)$$

for all $x \in X$.

5. Application to Banach module, I

In Sections 4 and 5, we consider the case that X and Y are Banach modules, and we take up the mapping $f : X \to Y$ satisfying $f(ax + bx') = c\,f(x) + d\,f(x')$ for all $x, x' \in X$.

Theorem 5.1. *Let A be a unital commutative Banach algebra and X a normed A-module. Let B be a commutative Banach algebra and Y a Banach B-module. Fix $a, b \in A$, $c, d \in B$, $p, q, r \in \mathbb{R}$ and $\lambda, \kappa \ge 0$. Suppose that $a + b$ is invertible and*

$$\|c+d\| \max_{t=p,q,r} \|(a+b)^{-1}\|^t < 1. \tag{5.1}$$

If two mappings $f, g : X \to Y$ satisfy

$$\|f(ax+bx') - c\,g(x) - d\,g(x')\| \le \lambda\big(\|x\|^p + \|x'\|^q\big) \qquad (x, x' \in X), \tag{5.2}$$

$$\|f(x) - g(x)\| \le \kappa\,\|x\|^r \qquad (x \in X), \tag{5.3}$$

then there exists a unique mapping $f_\infty : X \to Y$ such that

(5.4)

$$f_\infty(ax+bx') = c\,f_\infty(x) + d\,f_\infty(x') \qquad (x, x' \in X),$$

$$\|f(x) - f_\infty(x)\| \le \frac{\lambda \max_{t=p,q} \|(a+b)^{-1}\|^t\big(\|x\|^p + \|x\|^q\big) + \kappa\,\|(a+b)^{-1}\|^r \|c+d\|\,\|x\|^r}{1 - \|c+d\| \max_{t=p,q,r} \|(a+b)^{-1}\|^t},$$

$$\|g(x) - f_\infty(x)\| \le \frac{\lambda \max_{t=p,q} \|(a+b)^{-1}\|^t\big(\|x\|^p + \|x\|^q\big) + \kappa\|x\|^r}{1 - \|c+d\| \max_{t=p,q,r} \|(a+b)^{-1}\|^t} \qquad (x \in X).$$

If the mapping $f_\infty : X \to Y$ satisfy (5.4), then we say that f_∞ is an (a,b,c,d)-*module homomorphism* from X to Y.

Proof. Define binary operations $\circ$ on X and $\diamond$ on Y by

$$x \circ x' = ax + bx' \quad (x, x' \in X) \quad \text{and} \quad y \diamond y' = cy + dy' \quad (y, y' \in Y). \tag{5.5}$$

Obviously, $\circ$ and $\diamond$ are continuous. Also, the corresponding square operators $\hat{\sigma}$ and $\hat{\tau}$ are given by

$$\hat{\sigma}x = x \circ x = (a+b)x \quad (x \in X) \quad \text{and} \quad \hat{\tau}y = y \diamond y = (c+d)y \quad (y \in Y). \tag{5.6}$$

Hence the easy computation shows that $\hat{\sigma}$ and $\hat{\tau}$ are endomorphisms of X and Y respectively. In particular, $\hat{\sigma}$ is an automorphism of X, because $a+b$ is invertible and $\hat{\sigma}^{-1}x = (a+b)^{-1}x$ for all $x \in X$. Next, we put

$$\varepsilon(x, x') = \lambda\big(\|x\|^p + \|x'\|^q\big) \quad \text{and} \quad \delta(x) = \kappa\,\|x\|^r \qquad (x, x' \in X). \tag{5.7}$$

Under these circumstances, we have

$$\begin{aligned}
\varepsilon(\hat{\sigma}^{-1}x, \hat{\sigma}^{-1}x') &= \lambda\big(\|(a{+}b)^{-1}x\|^p + \|(a{+}b)^{-1}x'\|^q\big) \\
&\le \lambda\big(\|(a{+}b)^{-1}\|^p\|x\|^p + \|(a{+}b)^{-1}\|^q\|x'\|^q\big) \\
&\le \lambda \max_{t=p,q} \|(a{+}b)^{-1}\|^t \left(\|x\|^p + \|x'\|^q\right) = \max_{t=p,q} \|(a{+}b)^{-1}\|^t\,\varepsilon(x, x')
\end{aligned}$$

for all $x, x' \in X$, and hence

$$\alpha_{\hat{\sigma}^{-1},\varepsilon} \le \max_{t=p,q} \|(a+b)^{-1}\|^t. \tag{5.8}$$

Similarly, we easily see that

$$\beta_{\hat{\sigma}^{-1},\delta} \le \|(a+b)^{-1}\|^r \quad \text{and} \quad \gamma_{\hat{\tau}} \le \|c+d\|. \tag{5.9}$$

Hence the inequalities (5.8), (5.9) and (5.1) imply $\gamma_{\hat{\tau}} \max\{\alpha_{\hat{\sigma}^{-1},\varepsilon}, \beta_{\hat{\sigma}^{-1},\delta}\} < 1$. Applying Theorem 3.1, we get the theorem at once. □

The next two corollaries describe the case $\kappa = 0$ or the case $\lambda = 0$ in Theorem 5.1:

Corollary 5.2. *Let A, X and B, Y be as in Theorem 5.1. Fix $a, b \in A$, $c, d \in B$, $p, q \in \mathbb{R}$ and $\lambda \ge 0$. Suppose that $a+b$ is invertible and $\|c+d\| \max_{t=p,q} \|(a+b)^{-1}\|^t < 1$. If a mapping $f : X \to Y$ satisfies $\|f(ax+bx') - c\,f(x) - d\,f(x')\| \le \lambda\big(\|x\|^p + \|x'\|^q\big)$ for all $x, x' \in X$, then there exists a unique (a, b, c, d)-module homomorphism $f_\infty : X \to Y$ such that*

$$\|f(x) - f_\infty(x)\| \le \frac{\lambda \max_{t=p,q} \|(a+b)^{-1}\|^t}{1 - \|c+d\| \max_{t=p,q} \|(a+b)^{-1}\|^t}\big(\|x\|^p + \|x\|^q\big) \qquad (x \in X).$$

Corollary 5.3. *Let A, X and B, Y be as in Theorem 5.1. Fix $a, b \in A$, $c, d \in B$, $r \in \mathbb{R}$ and $\kappa \ge 0$, Suppose that $a+b$ is invertible and $\|(a+b)^{-1}\|^r\,\|c+d\| < 1$. If two mappings $f, g : X \to Y$ satisfy $f(ax+bx') = c\,g(x) + d\,g(x')$ and $\|f(x) - g(x)\| \le \kappa\,\|x\|^r$ for all $x, x' \in X$, then there exists a unique (a, b, c, d)-module homomorphism $f_\infty : X \to Y$ such that*

$$\left.\begin{aligned}
\|f(x) - f_\infty(x)\| &\le \frac{\kappa\,\|(a+b)^{-1}\|^r\|c+d\|}{1 - \|(a+b)^{-1}\|^r\|c+d\|}\,\|x\|^r \\
\|g(x) - f_\infty(x)\| &\le \frac{\kappa}{1 - \|(a+b)^{-1}\|^r\|c+d\|}\,\|x\|^r
\end{aligned}\right\} \qquad (x \in X).$$

The next theorem holds similarly.

Theorem 5.4. *Let A be a commutative Banach algebra and X a normed A-module. Let B be a unital commutative Banach algebra and Y a Banach B-module. Fix $a, b \in A$, $c, d \in B$, $p, q, r \in \mathbb{R}$ and $\lambda, \kappa \geq 0$. Suppose that $c + d$ is invertible and*

$$\max_{t=p,q,r} \|a+b\|^t < \frac{1}{\|(c+d)^{-1}\|}.$$

If two mappings $f, g : X \to Y$ satisfy (5.2) and (5.3), then there exists a unique (a, b, c, d)-module homomorphism $f_\infty : X \to Y$ such that

$$\left.\begin{aligned} \|f(x)-f_\infty(x)\| &\leq \frac{\lambda\,\|(c+d)^{-1}\|\big(\|x\|^p+\|x\|^q\big)+\kappa\,\|x\|^r}{1-\|(c+d)^{-1}\|\,\max\limits_{t=p,q,r}\|a+b\|^t} \\ \|g(x)-f_\infty(x)\| &\leq \frac{\|(c+d)^{-1}\|\,\big[\lambda\big(\|x\|^p+\|x\|^q\big)+\kappa\,\|a+b\|^r\|x\|^r\big]}{1-\|(c+d)^{-1}\|\,\max\limits_{t=p,q,r}\|a+b\|^t} \end{aligned}\right\} \quad (x \in X).$$

Proof. Recall the proof of Theorem 5.1. If we define two continuous binary operations $\circ$ on X and $\diamond$ on Y by (5.5), then the corresponding square operators $\hat{\sigma}$ and $\hat{\tau}$ are given by (5.6) and become endomorphisms of X and Y respectively. This time, $\hat{\tau}$ is an automorphism of Y with the inverse $\hat{\tau}^{-1}y = (c+d)^{-1}y$ for all $y \in Y$. If we define ε and δ by (5.7), then we have

$$\alpha_{\hat{\sigma},\varepsilon} \leq \max_{t=p,q} \|a+b\|^t, \quad \beta_{\hat{\sigma},\delta} \leq \|a+b\|^r \quad \text{and} \quad \gamma_{\hat{\tau}^{-1}} \leq \|(c+d)^{-1}\|.$$

Applying Theorem 3.4 instead of Theorem 3.1. we arrive at the theorem. □

The next two corollaries describe the case $\kappa = 0$ or the case $\lambda = 0$ in Theorem 5.4:

Corollary 5.5. *Let A, X and B, Y be as in Theorem 5.4. Fix $a, b \in A$, $c, d \in B$, $p, q \in \mathbb{R}$ and $\lambda \geq 0$, Suppose that $c + d$ is invertible and $\max_{t=p,q} \|a+b\|^t < 1/\|(c+d)^{-1}\|$. If a mapping $f : X \to Y$ satisfies $\|f(ax+bx') - c\,f(x) - d\,f(x')\| \leq \lambda\big(\|x\|^p + \|x'\|^q\big)$ for all $x, x' \in X$, then there exists a unique (a, b, c, d)-module homomorphism $f_\infty : X \to Y$ such that*

$$\|f(x) - f_\infty(x)\| \leq \frac{\lambda\,\|(c+d)^{-1}\|}{1 - \|(c+d)^{-1}\|\,\max_{t=p,q}\|a+b\|^t}\big(\|x\|^p + \|x\|^q\big) \qquad (x \in X).$$

Corollary 5.6. *Let A, X and B, Y be as in Theorem 5.4. Fix $a, b \in A$, $c, d \in B$, $r \in \mathbb{R}$ and $\kappa \geq 0$, Suppose that $c+d$ is invertible and $\|a+b\|^r\|(c+d)^{-1}\| < 1$. If two mappings $f, g : X \to Y$ satisfy $f(ax+bx') = c\,g(x) + d\,g(x')$ and $\|f(x) - g(x)\| \leq \kappa\,\|x\|^r$ for all $x, x' \in X$, then there exists a unique (a, b, c, d)-module homomorphism $f_\infty : X \to Y$ such that*

$$\left.\begin{aligned} \|f(x) - f_\infty(x)\| &\leq \frac{\kappa}{1 - \|a+b\|^r\|(c+d)^{-1}\|}\,\|x\|^r \\ \|g(x) - f_\infty(x)\| &\leq \frac{\kappa\,\|a+b\|^r\|(c+d)^{-1}\|}{1 - \|a+b\|^r\|(c+d)^{-1}\|}\,\|x\|^r \end{aligned}\right\} \quad (x \in X).$$

Let us consider the case that $A = B = \mathbb{C}$ and $a = b = c = d = 1$ and $p = q$. Then Corollary 5.2 leads to the result of Z. Gajda [7] : If $p > 1$ and if $\|f(x + x') - f(x) - f(x')\| \le \lambda(\|x\|^p + \|x'\|^p)$ for all $x, x' \in X$, then there exists a unique additive mapping f_∞ such that

$$\|f(x) - f_\infty(x)\| \le \frac{\lambda}{2^{p-1} - 1}\|x\|^p \qquad (x \in X).$$

Also, Corollary 5.5 yields the following result: If $p < 1$ and if $\|f(x + x') - f(x) - f(x')\| \le \lambda(\|x\|^p + \|x'\|^p)$ for all $x, x' \in X$, then there exists a unique additive mapping f_∞ such that

$$\|f(x) - f_\infty(x)\| \le \frac{\lambda}{1 - 2^{p-1}}\|x\|^p \qquad (x \in X).$$

This is due to D. H. Hyers [8] if $p = 0$, due to T. Aoki [2] or Th. M. Rassias [15] if $0 < p < 1$, and due to Miura, Hirasawa and Takahasi [14] if $p < 0$.

6. Application to Banach module, II

Even if we change the term $\|x\|^p + \|x'\|^q$ into $\|x\|^p\|x'\|^q$ in Section 4, we can discuss in the same way. The easy modification yields two theorems and two corollaries below.

Theorem 6.1. *Let A, X, B, Y, a, b, c, d, p, q, r and λ, κ be as in Theorem 5.1. Suppose that $a + b$ is invertible and*

$$\|c + d\| \max_{t=p+q,r} \|(a + b)^{-1}\|^t < 1.$$

If two mappings $f, g : X \to Y$ satisfy

$$\|f(ax + bx') - c\,g(x) - d\,g(x')\| \le \lambda\,\|x\|^p\|x'\|^q \qquad (x, x' \in X), \tag{6.1}$$

$$\|f(x) - g(x)\| \le \kappa\,\|x\|^r \qquad (x \in X), \tag{6.2}$$

then there exists a unique (a, b, c, d)-module homomorphism $f_\infty : X \to Y$ such that

$$\left.\begin{aligned} \|f(x)-f_\infty(x)\| &\le \frac{\lambda\,\|(a+b)^{-1}\|^{p+q}\|x\|^{p+q} + \kappa\|(a+b)^{-1}\|^r\|c+d\|\|x\|^r}{1 - \|c + d\| \max_{t=p+q,r} \|(a + b)^{-1}\|^t} \\ \|g(x)-f_\infty(x)\| &\le \frac{\lambda\,\|(a + b)^{-1}\|^{p+q}\|x\|^{p+q} + \kappa\,\|x\|^r}{1 - \|c + d\| \max_{t=p+q,r} \|(a + b)^{-1}\|^t} \end{aligned}\right\} \quad (x \in X).$$

Proof. Modify the proof of Theorem 5.1. In (5.7), we replace the first equation by $\varepsilon(x, x') = \lambda\,\|x\|^p\|x'\|^q$ for all $x, x' \in X$. Then the inequality (5.8) changes into $\alpha_{\hat{\sigma}^{-1},\varepsilon} \le \|(a + b)^{-1}\|^{p+q}$. Thus the theorem follows in the same way. □

Corollary 6.2. *Let A, X, B, Y, a, b, c, d, p, q and λ be as in Corollary 5.2. Suppose that $a + b$ is invertible and $\|(a + b)^{-1}\|^{p+q}\|c + d\| < 1$. If a mapping $f : X \to Y$ satisfies $\|f(ax + bx') - c\,f(x) - d\,f(x')\| \le \lambda\,\|x\|^p\|x'\|^q$ for all $x, x' \in X$, then there exists a unique (a, b, c, d)-module homomorphism $f_\infty : X \to Y$ such that*

$$\|f(x) - f_\infty(x)\| \le \frac{\lambda\,\|(a + b)^{-1}\|^{p+q}}{1 - \|(a + b)^{-1}\|^{p+q}\|c + d\|}\|x\|^{p+q} \qquad (x \in X).$$

Theorem 6.3. *Let A, X, B, Y, a, b, c, d, p, q, r and λ, κ be as in Theorem 5.4. Suppose that $c + d$ is invertible and*

$$\max_{t=p+q,r} \|(a+b)\|^t < \frac{1}{\|(c+d)^{-1}\|}.$$

If two mappings $f, g : X \to Y$ satisfy (6.1) and (6.2), then there exists a unique (a, b, c, d)-module homomorphism $f_\infty : X \to Y$ such that

$$\left.\begin{aligned} \|f(x) - f_\infty(x)\| &\le \frac{\lambda \|(c+d)^{-1}\| \, \|x\|^{p+q} + \kappa \, \|x\|^r}{1 - \|(c+d)^{-1}\| \max\limits_{t=p+q,r} \|a+b\|^t} \\ \|g(x) - f_\infty(x)\| &\le \frac{\|(c+d)^{-1}\| \left[\lambda \, \|x\|^{p+q} + \kappa \, \|a+b\|^r \|x\|^r\right]}{1 - \|(c+d)^{-1}\| \max\limits_{t=p+q,r} \|a+b\|^t} \end{aligned}\right\} \quad (x \in X).$$

Corollary 6.4. *Let A, X, B, Y, a, b, c, d, p, q and λ be as in Corollary 5.5. Suppose that $c + d$ is invertible and $\|a + b\|^{p+q} < 1/\|(c + d)^{-1}\|$. If a mapping $f : X \to Y$ satisfies $\|f(ax + bx') - c\, f(x) - d\, f(x')\| \le \lambda \, \|x\|^p \|x'\|^q$ for all $x, x' \in X$, then there exists a unique (a, b, c, d)-module homomorphism $f_\infty : X \to Y$ such that*

$$\|f(x) - f_\infty(x)\| \le \frac{\lambda \, \|(c+d)^{-1}\|}{1 - \|a+b\|^{p+q} \|(c+d)^{-1}\|} \|x\|^{p+q} \qquad (x \in X).$$

Let us consider the case that $A = B = \mathbb{C}$ and $a = b = c = d = 1$. Then Corollary 6.2 leads to the following result: If $p + q > 1$ and if $\|f(x + x') - f(x) - f(x')\| \le \lambda \, \|x\|^p \|x'\|^q$ for all $x, x' \in X$, then there exists a unique additive mapping f_∞ such that

$$\|f(x) - f_\infty(x)\| \le \frac{\lambda}{2^{p+q} - 2} \|x\|^{p+q} \qquad (x \in X).$$

Also, Corollary 6.4 yields the following result: If $p+q < 1$ and if $\|f(x+x') - f(x) - f(x')\| \le \lambda \, \|x\|^p \|x'\|^q$ for all $x, x' \in X$, then there exists a unique additive mapping f_∞ such that

$$\|f(x) - f_\infty(x)\| \le \frac{\lambda}{2 - 2^{p+q}} \|x\|^{p+q} \qquad (x \in X).$$

7. The other applications

Finally, we record two simple corollaries of Theorem 3.4.

Corollary 7.1. *Let X be a set with a binary operation $\circ$ satisfying $(x \circ x) \circ (x' \circ x') = (x \circ x') \circ (x \circ x')$ for all $x, x' \in X$. Write $[1, \infty) = \{x \in \mathbb{R} : x \ge 1\}$ and take $\varepsilon, \delta \ge 0$. If two functions $f, g : X \to [1, \infty)$ satisfy*

$$|f(x \circ x') - g(x)g(x')| \le \varepsilon \quad \text{and} \quad |f(s) - g(s)| \le \delta \qquad (x, x' \in X),$$

then there exists a unique function $f_\infty : X \to [1, \infty)$ such that

$$f_\infty(x \circ x') = f_\infty(x)\, f_\infty(x') \qquad (x, x' \in X),$$
$$|f(x) - f_\infty(x)| \le \varepsilon + 2\delta \quad \text{and} \quad |g(x) - f_\infty(x)| \le \varepsilon + \delta \qquad (x \in X).$$

Proof. By the property of $\circ$, the square operator $\hat{\sigma} : x \mapsto x \circ x$ is an endomorphism of X. If we put $y \diamond y' = yy'$ for all $y, y' \in [1, \infty)$, then $\diamond$ is a continuous binary operation on $[0, \infty)$, and the square operator $\hat{\tau} : y \mapsto y \diamond y = y^2$ is an automorphism of $[1, \infty)$ with the inverse $\hat{\tau}^{-1} y = \sqrt{y}$. Also, we understand ε and δ to be the constant functions on $S \times S$ and S, respectively. Then we can easily see that $\alpha_{\hat{\sigma},\varepsilon} \leq 1$, $\beta_{\hat{\sigma},\delta} \leq 1$ and

$$\gamma_{\hat{\tau}^{-1}} = \sup\left\{ \frac{|\hat{\tau}^{-1}y - \hat{\tau}^{-1}y'|}{|y - y'|} : \begin{matrix} y, y' \in [1, \infty) \\ y \neq y' \end{matrix} \right\} = \sup\left\{ \frac{1}{\sqrt{y} + \sqrt{y'}} : \begin{matrix} y, y' \in [1, \infty) \\ y \neq y' \end{matrix} \right\} = \frac{1}{2}.$$

These imply that $\gamma_{\hat{\tau}^{-1}} \max\{\alpha_{\hat{\sigma},\varepsilon}, \beta_{\hat{\sigma},\delta}\} \leq 1/2 < 1$. Thus the corollary follows from Theorem 3.4. □

Corollary 7.2. *Let X be a set with a binary operation $\circ$ satisfying $(x \circ x) \circ (x' \circ x') = (x \circ x') \circ (x \circ x')$ for all $x, x' \in X$. Take $\varepsilon, \delta \geq 0$. If two complex functions f, g on X satisfy*

$$|f(x \circ x') - g(x) - g(x')| \leq \varepsilon \quad \text{and} \quad |f(x) - g(x)| \leq \delta \qquad (x, x' \in X),$$

then there exists a unique complex function f_∞ on X such that

$$f_\infty(x \circ x') = f_\infty(x) + f_\infty(x') \qquad (x, x' \in X),$$
$$|f(x) - f_\infty(x)| \leq \varepsilon + 2\delta \quad \text{and} \quad |g(x) - f_\infty(x)| \leq \varepsilon + \delta \qquad (x \in X).$$

Proof. Take a binary operation $\diamond$ on $\mathbb{C}$ as the usual addition $+$ on $\mathbb{C}$. Then the square operator $\hat{\tau} : y \mapsto y \diamond y = y + y = 2y$ on $\mathbb{C}$ is clearly an automorphism of $(\mathbb{C}, +)$ and its inverse is given by $\hat{\tau}^{-1} y = y/2$. Employing the argument in the proof of Corollary 7.1, we obtain the desired result. □

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